PROFESSOR: So, finally, before I get started on the new stuff, questions from the previous lectures? No questions? Yeah.

AUDIENCE: I have a question. You might have said this last time, but when is the first exam?

PROFESSOR: Ah, excellent. Those will be posted on the Stellar page later today. Yeah.

AUDIENCE: OK, so we're associating operators with observables, right?

PROFESSOR: Yes.

AUDIENCE: And Professor [? Zugoff ?] mentioned that whenever we have done a wave function with an operator, it collapses.

PROFESSOR: OK, so let me rephrase the question. This is a very valuable question to talk through. So, thanks for asking it.

So, we've previously observed that observables are associated with operators-- and we'll review that in more detail in a second-- and the statement was then made, does that mean that acting on a wave function with an operator is like measuring the observable?

And it's absolutely essential that you understand that acting on a wave function with an operator has nothing whatsoever to do with measuring that associated observable. Nothing. OK? And we'll talk about the relationship and what those things mean.

But here's a very tempting thing to think. I have a wave function. I want to know the momentum. I will thus operate with the momentum operator. Completely wrong. So, before I even tell you what the right statement is, let me just get that out of your

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head, and then we'll talk through that in much more detail over the next lecture. Yeah.

AUDIENCE: Why doesn't it collapse by special relativity?

PROFESSOR: We're doing everything non-relativistically. Quantum Mechanics for 804 is going to be a universe in which there is no relativity. If you ask me that more precisely in my office hours, I will tell you a relativistic story. But it doesn't violate anything relativistic. At all.

We'll talk about that-- just to be a little more detailed-- that will be a very important question that we'll deal with in the last two lectures of the course, when we come back to Bell's inequality and locality. Other questions? OK.

So, let's get started. So, just to review where we are. In Quantum Mechanics according to 804, our first pass at the definition of quantum mechanics is that the configuration of any system-- and in particular, think about a single point particle-- the configuration of our particle is specified by giving a wave function, which is a function which may depend on time, but a function of position.

Observables-- and this is a complete specification of the state of the system. If I know the wave function, I neither needed nor have access to any further information about the system. All the information specifying the configuration system is completely contained in the wave function.

Secondly, observables in quantum mechanics are associated with operators. Something you can build an experiment to observe or to measure is associated with an operator. And by an operator, I mean a rule or a map, something that tells you if you give me a function, I will give you a different function back. OK? An operator is just a thing which eats a function and spits out another function.

Now, operators-- which I will denote with a hat, as long as I can remember to do so-- operators come-- and in particular, the kinds of operators we're going to care about, linear operators, which you talked about in detail last lecture-- linear

operators come endowed with a natural set of special functions called Eigenfunctions with the following property.

Your operator, acting on its Eigenfunction, gives you that same function back times a constant. So, that's a very special generically. An operator will take a function and give you some other random function that doesn't look all like the original function. It's a very special thing to give you the same function back times a constant.

So, a useful thing to think about here is just in the case of vector spaces. So, I'm going to consider the operation corresponding to rotation around the z-axis by a small angle. OK? So, under rotation around the z-axis by a small angle, I take an arbitrary vector to some other stupid vector.

Which vector is completely determined by the rule? I rotate by a small amount, right? I take this vector and it gives me this one. I take that vector, it gives me this one. Everyone agree with that? What are the Eigenvectors of the rotation by a small angle around the z-axis?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, it's got to be a vector that doesn't change its direction. It just changes by magnitude. So there's one, right? I rotate. And what's its Eigenvalue?

AUDIENCE: One.

PROFESSOR: One, because nothing changed, right? Now, let's consider the following operation. Rotate by small angle and double its length. OK, that's a different operator. I rotate and I double the length. I rotate and I double the length. I rotate and I double the length. Yeah, so what's the Eigenvalue under that operator?

AUDIENCE: Two.

PROFESSOR: Two. Right, exactly. So these are a very special set of functions. This is the same idea, but instead of having vectors, we have functions. Questions? I thought I saw a hand pop up. No? OK, cool.

Third, superposition. Given any two viable wave functions that could describe our system, that could specify states or configurations of our system, an arbitrary superposition of them-- arbitrary linear sum-- could also be a valid physical configuration. There is also a state corresponding to being in an arbitrary sum.

For example, if we know that the electron could be black and it could be white, it could also be in an arbitrary superposition of being black and white. And that is a statement in which the electron is not black. The electron is not white. It is in the superposition of the two. It does not have a definite color. And that is exactly the configuration we found inside our apparatus in the first lecture. Yeah.

AUDIENCE: Are those Phi-A arbitrary functions, or are they supposed to be Eigenfunctions?

PROFESSOR: Excellent. So, in general the superposition thank you. It's an excellent question. The question was are these Phi-As arbitrary functions, or are they specific Eigenfunctions of some operator?

So, the superposition principle actually says a very general thing. It says, given any two viable wave functions, an arbitrary sum, an arbitrary linear combination, is also a viable wave function. But here I want to mark something slightly different. And this is why I chose the notation I did.

Given an operator A, it comes endowed with a special set of functions, its Eigenfunctions, right? We saw the last time. And I claimed the following. Beyond just the usual superposition principle, the set of Eigenfunctions of operators corresponding to physical observables-- so, pick your observable, like momentum. That corresponds to an operator.

Consider the Eigenfunctions of momentum. Those we know what those are. They're plane waves with definite wavelength, right? u to the ikx. Any function can be expressed as a superposition of those Eigenfunctions of your physical observable. We'll go over this in more detail in a minute.

But here I want to emphasize that the Eigenfunctions have a special property that-- for observables, for operators corresponding to observables-- the Eigenfunctions

form a basis. Any function can be expanded as some linear combination of these basis functions, the classic example being the Fourier expansion. Any function, any periodic function, can be expanded as a sum of sines and cosines, and any function on the real line can be expanded as a sum of exponentials, e to the ikx.

This is the same statement. The Eigenfunctions of momentum are what? e to the ikx. So, this is the same that an arbitrary function-- when the observable is the momentum, this is the statement that an arbitrary function can be expanded as a superposition, or a sum of exponentials, and that's the Fourier theorem. Cool? Was there a question?

AUDIENCE: [INAUDIBLE]

PROFESSOR: OK, good. Other questions on these points? So, these should not yet be trivial and obvious to you. If they are, then that's great, but if they're not, we're going to be working through examples for the next several lectures and problem sets. The point now is to give you a grounding on which to stand.

Fourth postulate. What these expansion coefficients mean. And this is also an interpretation of the meaning of the wave function. What these expansion coefficients mean is that the probability that I measure the observable to be a particular Eigenvalue is the norm squared of the expansion coefficient. OK?

So, I tell you that any function can be expanded as a superposition of plane waves-- waves with definite momentum-- with some coefficients. And those coefficients depend on which function I'm talking about. What these coefficients tell me is the probability that I will measure the momentum to be the associated value, the Eigenvalue. OK? Take that coefficient, take its norm squared, that gives me the probability.

How do we compute these expansion coefficients? I think Barton didn't introduce to you this notation, but he certainly told you this. So let me introduce to you this notation which I particularly like. We can extract the expansion coefficient if we know the wave function by taking this integral, taking the wave function, multiplying by the

complex conjugate of the associated Eigenfunction, doing the integral. And that notation is this round brackets with Phi A and Psi is my notation for this integral. And again, we'll still see this in more detail later on.

And finally we have collapse, the statement that, if we go about measuring some observable A, then we will always, always observe precisely one of the Eigenvalues of that operator. We will never measure anything else. If the Eigenvalues are one, two, three, four, and five, you will never measure half, 13 halves. You will always measure an Eigenvalue.

And upon measuring that Eigenvalue, you can be confident that that's the actual value of the system. I observe that it's a white electron, then it will remain white if I subsequently measure its color. What that's telling you is it's no longer a superposition of white and black, but it's wave function is that corresponding to a definite value of the observable.

So, somehow the process of measurement-- and this is a disturbing statement, to which we'll return-- somehow the process of measuring the observable changes the wave function from our arbitrary superposition to a specific Eigenfunction, one particular Eigenfunction of the operator we're measuring.

And this is called the collapse of the wave function. It collapses from being a superposition over possible states to being in a definite state upon measurement. And the definite state is that state corresponding to the value we observed or measured. Yeah.

AUDIENCE: So, when the wave function collapses, does it instantaneously not become a function of time anymore? Because originally we had Psi of (x,t).

PROFESSOR: Yeah, that's a really good question. So I wrote this only in terms of position, but I should more precisely write. So, the question was, does this happen instantaneously, or more precisely, does it cease to be a function of time? Thank you. It's very good question.

So, no, it doesn't cease to be a function of time. It just says that Psi at x-- what you

know upon doing this measurement is that Psi, as a function of x, at the time which I'll call T star, at what you've done the measurement is equal to this wave function. And so that leaves us with the following question, which is another way of asking the question you just asked. What happens next? How does the system evolve subsequently?

And at the very end of the last lecture, we answered that-- or rather, Barton answered that-- by introducing the Schrodinger equation. And the Schrodinger equation, we don't derive, we just posit. Much like Newton posits f equals ma. You can motivate it, but you can't derive it. It's just what we mean by the quantum mechanical model.

And Schrodinger's equation says, given a wave function, I can determine the time derivative, the time rate of changes of that wave function, and determine its time evolution, and its time derivative, its slope-- its velocity, if you will-- is one upon I h bar, the energy operator acting on that wave function.

So, suppose we measure that our observable capital A takes the value of little a, one of the Eigenvalues of the associated operators. Suppose we measure that A equals little a at some particular moment T start. Then we know that the wave function is Psi of x at that moment in time. We can then compute the time derivative of the wave function at that moment in time by acting on this wave function with the operator e hat, the energy operator. And we can then integrate that differential equation forward in time and determine how the wave function evolves.

The point of today's lecture is going to be to study how time evolution works in quantum mechanics, and to look at some basic examples and basic strategies for solving the time evolution problem in quantum mechanics. One of the great surprises in quantum mechanics-- hold on just one sec-- one of the real surprises in quantum mechanics is that time evolution is in a very specific sense trivial in quantum mechanics. It's preposterously simple.

In particular, time evolution is governed by a linear equation. How many of you have studied a classical mechanical system where the time evolution is governed by a

linear equation? Right. OK, all of you. The harmonic oscillator. But otherwise, not at all.

Otherwise, the equations in classical mechanics are generically highly nonlinear. The time rate of change of position of a particle is the gradient of the force, and the force is generally some complicated function of position. You've got some capacitors over here, and maybe some magnetic field. It's very nonlinear.

Evolution in quantum mechanics is linear, and this is going to be surprising. It's going to lead to some surprising simplifications. And we'll turn back to that, but I want to put that your mind like a little hook, that that's something you should mark on to as different from classical mechanics. And we'll come back to that. Yeah.

AUDIENCE: If a particle is continuously observed as a not evolving particle?

PROFESSOR: That's an awesome question. The question is, look, imagine I observe-- I'm going to paraphrase-- imagine I observe a particle and I observe that it's here. OK? Subsequently, its wave function will evolve in some way-- and we'll actually study that later today-- its wave function will evolve in some way, and it'll change. It won't necessarily be definitely here anymore.

But if I just keep measuring it over and over and over again, I just keep measure it to be right there. It can't possibly evolve. And that's actually true, and it's called the Quantum Zeno problem. So, it's the observation that if you continuously measure a thing, you can't possibly have its wave function evolve significantly.

And not only is it a cute idea, but it's something people do in the laboratory. So, Martin-- well, OK. People do it in a laboratory and it's cool. Come ask me and I'll tell you about the experiments. Other questions? There were a bunch. Yeah.

AUDIENCE: So after you measure, the Schrodinger equation also gives you the evolution backwards in time?

PROFESSOR: Oh, crap! Yes. That's such a good question. OK. I hate it when people ask that at this point, because I had to then say more words. That's a very good question. So

the question goes like this. So this was going to be a punchline later on in the in the lecture but you stole my thunder, so that's awesome.

So, here's the deal. We have a rule for time evolution of a wave function, and it has some lovely properties. In particular-- let me talk through this-- in particular, this equation is linear. So what properties does it have? Let me just-- I'm going to come back to your question in just a second, but first I want to set it up so we have a little more meat to answer your question precisely.

So we note some properties of this equation, this time evolution equation. The first is that it's a linear equation. The derivative of a sum of function is a sum of the derivatives. The energy operator's a linear operator, meaning the energy operator acting on a sum of functions is a sum of the energy operator acting on each function. You guys studied linear operators in your problem set, right?

So, these are linear. What that tells you is if Psi 1 of x and t solves the Schrodinger equation, and Psi 2 of x and t-- two different functions of position in time-- both solve the Schrodinger equation, then any combination of them-- alpha Psi 1 plus Beta Psi 2-- also solves-- which I will call Psi, and I'll make it a capital Psi for fun-- solves the Schrodinger equation automatically. Given two solutions of the Schrodinger equation, a superposition of them-- an arbitrary superposition-- also solves the Schrodinger equation. This is linearity. Cool?

Next property. It's unitary. What I mean by unitary is this. It concerns probability. And you'll give a precise derivation of what I mean by unitary and you'll demonstrate that, in fact, Schrodinger evolution is unitary on your next problem set. It's not on the current one. But what I mean by unitary is that conserves probability. Whoops, that's an o. Conserves probability.

IE, if there's an electron here, or if we have an object, a piece of chalk-- which I'm treating as a quantum mechanical point particle-- it's described by the wave function. The integral, the probability distribution over all the places it could possibly be had better be one, because it had better be somewhere with probability one. That had better not change in time.

If I solve the Schrodinger equation evolve the system forward for half an hour, it had better not be the case that the total probability of finding the particle is one half. That means things disappear in the universe. And much as my socks would seem to be a counter example of that, things don't disappear, right? It just doesn't happen.

So, quantum mechanics is demonstrably-- well, quantum mechanics is unitary, and this is a demonstrably good description of the real world. It fits all the observations we've ever made. No one's ever discovered an experimental violation of unitarity of quantum mechanics.

I will note that there is a theoretical violation of unitarity in quantum mechanics, which is dear to my heart. It's called the Hawking Effect, and it's an observation that, due quantum mechanics, black holes in general relativity-- places from which light cannot escape-- evaporate.

So you throw stuff and you form a black hole. It's got a horizon. If you fall through that horizon, we never see you again. Surprisingly, a black hole's a hot object like an iron, and it sends off radiation. As it sends off radiation, it's losing its energy. It's shrinking. And eventually it will, like the classical atom, collapse to nothing. There's a quibble going on right now over whether it really collapses to nothing, or whether there's a little granule nugget of quantum goodness.

[LAUGHTER]

We argue about this. We get paid to argue about this.

[LAUGHTER]

So, but here's the funny thing. If you threw in a dictionary and then the black hole evaporates, where did the information about what made the black hole go if it's just thermal radiation coming out? So, this is a classic calculation, which to a theorist says, ah ha! Maybe unitarity isn't conserved.

But, look. Black holes, theorists. There's no experimental violation of unitarity anywhere. And if anyone ever did find such a violation, it would shatter the basic

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tenets of quantum mechanics, in particular the Schrodinger equation. So that's something we would love to see but never have. It depends on your point of view. You might hate to see it.

And the third-- and this is, I think, the most important-- is that the Schrodinger evolution, this is a time derivative. It's a differential equation. If you know the initial condition, and you know the derivative, you can integrate it forward in time. And they're existence and uniqueness theorems for this.

The system is deterministic. What that means is that if I have complete knowledge of the system at some moment in time, if I know the wave function at some moment in time, I can determine unambiguously the wave function in all subsequent moments of time. Unambiguously.

There's no probability, there's no likelihood, it's determined. Completely determined. Given full knowledge now, I will have full knowledge later. Does everyone agree that this equation is a deterministic equation in that sense? Question.

AUDIENCE: It's also local?

PROFESSOR: It's all-- well, OK. This one happens to be-- you need to give me a better definition of local. So give me a definition of local that you want.

AUDIENCE: The time evolution of the wave function happens only at a point that depends only on the value of the derivatives of the wave function and its potential energy at that point.

PROFESSOR: No. Unfortunately, that's not the case. We'll see counter examples of that. The wave function-- the energy operator. So let's think about what this equation says. What this says is the time rate of change of the value of the wave function at some position and some moment in time is the energy operator acting on Psi at x of t.

But I didn't tell you what the energy operator is. The energy operator just has to be linear. But it doesn't have to be-- it could know about the wave function everywhere. The energy operator's a map that takes the wave function and tells you what it

should be later. And so, at this level there's nothing about locality built in to the energy operator, and we'll see just how bad that can be.

So, this is related to your question about special relativity, and so those are deeply intertwined. We don't have that property here yet. But keep that in your mind, and ask questions when it seems to come up. Because it's a very, very, very important question when we talk about relativity. Yeah.

AUDIENCE: Are postulates six and three redundant if the Schrodinger equation has superposition in it?

PROFESSOR: No. Excellent question. That's a very good question. The question is, look, there's postulate three, which says, given any two wave functions that are viable wave functions of the system, then there's another state which is a viable wave function at some moment in time, which is also a viable wave function.

But number six, the Schrodinger equation-- or sorry, really the linearity property of the Schrodinger equation-- so it needs to be the case for the Schrodinger question, but it says something slightly different. It doesn't just say that any any plausible or viable wave function and another can be superposed. It says that, specifically, any solution of the Schrodinger equation plus any other solution of the Schrodinger equation is again the Schrodinger operation.

So, it's a slightly more specific thing than postulate three. However, your question is excellent because could it have been that the Schrodinger evolution didn't respect superposition?

Well, you could imagine something, sure. We could've done a differ equation, right? It might not have been linear. We could have had that Schrodinger equation was equal to dt Psi. So imagine this equation. How do we have blown linearity while preserving determinism? So we could have added plus, I don't know, PSI squared of x.

So that would now be a nonlinear equation. It's actually refer to as the nonlinear Schrodinger equation. Well, people mean many different things by the nonlinear

Schrodinger equation, but that's a nonlinear Schrodinger equation. So you could certainly write this down. It's not linear.

Does it violate the statement three that any two states of the system could be superposed to give another viable state at a moment in time? No, right? It doesn't directly violate. It violates the spirit of it.

And as we'll see later, it actually would cause dramatic problems. It's something we don't usually emphasize-- something I don't usually emphasize in lectures of 804, but I will make a specific effort to mark the places where this would cause disasters. But, so this is actually a logically independent, although morally-- and in some sense is a technically related point to the superposition principle number three. Yeah.

AUDIENCE: For postulate three, can that sum be infinite sum?

PROFESSOR: Absolutely.

AUDIENCE: Can you do bad things, then, like creating discontinuous wave functions?

PROFESSOR: Oh yes. Oh, yes you can. So here's the thing. Look, if you have two functions and you add them together-- like two smooth continuous functions, you add them together-- what do you get? You get another smooth continuous function, right? Take seven. You get another. But if you take an infinite number-- look, mathematicians are sneaky. There's a reason we keep them down that hall, far away from us.

[LAUGHTER]

They're very sneaky. And if you give them an infinite number of continuous functions, they'll build for you a discontinuous function, right? Sneaky. Does that seem terribly physical? No. It's what happens when you give a mathematician too much paper and time, right?

So, I mean this less flippantly than I'm saying it, but it's worth being a little flippant here. In a physical setting, we will often find that there are effectively an infinite

number of possible things that could happen. So, for example in this room, where is this piece of chalk? It's described by a continuous variable. That's an uncountable infinite number of positions.

Now, in practice, you can't really build an experiment that does that, but it is in principle an uncountable infinity of possible positions, right? You will never get a discontinuous wave function for this guy, because it would correspond to divergent amounts of momentum, as you showed on the previous problem set.

So, in general, we will often be in a situation as physicists where there's the possibility of using the machinery-- the mathematical machinery-- to create pathological examples. And yes, that is a risk. But physically it never happens. Physically it's extraordinarily rare that such infinite divergences could matter.

Now, I'm not saying that they never do. But we're going to be very carefree and casual in 804 and just assume that when problems can arise from, say, superposing an infinite number of smooth functions, leading potentially to discontinuities or singularities, that they will either not happen for us-- not be relevant-- or they will happen because they're forced too, so for physical reasons we'll be able to identify.

So, this is a very important point. We're not proving mathematical theorems. We're not trying to be rigorous. To prove a mathematical theorem you have to look at all the exceptional cases and say, those exceptional cases, we can deal with them mathematically.

To a physicist, exceptional cases are exceptional. They're irrelevant. They don't happen. It doesn't matter. OK? And it doesn't mean that we don't care about the mathematical precision, right? I mean, I publish papers in math journals, so I have a deep love for these questions.

But they're not salient for most of the physical questions we care about. So, do your best to try not to let those special cases get in the way of your understanding of the general case. I don't want you to not think about them, I just want you not let them stop you, OK? Yeah.

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AUDIENCE: So, in postulate five, you mentioned that [? functions ?] in effect was a experiment that more or less proves this collapse [INAUDIBLE]

But, so I read that it is not [? complicit. ?]

PROFESSOR: Yeah, so as with many things in quantum mechanics-- that's a fair question. So, let me make a slightly more general statement than answering that question directly.

Many things will-- how to say-- so, we will not prove-- and experimentally you almost never prove a positive thing. You can show that a prediction is violated by experiment. So there's always going to be some uncertainty in your measurements, there's always going to be some uncertainty in your arguments.

However, in the absence of a compelling alternate theoretical description, you cling on to what you've got it as long as it fits your data, and this fits the data like a champ. Right? So, does it prove? No. It fits pretty well, and nothing else comes even within the ballpark. And there's no explicit violation that's better than our experimental uncertainties.

So, I don't know if I'd say, well, we could prove such a thing, but it fits. And I'm a physicist. I'm looking for things that fit. I'm not a metaphysicist. I'm not trying to give you some ontological commitment about what things are true and exist in the world, right? That's not my job.

OK. So much for our review. But let me finally come back to-- now that we've observed that it's determinist, let me come back to the question that was asked a few minutes ago, which is, look, suppose we take our superposition.

We evolve it forward for some time using the Schrodinger evolution. Notice that it's time reversal. If we know it's time reverted, we could run it backwards just as well as we could run it forwards, right? We could integrate that in time back, or we could integrate that in time forward.

So, if we know the wave function at some moment in time, we can integrate it forward, and we can integrate it back in time. But, If at some point we measure,

then the wave function collapses. And subsequently, the system evolves according to the Schrodinger equation, but with this new initial condition.

So now we seem to have a problem. We seem to have-- and I believe this was the question that was asked. I don't remember who asked it. Who asked it? So someone asked it. It was a good question.

We have this problem that there seem to be two definitions of time evolution in quantum mechanics. One is the Schrodinger equation, which says that things deterministically evolve forward in time. And the second is collapse, that if you do a measurement, things non-deterministically by probabilities collapse to some possible state.

Yeah? And the probability is determined by which wave function you have. How can these things both be true? How can you have two different definitions of time evolution?

So, this sort of frustration lies at the heart of much of the sort of spiel about the interpretation of quantum mechanics. On the one hand, we want to say, well, the world is inescapably probabilistic. Measurement comes with probabilistic outcomes and leads to collapse of the wave function.

On the other hand, when you're not looking, the system evolves deterministically. And this sounds horrible. It sounds horrible to a classical physicist. It sounds horrible to me. It just sounds awful. It sounds arbitrary. Meanwhile, it makes it sound like the world cares. It evolves differently depending on whether you're looking or not. And that-- come on. I mean, I think we can all agree that that's just crazy. So what's going on?

So for a long time, physicists in practice-- and still in practice-- for a long time physicists almost exclusively looked at this problem and said, look, don't worry about. It fits the data. It makes good predictions. Work with me here. Right? And it's really hard to argue against that attitude. You have a set of rules. It allows you to compute things. You compute them. They fit the data. Done. That is triumph. But it's

deeply disconcerting.

So, over the last, I don't know, in the second or the last quarter, roughly, the last third of the 20th century, various people started getting more upset about this. So, this notion of just shut up and calculate, which has been enshrined in the physics literature, goes under the name of the Copenhagen interpretation, which roughly says, look, just do this. Don't ask. Compute the numbers, and get what you will.

And people have questioned the sanity or wisdom of doing that. And in particular, there's an idea-- so I refer to the Copenhagen interpretation with my students as the cop out, because it's basically disavowal of responsibility. Look, it doesn't make sense, but I'm not responsible for making sense. I'm just responsible for making predictions. Come on.

So, more recently has come the theory of decoherence. And we're not going to talk about it in any detail until the last couple lectures of 804. Decoherence. I can't spell to save my life. So, the theory of decoherence. And here's roughly what the theory says.

The theory says, look, the reason you have this problem between on the one hand, Schrodinger evolution of a quantum system, and on the other hand, measurement leading to collapse, is that in the case of measurement meaning to collapse, you're not really studying the evolution of a quantum system.

You're studying the evolution of a quantum system-- ie a little thing that you're measuring-- interacting with your experimental apparatus, which is made up of 10 to the 27th particles, and you made up of 10 to the 28 particles. Whatever. It's a large number. OK, a lot more than that. You, a macroscopic object, where classical dynamics are a good description.

In particular, what that means is that the quantum effects are being washed out. You're washing out the interference of fringes, which is why I can catch this thing and not have it split into many different possible wave functions and where it went. So, dealing with that is hard, because now if you really want to treat the system with

Schrodinger evolution, you have to study the trajectory and the motion, the dynamics, of every particle in the system, every degree of freedom in the system.

So here's the question that decoherence is trying to ask. If you take a system where you have one little quantum subsystem that you're trying to measure, and then again a gagillion other degrees of freedom, some of which you care about-- they're made of you-- some of which you don't, like the particles of gas in the room, the environment. If you take that whole system, does Schrodinger evolution in the end boil down to collapse for that single quantum microsystem?

And the answer is yes. Showing that take some work, and we'll touch on it at the end of 804. But I want to mark right here that this is one of the most deeply unsatisfying points in the basic story of quantum mechanics, and that it's deeply unsatisfying because of the way that we're presenting it.

And there's a much more satisfying-- although still you never escape the fact that quantum mechanics violates your intuition. That's inescapable. But at least it's not illogical. it doesn't directly contradict itself. So that story is the story of decoherence. And if we're very lucky, I think we'll try to get one of my friends who's a quantum computing guy to talk about it. Yeah.

AUDIENCE: [INAUDIBLE] Is it possible that we get two different results?

PROFESSOR: No. No. No. There's never any ambiguity about what result you got. You never end up in a state of-- and this is also something that decoherence is supposed to explain. You never end up in a situation where you go like, wait, wait. I don't know. Maybe it was here, maybe it was there. I'm really confused.

I mean, you can get up in that situation because you did a bad job, but you don't end up in that situation because you're in a superposition state. You always end up when you're a classical beast doing a classical measurement, you always end up in some definite state.

Now, what wave function describes you doesn't necessarily correspond to you being in a simple state. You might be in a superposition of thinking this and thinking that.

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But, when you think this, that's in fact what happened. And when you think that, that's in fact what happened.

OK. So I'm going to leave this alone for the moment, but I just wanted to mark that as an important part of the quantum mechanical story.

OK. So let's go on to solving the Schrodinger equation. So what I want to do for the rest of today is talk about solving the Schrodinger equation.

So when we set about solving the Schrodinger equation, the first thing we should realize is that at the end of the day, the Schrodinger equation is just some differential equation. And in fact, it's a particularly easy differential equation. It's a first order linear differential equation. Right? We know how to solve those.

But, while it's first order in time, we have to think about what this energy operator is. So, just like the Newton equation f equals ma, we have to specify the energy operative before we can actually solve the dynamics of the system. In f equals ma, we have to tell you what the force is before we can solve for p, from p is equal to f.

So, for example. So one strategy to solve the Schrodinger equation is to say, look, it's just a differential equation, and I'll solve it using differential equation techniques. So let me specify, for example, the energy operator.

What's an easy energy operator? Well, imagine you had a harmonic oscillator, which, you know, physicists, that's your go-to. So, harmonic oscillator has energy p squared over 2m plus M Omega squared upon 2x squared. But we're going quantum mechanics, so we replace these guys by operators. So that's an energy operator. It's a perfectly viable operator.

And what is the differential equation that this leads to? What's the Schrodinger equation leads to? Well, I'm going to put the ih bar on the other side. ih bar derivative with respect to time of Psi of x and t is equal to p squared. Well, we remember that p is equal to h bar upon i, derivative with respect to x. So p squared is minus h bar squared derivative with respect to x squared upon 2m, or minus h bar squared upon 2m. Psi prime prime. Let me write this as dx squared. Two spatial

derivatives acting on Psi of x and t plus m omega squared upon 2x squared Psi of x and t.

So here's a differential equation. And if we want to know how does a system evolve in time, ie given some initial wave function, how does it evolve in time, we just take this differential equation and we solve it. And there are many tools to solve this partial differential equation.

For example, you could put it on Mathematica and just use NDSolve, right? This wasn't available, of course, to the physicists at the turn of the century, but they were less timid about differential equations than we are, because they didn't have Mathematica.

So, this is a very straightforward differential equation to solve, and we're going to solve it in a couple of lectures. We're going to study the harmonic oscillator in detail. What I want to emphasize for you is that any system has have some specified energy operator, just like any classical system, has some definite force function. And given that energy operator, that's going to lead to a differential equation.

So one way to solve the differential equation is just to go ahead and brute force solve it. But, at the end of the day, solving the Schrodinger equation is always, always going to boil down to some version morally of solve this differential equation. Questions about that? OK.

But when we actually look at a differential equation like this-- so, say we have this differential equation. It's got a derivative with respect to time, so we have to specify some initial condition. There are many ways to solve it. So given E, given some specific E, given some specific energy operator, there are many ways to solve.

The resulting differential equation. And I'm just going to mark that, in general, it's a PDE, because it's got derivatives with respect to time and derivatives with respect to space.

And roughly speaking, all these techniques fall into three camps. The first is just

brute force. That means some analog of throw it on Mathematica, go to the closet and pull out your mathematician and tie them to the chalkboard until they're done, and then put them back. But some version of a brute force, which is just use, by hook or by crook, some technique that allows you to solve the differential equation. OK.

The second is extreme cleverness. And you'd be amazed how often this comes in handy. So, extreme cleverness-- which we'll see both of these techniques used for the harmonic oscillator. That's what we'll do next week. First, the brute force, and secondly, the clever way of solving the harmonic oscillator.

When I say extreme cleverness, what I really mean is a more elegant use of your mathematician. You know something about the structure, the mathematical structure of your differential equation. And you're going to use that structure to figure out a good way to organize the differential equation, the good way to organize the problem. And that will teach you physics.

And the reason I distinguish brute force from cleverness in this sense is that brute force, you just get a list of numbers. Cleverness, you learn something about the way the physics of the system operates. We'll see this at work in the next two lectures.

And see, I really should separate this out numerically. And here I don't just mean sticking it into MATLAB. Numerically, it can be enormously valuable for a bunch of reasons. First off, there are often situations where no classic technique in differential equations or no simple mathematical structure that would just leap to the imagination comes to use. And you have some horrible differential you just have to solve, and you can solve it numerically.

Very useful lesson, and a reason to not even-- how many of y'all are thinking about being theorists of some stripe or other? OK. And how many of y'all are thinking about being experimentalists of some stripe or another? OK, cool.

So, look, there's this deep, deep prejudice in theory against numerical solutions of problems. It's myopia. It's a terrible attitude, and here's the reason.

Computers are stupid. Computers are breathtakingly dumb. They will do whatever you tell them to do, but they will not tell you that was a dumb thing to do. They have no idea. So, in order to solve an interesting physical problem, you have to first extract all the physics and organize the problem in such a way that a stupid computer can do the solution.

As a consequence, you learn the physics about the problem. It's extremely valuable to learn how to solve problems numerically, and we're going to have problem sets later in the course in which you're going to be required to numerically solve some of these differential equations. But it's useful because you get numbers, and you can check against data, but also it lets you in the process of understanding how to solve the problem. You learn things about the problem. So I want to mark that as a separate logical way to do it.

So today, I want to start our analysis by looking at a couple of examples of solving the Schrodinger equation. And I want to start by looking at energy Eigenfunctions. And then once we understand how a single energy Eigenfunction evolves in time, once we understand that solution to the Schrodinger equation, we're going to use the linearity of the Schrodinger equation to write down a general solution of the Schrodinger equation. OK.

So, first. What happens if we have a single energy Eigenfunction? So, suppose our wave function as a function of x at time t equals zero is in a known configuration, which is an energy Eigenfunction Phi sub E of x. What I mean by Phi sub E of x is if I take the energy operator, and I act on Phi sub E of x, this gives me back the number E Phi sub E of x. OK? So it's an Eigenfunction of the energy operator, the Eigenvalue E.

So, suppose our initial condition is that our system began life at time t equals zero in this state with definite energy E. Everyone cool with that? First off, question. Suppose I immediately at time zero measure the energy of this system. What will I get?

AUDIENCE: E.

PROFESSOR: With what probability?

AUDIENCE: 100%

PROFESSOR: 100%, because this is, in fact, of this form, it's a superposition of energy Eigenstates, except there's only one term. And the coefficient of that one term is one, and the probability that I measure the energy to be equal to that value is the coefficient norm squared, and that's one norm squared. Everyone cool with that?

Consider on the other hand, if I had taken this wave function and I had multiplied it by phase E to the i Alpha. What now is the probability where alpha is just a number? What now is the probability that I measured the state to have energy E?

AUDIENCE: One.

PROFESSOR: It's still one, because the norm squared of a phase is one. Right? OK. The overall phase does not matter.

So, suppose I have this as my initial condition. Let's take away the overall phase because my life will be easier. So here's the wave function. What is the Schrodinger equation? Well, the Schrodinger equation says that ih bar time derivative of Psi is equal to the energy operator acting on Psi. And I should be specific. This is Psi at x at time t, Eigenvalued at this time zero is equal to the energy operator acting on this wave function. But what's the energy operator acting on this wave function?

AUDIENCE: E.

PROFESSOR: E. E on Psi is equal to E on Phi sub E, which is just E the number. This is the number E, the Eigenvalue E times Psi at x zero. And now, instead of having an operator on the right hand side, we just have a number. So, I'm going to write this differential equation slightly differently, ie time derivative of Psi is equal to E upon ih bar, or minus ie over h bar Psi. Yeah? Everyone cool with that?

This is the easiest differential equation in the world to solve. So, the time derivative is a constant. Well, times itself. That means that therefore Psi at x and t is equal to E

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to the minus i ET over h bar Psi at x zero. Where I've imposed the initial condition that at time t equals zero, the wave function is just equal to Psi of x at zero. And in particular, I know what Psi of x and zero is. It's Phi E of x. So I can simply write this as Phi E of x. Are we cool with that?

So, what this tells me is that under time evolution, a state which is initially in an energy Eigenstate remains in an energy Eigenstate with the same energy Eigenvalue. The only thing that changes about the wave function is that its phase changes, and its phase changes by rotating with a constant velocity. E to the minus i, the energy Eigenvalue, times time upon h bar.

Now, first off, before we do anything else as usual, we should first check the dimensions of our result to make sure we didn't make a goof. So, does this make sense dimensionally? Let's quickly check. Yeah, it does. Let's just quickly check.

So we have that the exponent there is Et over h bar. OK? And this should have dimensions of what in order to make sense?

AUDIENCE: Nothing.

PROFESSOR: Nothing, exactly. It should be dimensionless. So what are the dimensions of h bar?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Oh, no, the dimensions, guys, not the units. What are the dimensions?

AUDIENCE: [INAUDIBLE]

PROFESSOR: It's an action, which is energy of time. So the units of the dimensions of h are an energy times a time, also known as a momentum times a position. OK? So, this has dimensions of action or energy times time, and then upstairs we have dimensions of energy times time. So that's consistent. So this in fact is dimensionally sensible, which is good.

Now, this tells you a very important thing. In fact, we just answered this equation. At time t equals zero, what will we get if we measure the energy? E. At time t prime--

some subsequent time-- what energy will we measure?

AUDIENCE: E.

PROFESSOR: Yeah. Does the energy change over time? No. When I say that, what I mean is, does the energy that you expect to measure change over time? No. Does the probability that you measure energy E change? No, because it's just a phase, and the norm squared of a phase is one. Yeah? Everyone cool with that? Questions at this point. This is very simple example, but it's going to have a lot of power. Oh, yeah, question. Thank you.

AUDIENCE: Are we going to deal with energy operators that change over time?

PROFESSOR: Excellent question. We will later, but not in 804. In 805, you'll discuss it in more detail. Nothing dramatic happens, but you just have to add more symbols. There's nothing deep about it. It's a very good question.

The question was, are we going to deal with energy operators that change in time? My answer was no, not in 804, but you will in 805. And what you'll find is that it's not a big deal. Nothing particularly dramatic happens.

We will deal with systems where the energy operator changes instantaneously. So not a continuous function, but we're at some of them you turn on the electric field, or something like that. So we'll deal with that later on.

But we won't develop a theory of energy operators that depend on time. But you could do it, and you will do in 805. There's nothing mysterious about it. Other questions? OK.

So, these states-- a state Psi of x and t, which is of the form e to the minus i Omega t, where Omega is equal to E over h bar. This should look familiar. It's the de Broglie relation, [INAUDIBLE] relation, whatever. Times some Phi E of x, where this is an energy Eigenfunction. These states are called stationary states.

And what's the reason for that? Why are they called stationary states? I'm going to erase this. Well, suppose this is my wave function as a function of time. What is the

probability that at time t I will measure the particle to be at position x, or the probability density?

Well, the probability density we know from our postulates, it's just the norm squared of the wave function. This is Psi at x t norm squared. But this is equal to the norm squared of e to the minus Psi Omega t Phi E by the Schrodinger equation. But when we take the norm squared, this phase cancels out, as we already saw. So this is just equal to Phi E of x norm squared, the energy Eigenfunction norm squared independent of time.

So, if we happen to know that our state is in an energy Eigenfunction, then the probability density for finding the particle at any given position does not change in time. It remains invariant. The wave function rotates by an overall phase, but the probability density is the norm squared. It's insensitive to that overall phase, and so the probability density just remains constant in whatever shape it is. Hence it's called a stationary state.

Notice its consequence. What can you say about the expectation value of the position as a function of time? Well, this is equal to the integral dx in the state Psi of x and t. And I'll call this Psi sub E just to emphasize. It's the integral of the x, integral over all possible positions of the probability distribution, probability of x at time t times x. But this is equal to the integral dx of Phi E of x squared x. But that's equal to expectation value of x at any time, or time zero. t equals zero. And maybe the best way to write this is as a function of time. So, the expectation value of x doesn't change. In a stationary state, expected positions, energy-- these things don't change. Everyone cool with that? And it's because of this basic fact that the wave function only rotates by a phase under time evolution when the system is an energy Eigenstate. Questions? OK.

So, here's a couple of questions for you guys. Are all systems always in energy Eigenstates? Am I in an energy Eigenstate?

AUDIENCE: No.

PROFESSOR: No, right? OK, expected position of my hand is changing in time. I am not in-- so obviously, things change in time. Energies change in time. Positions-- expected typical positions-- change in time. We are not in energy Eigenstates. That's a highly non-generic state.

So here's another question. Are any states ever truly in energy Eigenstates? Can you imagine an object in the world that is truly described precisely by an energy Eigenstate in the real world?

AUDIENCE: No.

PROFESSOR: Ok, there have been a few nos. Why? Why not? Does anything really remain invariant in time? No, right? Everything is getting buffeted around by the rest of the universe. So, not only are these not typical states, not only are stationary states not typical, but they actually never exist in the real world. So why am I talking about them at all?

So here's why. And actually I'm going to do this here. So here's why. The reason is this guy, the superposition principle, which tells me that if I have possible states, I can build superpositions of them. And this statement-- and in particular, linearity-- which says that given any two solutions of the Schrodinger equation, I can take a superposition and build a new solution of the Schrodinger equation.

So, let me build it. So, in particular, I want to exploit the linearity of the Schrodinger equation to do the following.

Suppose Psi. And I'm going to label these by n. Psi n of x and t is equal to e to the minus i Omega nt Phi sub En of x, where En is equal to h bar Omega n. n labels the various different energy Eigenfunctions. So, consider all the energy Eigenfunctions Phi sub En. n is a number which labels them. And this is the solution to the Schrodinger equation, which at time zero is just equal to the energy Eigenfunction of interest. Cool? So, consider these guys.

So, suppose we have these guys such that they solve the Schrodinger equation. Solve the Schrodinger equation. Suppose these guys solve the Schrodinger

equation. Then, by linearity, we can take Psi of x and t to be an arbitrary superposition sum over n, c sub n, Psi sub n of x and t. And this will automatically solve the Schrodinger equation by linearity of the Schrodinger equation. Yeah.

AUDIENCE: But can't we just get n as the sum of the energy Eigenstate by just applying that and by just measuring that?

PROFESSOR: Excellent. So, here's the question. The question is, look, a minute ago you said no system is truly in an energy Eigenstate, right? But can't we put a system in an energy Eigenstate by just measuring the energy? Right? Isn't that exactly what the collapse postulate says? So here's my question. How confident are you that you actually measure the energy precisely? With what accuracy can we measure the energy?

So here's the unfortunate truth, the unfortunate practical truth. And I'm not talking about in principle things. I'm talking about it in practice things in the real universe. When you measure the energy of something, you've got a box, and the box has a dial, and the dial has a needle, it has a finite width, and your current meter has a finite sensitivity to the current. So you never truly measure the energy exactly. You measure it to within some tolerance. And

In fact, there's a fundamental bound-- there's a fundamental bound on the accuracy with which you can make a measurement, which is just the following. And this is the analog of the uncertainty equation. We'll talk about this more later, but let me just jump ahead a little bit.

Suppose I want to measure frequency. So I have some signal, and I look at that signal for 10 minutes. OK? Can I be absolutely confident that this signal is in fact a plane wave with the given frequency that I just did?

No, because it could change outside that. But more to the point, there might have been small variations inside. There could've been a wavelength that could change on a time scale longer than the time that I measured. So, to know that the system doesn't change on a arbitrary-- that it's strictly fixed Omega, I have to wait a very

long time.

And in particular, how confident you can be of the frequency is bounded by the time over which-- so your confidence, your uncertainty in the frequency, is bounded in the following fashion. Delta Omega, Delta t is always greater than or equal to one, approximately. What this says is that if you want to be absolute confident of the frequency, you have to wait an arbitrarily long time.

Now if I multiply this whole thing by h bar, I get the following. Delta E-- so this is a classic equation that signals analysis-- Delta E, Delta t is greater than or approximately equal to h bar. This is a hallowed time- energy uncertainty relation, which we haven't talked about.

So, in fact, it is possible to make an arbitrarily precise measurement of the energy. What do I have to do? I have to wait forever. How patient are you, right? So, that's the issue. In the real world, we can't make arbitrarily long measurements, and we can't isolate systems for an arbitrarily long amount of time. So, we can't put things in a definite energy Eigenstate by measurement. That answer your question?

AUDIENCE: Yes.

PROFESSOR: Great. How many people have seen signals in this expression, the bound on the frequency? Oh, good. So we'll talk about that later in the course. OK, so coming back to this. So, we have our solutions of the Schrodinger equation that are initially energy Eigenstates. I claim I can take an arbitrary superposition of them, and by linearity derive that this is also a solution to the Schrodinger equation.

And in particular, what that tells me is-- well, another way to say this is that if I know that Psi of x times zero is equal to sum over n-- so if sum Psi of x-- if the wave function at some particular moment in time can be expanded as sum over n Cn Phi E of x, if this is my initial condition, my initial wave function is some superposition, then I know what the wave function is at subsequent times. The wave function by superposition Psi of x and t is equal to sum over n Cn e to the minus i Omega nt Phi n-- sorry, this should've been Phi sub n-- Phi n of x.

And I know this has to be true because this is a solution to the Schrodinger equation by construction, and at time t equals zero, it reduces to this. So, this is a solution to the Schrodinger equation, satisfying this condition at the initial time t equals zero. Don't even have to do a calculation.

So, having solved the Schrodinger equation once for energy, Eigenstates allows me to solve it for general superposition. However, what I just said isn't quite enough. I need one more argument. And that one more argument is really the stronger version of three that we talked about before, which is that, given an energy operator E, we find the set of wave functions Phi sub E, the Eigenfunctions of the energy operator, with Eigenvalue E.

So, given the energy operator, we find its Eigenfunctions. Any wave function Psi at x-- we'll say at time zero-- any function of x can be expanded as a sum. Specific superposition sum over n Cn Phi E sub n of x. And if any function can be expanded as a superposition of energy Eigenfunctions, and we know how to take a superposition, an arbitrary superposition of energy Eigenfunctions, and find the corresponding solution to the Schrodinger equation.

What this means is, we can take an arbitrary initial condition and compute the full solution of the Schrodinger equation. All we have to do is figure out what these coefficients Cn are. Everyone cool with that? So, we have thus, using superposition and energy Eigenvalues, totally solved the Schrodinger equation, and reduced it to the problem of finding these expansion coefficients.

Meanwhile, these expansion coefficients have a meaning. They correspond to the probability that we measure the energy to be equal to the corresponding energy E sub n. And it's just the norm squared of that coefficient. So those coefficients mean something. And they allow us to solve the problem. Cool?

So this is fairly abstract. So let's make it concrete by looking at some examples. So, just as a quick aside. This should sound an awful lot like the Fourier theorem. And let me comment on that.

This statement originally was about a general observable and general operator. Here I'm talking about the energy. But let's think about a slightly more special example, or more familiar example. Let's consider the momentum.

Given the momentum, we can find a set of Eigenstates. What are the set of good, properly normalized Eigenfunctions of momentum? What are the Eigenfunctions of the momentum operator?

AUDIENCE: E to the ikx.

PROFESSOR: E to the ikx. Exactly. In particular, one over 2 pi e to the ikx.

So I claim that, for every different value of k, I get a different value of p, and the Eigenvalue associated to this guy is p is equal to h bar k. That's the Eigenvalue. And we get that by acting with the momentum, which is h bar upon i, h bar times derivative with respect to x. Derivative with respect to x pulls down an ik times the same thing. H bar multiplies the k over i, kills the i, and leaves us with an overall coefficient of h bar k. This is an Eigenfunction of the momentum operator with Eigenvalue h bar k.

And that statement three is the statement that an arbitrary function f of x can be expanded as a superposition of all possible energy Eigenvalues. But k is continuously valued and the momentum, so that's an integral dk one over 2 pi, e to the ikx times some coefficients. And those coefficients are labeled by k, but since k is continuous, I'm going to call it a function. And just to give it a name, instead of calling C sub k, I'll call it f tilde of k.

This is of exactly the same form. Here is the expansion-- there's the Eigenfunction, here is the Eigenfunction, here is the expansion coefficient, here is expansion coefficient. And this has a familiar name. It's the Fourier theorem.

So, we see that the Fourier theorem is this statement, statement three, the superposition principal, for the momentum operator. We also see that it's true for the energy operator. And what we're claiming here is that it's true for any observable. Given any observable, you can find its Eigenfunctions, and they form a

basis on the space of all good functions, and an arbitrary function can be expanded in that basis.

So, as a last example, consider the following. We've done energy. We've done momentum. What's another operator we care about? What about position? What are the Eigenfunctions of position? Well, x hat on Delta of x minus y is equal to y Delta x minus y.

So, these are the states with definite value of position x is equal to y. And the reason this is true is that when x is equal to y, x is the operator that multiplies by the variable x. But it's zero, except at x is equal to y, so we might as well replace x by y.

So, there are the Eigenfunctions. And this statement is a statement that we can represent an arbitrary function f of x in a superposition of these states of definite x. f of x is equal to the integral over all possible expansion coefficients dy delta x minus y times some expansion coefficient.

And what's the expansion coefficient? It's got to be a function of y. And what function of y must it be? Just f of y. Because this integral against this delta function had better give me f of x. And that will only be true if this is f of x.

So here we see, in some sense, the definition of the delta function. But really, this is a statement of the superposition principle, the statement that any function can be expanded as a superposition of Eigenfunctions of the position operator. Any function can be expanded as a superposition of Eigenfunctions of momentum. Any function can be expanded as a superposition of Eigenfunctions of energy. Any function can be expanded as a superposition of Eigenfunctions of any operator of your choice. OK?

The special cases-- the Fourier theorem, the general cases, the superposition postulate. Cool? Powerful tool. And we've used this powerful tool to write down a general expression for a solution to the Schrodinger equation. That's good. That's progress. So let's look at some examples of this. I can leave this up.

So, our first example is going to be for the free particle. So, a particle whose energy

operator has no potential whatsoever. So the energy operator is going to be just equal to p squared upon 2m. Kinetic energy. Yeah.

AUDIENCE: When you say any wave function can be expanded in terms of--

PROFESSOR: Energy Eigenfunctions, position Eigenfunctions, momentum Eigenfunctions--

AUDIENCE: Eigenbasis, does the Eigenbasis have to come from an operator corresponding to an observable?

PROFESSOR: Yes. Absolutely. I'm starting with that assumption.

AUDIENCE: OK.

PROFESSOR: So, again, this is a first pass of the axioms of quantum mechanics. We'll make this more precise, and we'll make it more general, later on in the course, as we go through a second iteration of this. And there we'll talk about exactly what we need, and what operators are appropriate operators. But for the moment, the sufficient and physically correct answer is, operators correspond to each observable values. Yeah.

AUDIENCE: So are the set of all reasonable wave functions in the vector space that is the same as the one with the Eigenfunctions?

PROFESSOR: That's an excellent question. In general, no. So here's the question. The question is, look, if this is true, shouldn't it be that the Eigenfunctions, since they're our basis for the good functions, are inside the space of reasonable functions, they should also be reasonable functions, right?

Because if you're going to expand-- for example, consider two dimensional vector space. And you want to say any vector can be expanded in a basis of pairs of vectors in two dimensions, like x and y. You really want to make sure that those vectors are inside your vector space. But if you say this vector in this space can be expanded in terms of two vectors, this vector and that vector, you're in trouble, right? That's not going to work so well. So you want to make sure that your vectors,

your basis vectors, are in the space.

For position, the basis vector's a delta function. Is that a smooth, continuous normalizable function? No. For momentum, the basis functions are plane waves that extend off to infinity and have support everywhere. Is that a normalizable reasonable function? No. So, both of these sets are really bad. So, at that point you might say, look, this is clearly nonsense.

But here's an important thing. So this is a totally mathematical aside, and for those of you who don't care about the math, don't worry about it. Well, these guys don't technically live in the space of non-stupid functions-- reasonable, smooth, normalizable functions. What you can show is that they exist in the closure, in the completion of that space. OK? So, you can find a sequence of wave functions that are good wave functions, an infinite sequence, that eventually that infinite sequence converges to these guys, even though these are silly.

So, for example, for the position Eigenstates, the delta function is not a continuous smooth function. It's not even a function. Really, it's some god- awful thing called a distribution. It's some horrible thing. It's the thing that tells you, give it an integral, it'll give you a number. Or a function.

But how do we build this as a limit of totally reasonable functions? We've already done that. Take this function with area one, and if you want, you can round this out by making it hyperbolic tangents. OK? We did it on one of the problem sets. And then just make it more narrow and more tall. And keep making it more narrow and more tall, and more narrow and more tall, keeping its area to be one.

And I claim that eventually that series, that sequence of functions, converges to the delta function. So, while this function is not technically in our space, it's in the completion of our space, in the sense that we take a series and they converge to it.

And that's what you need for this theorem to work out. That's what you need for the Fourier theorem. And in some sense, that observation was really the genius of Fourier, understanding that that could be done. That was totally mathematical

aside. But that answer your question?

AUDIENCE: Yes.

PROFESSOR: OK. Every once in a while I can't resist talking about these sort of details, because I really like them. But it's good to know that stupid things like this can't matter for us, and they don't. But it's a very good question.

If you're confused about some mathematical detail, no matter how elementary, ask. If you're confused, someone else the room is also confused. So please don't hesitate.

OK, so our first example's going to be the free particle. And this operator can be written in a nice way. We can write it as minus-- so p is h bar upon iddx, so this line is minus h bar squared upon 2m to the derivative with respect to x. There's the energy operator.

So, we want to solve for the wave functions. So let's solve it using an expansion in terms of energy Eigenfunctions. So what are the energy Eigenfunctions? We want to find the functions E on Phi sub E such that this is equal to-- whoops. That's not a vector. That's a hat-- such as this is equal to a number E Phi sub E.

But given this energy operator, this says that minus h bar squared over 2m-- whoops, that's a 2. 2m-- Phi prime prime of x is equal to E Phi of x. Or equivalently, Phi prime prime of x plus 2me over h bar squared Phi of x is equal to zero.

So I'm just going to call 2me-- because it's annoying to write it over and over again-- over h bar squared. Well, first off, what are its units? What are the units of this coefficient? Well, you could do it two ways. You could either do dimensional analysis of each thing here, or you could just know that we started with a dimensionally sensible equation, and this has units of this divided by length twice. So this must have to whatever length squared. So I'm going to call this something like k squared, something that has units of one over length squared.

And the general solution of this is that phi E E of x-- well, this is a second order

differential equation that will have two solutions with two expansion coefficients-- A e to the ikx plus B e to the minus iks. A state with definite momentum and definite negative momentum where such that E is equal to h bar squared k squared upon 2m. And we get that just from this. So, this is the solution of the energy Eigenfunction equation.

Just a note of terminology. People sometimes call the equation determining an energy Eigenfunction-- the energy Eigenfunction equation-- sometimes that's refer to as the Schrodinger equation. That's a sort of cruel thing to do to language, because the Schrodinger's equation is about time evolution, and this equation is about energy Eigenfunctions. Now, it's true that energy Eigenfunctions evolve in a particularly simple way under time evolution, but it's a different equation. This is telling you about energy Eigenstates, OK?

And then more discussion of this is done in the notes, which I will leave aside for the moment. But I want to do one more example before we take off. Wow. We got through a lot less than expected today. And the one last example is the following. It's a particle in a box And this is going to be important for your problem sets, so I'm going to go ahead and get this one out of the way as quickly as possible.

So, example two. Particle in a box. So, what I mean by particle in a box. I'm going to take a system that has a deep well. So what I'm drawing here is the potential energy U of x, where this is some energy E naught, and this is the energy zero, and this is the position x equals zero, and this is position x equals l.

And I'm going to idealize this by saying look, I'm going to be interested in low energy physics, so I'm going to just treat this as infinitely deep. And meanwhile, my life is easier if I don't think about curvy bottoms but I just think about things as being constant. So, my idealization is going to be that the well is infinitely high and square. So out here the potential is infinite, and in here the potential is zero. U equals inside, between zero and l for x. So that's my system particle in a box.

So, let's find the energy Eigenfunctions. And again, it's the same differential equations as before. So, first off, before we even solve anything, what's the

probability that I find x less than zero, or find the particle at x greater than l?

AUDIENCE: Zero.

PROFESSOR: Right, because the potential is infinitely large out there. It's just not going to happen. If you found it there, that would correspond to a particle of infinite energy, and that's not going to happen. So, our this tells us effectively the boundary condition Psi of x is equal to zero outside the box. So all we have to do is figure out what the wave function is inside the box between zero and l.

And meanwhile, what must be true of the wave function at zero and at l? It's got to actually vanish at the boundaries. So this gives us boundary conditions outside the box and at the boundaries x equals zero, x equals l.

But, what's our differential equation inside the box? Inside the box, well, the potential is zero. So the equation is the same as the equation for a free particle. It's just this guy. And we know what the solutions are. So the solutions can be written in the following form.

Therefore inside the wave function-- whoops. Let me write this as Phi sub E-- Phi sub E is a superposition of two. And instead of writing it as exponentials, I'm going to write it as sines and cosines, because you can express them in terms of each other. Alpha cosine of kx plus Beta sine of kx, where again Alpha and Beta are general complex numbers.

But, we must satisfy the boundary conditions imposed by our potential at x equals zero and x equals l. So from x equals zero, we find that Phi must vanish when x equals zero. When x equals zero, this is automatically zero. Sine of zero is zero. Cosine of zero is one. So that tells us that Alpha is equal to zero.

Meanwhile, the condition that at x equals l-- the wave function must also vanish-- tells us that-- so this term is gone, set off with zero-- this term, when x is equal to l, had better also be zero. We can solve that by setting Beta equal to zero, but then our wave function is just zero. And that's a really stupid wave function.

So we don't want to do that. We don't want to set Beta to zero. Instead, what must we do? Well, we've got a sine, and depending on what k is, it starts at zero and it ends somewhere else. But we need it hit zero.

So only for a very special value of k will it actually hit zero at the end of l. We need kl equals zero. Or really, kl is a multiple of pi. Kl is equal-- and we want it to not be zero, so I'll call it n plus 1, an integer, times pi. Or equivalently, k is equal to sub n is equal to n plus 1, where n goes from zero to any large positive integer, pi over l.

So the energy Eigenfunction's here. The energy Eigenfunction is some normalization-- whoops-- a sub n sine of k and x. And where kn is equal to this-- and as a consequence, E is equal to h bar squared kn can squared E sub n is h bar squared kn squared over 2m, which is equal to h bar squared-- just plugging in-- pi squared n plus 1 squared over 2ml squared.

And what we found is something really interesting. What we found is, first off, that the wave functions look like-- well, the ground state, the lowest possible energy there is n equals zero. For n equals zero, this is just a single half a sine wave. It does this. This is the n equals zero state. And it has some energy, which is E zero. And in particular, E zero is not equal to zero. E zero is equal to h bar squared pi squared over 2ml squared.

It is impossible for a particle in a box to have an energy lower than some minimal value E naught, which is not zero. You cannot have less energy than this. Everyone agree with that? There is no such Eigenstate with energy less than this.

Meanwhile, it's worse. The next energy is when n is equal to 1, because if we decrease the wavelength or increase k a little bit, we get something that looks like this, and that doesn't satisfy our boundary condition. In order to satisfy our boundary condition, we're going to have to eventually have it cross over and get to zero again.

And if I could only draw-- I'll draw it up here-- it looks like this. And this has an energy E one, which you can get by plugging one in here. And that differs by one,

two, four, a factor of four from this guy. E one is four E zero. And so on and so forth.

The energies are gaped. They're spread away from each other. The energies are discrete. And they get further and further away from each other as we go to higher and higher energies. So this is already a peculiar fact, and we'll explore some of its consequences later on.

But here's that I want to emphasize for you. Already in the first most trivial example of solving a Schrodinger equation, or actually even before that, just finding the energy Eigenvalues and the energy of Eigenfunctions of the simplest system you possibly could, either a free particular, or a particle in a box, a particle trapped inside a potential well, what we discovered is that the energy Eigenvalues, the allowed values of the energy, are discrete, and that they're greater than zero. You can never have zero energy.

And if that doesn't sound familiar, let me remind you of something. The spectrum of light coming off of a gas of hot hydrogen is discrete. And no one's ever found a zero energy beam of light coming out of it. And we're going to make contact with that experimental data. That's going to be part of the job for the rest. See you next time.

[APPLAUSE]