8.04: Quantum Mechanics | Professor Allan Adams
Massachusetts Institute of Technology | February 5, 2013

Problem Set 1

Due Tuesday Feb 12 at 11.00AM

Readings:

E&R	1-(6,7)	2-(1,2,3,4,5)	3-(all)	NOT 4-(all)!!
Li.	1-(all)	2-(3,5,6)		NOT 2-4!!!
Ga.	1-(2,3,4)			NOT 1-5!!!
Sh.	3			

1. **(15 points) Radiative collapse of a classical atom**

Suppose the world was actually governed by classical mechanics. In such a classical universe, we might try to build a Hydrogen atom by placing an electron in a circular orbit around a proton. However, we know from 8.03 that a non-relativistic, accelerating electric charge radiates energy at a rate given by the Larmor formula,

$$\frac{dE}{dt} = -\frac{2}{3}\frac{q^2 a^2}{c^3}$$

(in cgs units) where q is the electric charge and a is the magnitude of the acceleration. So the classical atom has a stability problem. How big is this effect?

(a) Show that the energy lost per revolution is small compared to the electron's kinetic energy. Hence, it is an excellent approximation to regard the orbit as circular at any instant, even though the electron eventually spirals into the proton.

(b) Using the typical size of an atom (1Å) and a nucleus (1 fm), calculate how long it would take for the electron to spiral into the proton.

(c) Compare the velocity of the electron (assuming an orbital radius of 0.5 Å) to the speed of light – will relativistic corrections materially alter your conclusions?

(d) As the electron approaches the proton, what happens to its energy? Is there a minimum value of the energy the electron can have?

2. **(25 points) Dimensional Analysis: Two Kinds of Quantum Gravity**

(a) Gravitational bound states

Consider a particle sitting on a table which is kept from floating away only by the force of gravity. This system is characterized by just three physical parameters, the mass of the particle, m, the acceleration of gravity on Earth, $g = 9.8\frac{m}{s^2}$, and Planck's constant, $\hbar = \frac{1}{2\pi}h$. The energy given by $E = \frac{1}{2}mv^2 + mgx$.

i. Using only dimensional analysis, find the product of powers of m, g, $\hbar$ which give a characteristic energy, E. (*i.e.*, write $E \sim m^\alpha g^\beta \hbar^\gamma$ and solve for α, β, γ) Can you find such a characteristic energy without using the Planck constant?

ii. Repeat to find characteristic length, time, and speeds (l, t, v) for this system.

iii. Classically, putting the system in its lowest energy configuration (E=0) would require the particle to sit perfectly still ($v = 0$) precisely on the surface ($x = 0$). Use the uncertainty relation, $\Delta x\, \Delta p \geq \frac{\hbar}{2}$, to argue (briefly!) that the particle cannot have $E = 0$ while respecting the uncertainty principle.

ASIDE: Quantum mechanically, then, there must be some minimum energy this system can have which cannot be predicted classically! For a particle on a table, this may not seem so important – but for Hydrogen, which you've just shown to be classically unstable, this is absolutely key. We will soon learn how to calculate the minimum ("ground state") energy of such systems.

iv. Use your dimensional analysis results to give a simple estimate for the ground state energy of this system. How does your estimate behave as $h \to 0$? Does this make sense? Explain why or why not.

v. Evaluate E, l, t and v numerically for a neutron ($m_N = 1.7 \cdot 10^{-27} kg$). How high above the surface will the particle typically be found?[1]

(b) The Planck Scale

The scale at which gravity (characterized by the Newton constant, G_N), quantum mechanics ($\hbar$), and relativity (c) are all important is called the Planck scale.

i. Using dimensional analysis, find the combination of powers of G_N, $\hbar$ and c which make a length – we call this the Planck length, L_P.

ii. Evaluate L_P numerically, and compare to a typical scale for nuclear or particle physics, namely $1F = 10^{-15}m$.

iii. Repeat to find the Planck mass, M_P, evaluate it numerically, and compare to the mass of a typical nuclear constituent (like the proton mass). Do we need to understand Quantum Gravity to study nuclear physics?

[1] This system has been studied experimentally using neutrons in: Quantum states of neurons in the Earth's gravitational Field, V. V. Nesvizhevsky et al., *Nature* **415**, 297 (2002).

3. **(20 points) deBroglie Relations and the Scale of Quantum Effects**

   (a) Light Waves as Particles

   The Photoelectric effect suggests that light of frequency ν can be regarded as consisting of photons of energy $E = h\nu$, where $h = 6.63 \cdot 10^{-27} erg \cdot s$.

   i. Visible light has a wavelength in the range of 400-700 nm. What are the energy and frequency of a photon of visible light?

   ii. The microwave in my kitchen operates at roughly 2.5 GHz at a max power of $7.5 \cdot 10^9 \frac{erg}{s}$. How many photons per second can it emit? What about a low-power laser ($10^4 \frac{erg}{s}$ at 633 nm), or a cell phone ($4 \cdot 10^6 \frac{erg}{s}$ at 850 MHz)?

   iii. How many such microwave photons does it take to warm a 200ml glass of water by $10°C$? (The heat capacity of water is roughly $4.18 \cdot 10^7 \frac{erg}{g\,°K}$.)

   iv. At a given power of an electromagnetic wave, do you expect a classical wave description to work better for radio frequencies, or for X-rays?

   (b) Matter Particles as Waves

   If a wavelength can be associated with every moving particle, then why are we not forcibly made aware of this property in our everyday experience? In answering, calculate the de Broglie wavelength $\lambda = \frac{h}{p}$ of each of the following particles:

   i. an automobile of mass 2 metric tons (2000 kg) traveling at a speed of 50 mph (22 $\frac{m}{s}$),

   ii. a marble of mass 10 g moving with a speed of 10 $\frac{cm}{s}$,

   iii. a smoke particle of diameter $10^{-5} cm$ and a density of, say, .2 $\frac{g}{cm^3}$) being jostled about by air molecules at room temperature (T=300K) (assume that the particle has the same translational kinetic energy as the thermal average of the air molecules, $KE = \frac{3}{2} k_B T$, with $k_B = 1.38 \cdot 10^{-16} \frac{erg}{°K}$),

   iv. an ^{87}Rb atom that has been laser cooled to a temperature of $T = 100 \mu K$. Again, assume $KE = \frac{3}{2} k_B T$.

4. **(15 points) Double-slit interference of electrons**

   (a) Electrons of momentum p fall normally on a pair of slits separated by a distance d. What is the distance, w, between adjacent maxima of the interference fringe pattern formed on a screen a distance D beyond the slits? *note: You may assume that the width of the slits is much less than the electron de Broglie wavelength.*

   (b) In an experiment performed by Jönsson in 1961 (!!!), electrons were accelerated through a $50kV$ potential towards two slits separated by a distance $d = 2\,10^{-4}cm$, then detected on a screen $D = 35$cm beyond the slits. Calculate the electron's de Broglie wavelength, λ, and the fringe spacing, w.

   (c) What values would d, D, and w take if Jönsson's apparatus were simply scaled up for use with visible light rather than electrons?

5. **(15 points) Electron Diffraction**

   (a) Watch the video on Matter Waves,

   `http://tsgphysics.mit.edu/front/?page=demo.php&letnum=Z%2047`

   (b) Explain, using diagrams and/or equations, why there are *diagonal* lines in the diffraction pattern which appears at (12:59) in the video.

   (c) In their classic experiment, Davisson and Germer (see paper) directed an electron beam into a nickel crystal at 90 degree incidence and placed a detector at an angle θ from the beam. When the electrons were accelerated by a voltage of 54 volts, they observed strong reflection or these electrons into an angle $\theta = 50°$. Using the de Broglie relation and the Bragg relation, compute the lattice spacing in the nickel crystal. How does this to compare with the value (0.215 nm) measured by X-ray diffraction experiments?

6. **(15 points) Single-slit Diffraction and Uncertainty**

   Visible light with a wavelength λ is incident from a distant source onto a single slit of width δx; denote the propagation direction as z, and the direction transverse to it with x. Assume δx is a few times larger than λ.

   (a) Estimate the width w of the pattern observed on a screen that is a distance $D >> \lambda$ away (e.g., by assuming that the slit acts as a collection of emitters all oscillating in phase).

   (b) In the photon picture of light, the light beam after the slit comprises a large number of photons with a range of values of transverse momentum. Due to this range of propagation speeds in the x direction, different photons hit the screen at different spots. Using the knowledge of w, estimate the range of transverse values of momentum δp_x , assuming $\delta p_x << p$, where p is the photon momentum.

   (c) Since we know that each of the photons went through the slit, we have effectively measured the x-position of the photons at that point; the experimental uncertainty associated with this particular measurement is δx. Since the momentum of the photons in the transverse direction is conserved between the time they go through the slit, and the time they hit the screen, we have effectively also measured the uncertainty in their transverse momentum values just after the slit to be δp_x. Using the relation $E = h\nu = hc/\lambda$, together with the expression for the photon momentum $E = pc$, show $\delta x \cdot \delta p_x \sim h$.

   ASIDE: This is a heuristic realization of the uncertainty relation, which lies at the heart of Quantum Mechanics. We will derive it in a number of (increasingly general) ways in the next few weeks. Note that, in the above, the uncertainty relation is a consequence of the wave nature of light (c.f. part (a) of the problem).

8.04 Quantum Physics I
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