PROFESSOR: So today is going to be our last pass at bound states. So starting next week or actually starting next lecture, we're going to look at scattering. Scattering's going to be great. But we need to close out bound states.

So today's topic is the finite well, finite the potential well. We've sort of sketched this when we looked at qualitative structure of wave functions of energy eigenstates. But we're going to solve the system today.

So good. So the system we're interested in is going to be--

The system we're interested in is going to be a system with a finite depth and a finite width. And I'll go into detail and give you parameters in a bit. But first I want to just think about how do we find energy eigenfunctions of a potential of this form, v of x, which is piecewise constant.

So first off, is this is a terribly realistic potential? Will you ever in the real world find a system that has a potential which is piecewise constant? Probably not. It's discontinuous. Right? So it's rather unphysical. But it's a very useful toy model.

So for example, if you take a couple of capacitor plates, then you can induce a situation where the electric field is nonzero in between the capacitor plates and zero outside of the capacitor plates. Right. So at a superficial level, this looks discontinuous. It looks like the electric field is--

But actually, you know that microscopically there are a bunch of charges, and everything is nice and continuous except for the behavior right at the charges. So but it's reasonable to model this as a step function for an electric field. So this is going to be an idealization, but it's going to be a very useful idealization, the

constant potential.

OK. So what's the equation? What are we trying to do? We want to find the energy eigenstates for this because we want to study the time evolution. And the easiest way to solve the Schrodinger equation, the time evolution equation, is to expand an energy eigenstates.

So the equation we want to solve is energy eigenvalue times phi e of x is equal to minus h bar squared on 2m phi e phi prime plus v of x. And I'm going to put it in the form we've been using. Phi prime prime e of x is equal to 2m upon h bar squared v of x minus e. OK. So this is the form that I'm going to use today to solve for the energy eigenvalue equations.

e is some constant. Do you expect the allowed energies to be arbitrary? No. They should be discrete. Yeah, exactly.

So we expect that there should be discrete lowest energy state, some number of bound states. And then, eventually, if the energy is greater than the potential everywhere, the energies will be continuous. Any energy will be allowed above the potential. So we'll have a continuum of states above the potential. And we'll have a discrete set of bound states--

Probably, it's reasonable to expect some finite number of bound states just by intuition.

--from the infinite well. So we expect to have a finite number of discrete energies and then a continuous set of energies above zero. So if this is the asymptotic value potential of zero. OK. And this is intuition gained from our study of qualitative structure of energy eigenfunctions.

So we are going to talk today about the bound states. And in recitation, leaders should discuss the continuum above zero energy. OK. So to solve for the actual energy eigenfunctions and the energy eigenvalues, what we need to do is we need to solve this equation subject to some boundary conditions. And the boundary conditions we're going to want to solve are going to be finite. So it's normalizable

infinity. The solution should be vanishing far away. And the wave function should be everywhere smooth. Well, at least it should be continuous.

So let's talk about what exactly boundary conditions we want to impose. And so in particular, we're going to want to solve for the energy eigenfunctions in the regions where the potential is constant and then patch together solutions at these boundaries. We know how to solve for the energy eigenfunctions when the potential is constant. What are the energy eigenfunctions?

Yeah. Suppose I have a potential, which is constant. v is equal to 0. What are the energy eigenfunctions of this potential?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, e to the ikx. Yeah. Now, what if I happen to tell you--

So we're at h bar squared k squared upon 2m is equal to the energy. Suppose I happen to tell you that here's the potential. And I want to find the solution in this region where the energy is here less than the potential. What are the solutions?

That's a second order differential equation. There should be two solutions. What are the solutions to that differential equation when the energy is less than the potential?

AUDIENCE: Decaying and growing.

PROFESSOR: Decaying and growing exponentials. Exactly. e to the plus alpha x and e to the minus alpha x. And the reason is these are sinusoidal, and these have the opposite concavity. They are growing and dying exponentials. Cool?

OK. So we've studied that. Yeah?

AUDIENCE: Shouldn't that be a phi e of x?

PROFESSOR: Sorry. Oh, oh, yes, indeed. Sorry. Thank you. Phi e of x. It's early, and I'm still working on the coffee. It won't take long. Good. So we know how to solve the energy eigenvalue equation in all these regions where the potential is constant. So

our job is going to be to find a solution where we patch them together at these interfaces. We patch them together.

And what condition should we impose? So the basic condition is going to be continuity of the wave functions phi e of x. And so what are the conditions that we need?

Well, if v of x--

Here's the way I like to think about this. Suppose v of x is continuous. So if the potential is continuous, then what can you say? You can say that phi prime prime is a continuous function times phi of x.

So very roughly, if we look at a region where phi isn't varying very much, so if we have a potential that's varying in some way, then phi prime prime, in a region where it's not varying much compared to its value as a function of x, does something smooth because it's varying with the potential. And so phi prime, which is just going to be the integral of this--

The integral of a smooth function is again a smooth function.

--and phi, the integral of that function is also going to be a nice, smooth function. OK? I've drawn it badly, but--

So the key thing here is that if the potential is continuous, then the energy eigenfunction has a second derivative, which is also continuous. That means its first derivative is continuous. So that means the function itself is continuous. Everyone agree with this? Questions? So in regions where the potential's continuous, the wave function and its first two derivatives all have to be continuous.

On the other hand, suppose the potential has a step. v of x has a step discontinuity. OK. So the potential does one of those. So what does that tell you about phi prime prime? It's a function of x.

So for example, let's look at that first step. Suppose the potential is that first step down by some amount. Then phi prime prime is going to decrease precipitously at

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some point. And the actual amount that it decreases depends on the value of phi because the change in the potential time is the actual value of phi.

And if that's true of phi prime prime what can you say of phi prime? So this is discontinuous. Phi prime of x, however, is the integral of this discontinuous function. And what does it do?

Well, it's linearly increasing in this region because its derivative is constant. And here, it's linearly increasing less. So it's not differentiably smooth, but it's continuous.

And then let's look at the actual function phi of x. OK. What is it doing? Well, it's quadratic. And then here it's quadratic a little less afterwards. But that still continuous because it's the integral of a continuous function. Everyone we cool with that? So even when we have a step discontinuity in our potential, we still have that our derivative and the value of the function are continuous. Yeah?

However, imagine the potential has this delta function. Let's just really push it. What happens if our potential has a delta function singularity? Really badly discontinuous.

Then what can you say about phi prime prime as a function of x? So it has a delta function, right? So the phi prime prime has to look like something relatively slowly varying, and then a step delta function. So what does that tell you about the derivative of the wave function?

It's got a step function. Exactly. Because it's the integral of this, and the integral is 0, 1. Whoops. I missed.

So this is a delta. This is a step. And then the wave function itself is, well, it's the integral of a step, so it's continuous. Sorry.

It's certainly not differentiable. Its derivative is discontinuous. Well, it's differentiable but its derivative's not supposed to be. It is not continuous, indeed.

So this is continuous. So we've learned something nice, that unless our potential is

so stupid as to have delta functions, which sounds fairly unphysical--

We'll come back to that later in today's lecture.

--unless our potential has delta functions, the wave function and its first derivative must be smooth. Yeah. This is just from the energy eigenvalue equation.

Now, we actually argued this from the well definedness and the finiteness of the expectation value of the momentum earlier in the semester. But I wanted to give you this argument for it because it's going to play a useful role. And it also tells us that if we do have a delta function singularity in the potential, then the upshot is that the wave function is going to be continuous, but its first derivative will not. Its first derivative will jump at the wave function.

And that means that the wave function--

Let me draw this slightly differently.

--the wave function will have a kink. Its derivative will not be continuous. OK. So anywhere where we have a delta function in the potential, we will have a kink in the wave function where the first derivative is discontinuous. Cool? Yeah.

AUDIENCE: What does that mean as far as the expectation value of the momentum?

PROFESSOR: Ah, that's an excellent question. So what do you expect to happen at such a point?

AUDIENCE: Well, your momentum blows up.

PROFESSOR: Yeah, exactly. So we're going to have some pathologies with expectation values in the momentum. Let's come back to that when we talk about the delta function potential, which should be at the end of today. Hold that question in the back of your head. It's a good question. Other questions?

OK. So what we're going to do now is we're going to use, you know, math to find the energy eigenfunctions and eigenvalues for the finite well potential, i.e. we're going to solve this equation contingent on the boundary conditions that the wave function

and its derivative are smooth everywhere. And in particular, they must be smooth here and here. I mean they've got to be smooth everywhere.

We know what the solution is inside here. We know what the solution is in here. All we have to worry about is what happens at the interface. And we're going to use smoothness of the wave function and its derivative to impose conditions that allow us to match across that step. Everyone cool with that?

OK. So let's do it. By the way, just a quick side note. Let me give you a definition. I've used this phrase many times, but I haven't given you a definition of it.

So I've used the phrase, bound state. And its opposite is called a scattering state. So here's what I mean by a bound state. Intuitively, a bound state, if you think about this classically, imagine I have a potential well. And I have a marble, and I let go from here. This marble is bound. Right? It's never getting out of the well. It's stuck. Yeah?

And so I would call that a bound marble. On the other hand, a marble that I give a huge kick to, big velocity so that it can get out, well then it's not bound to this potential well. So I'll call that a scattering state just to give it a name. And next lecture we'll see why we call it a scattering state.

The important thing is that bound configurations of a classical potential are things basically in a well that are stuck. OK. So the quantum version of that statement is the following.

Suppose I take this potential, and I treat it quantum mechanically, and I consider a state with total energy like this. Well, among other things, the total energy is less than the value of the potential asymptotically far away. So what is the form of a wave function in this region with this energy? Exponential, right? It's e to the minus some alpha x, where alpha squared is roughly the difference, alpha x.

And out here it's going to be e to the plus alpha x. And that's for normalizability. We want to have a single particle in this state.

So what that tells us is that the wave function falls off in these classically disallowed regions exponentially. And so the probability of measuring the particle at an arbitrarily far position goes to zero. And it goes to zero exponentially rapidly. Cool?

So I will call a bound state, a quantum bound state, an energy eigenstate, such that the probability falls off exponentially as we go far away from wherever we think is an interesting point, like the bottom of the well. Cool? A bound state is just a state which is exponentially localized. If you put it there, it will stay there. Yeah.

And it's important that when I say a bound state I'm talking about energy eigenstates. And the reason is this. Bound state equals energy eigenstate. The reason is that, consider by contrast a free particle, so a free particle with constant potential. What are the wave functions? What are the energy eigenfunctions?

Well, they're plane waves. Right. So are these bound states? No. Good.

OK. On the other hand, I claimed that I can build a wave packet, a perfectly reasonable wave packet, which is a Gaussian. OK. This is some wave function, psi of x times 0. It's a Gaussian. It's nice and narrow. Is that a bound state?

Well, it's localized at this moment in time. But will it remain, and in particular its probability distribution, which is this thing norm squared is localized in space--

Sorry. This was zero.

--it's localized in space. The probability of it being out here is not just exponentially small, it's Gaussian so it's e to the minus x squared. It's really not out here. But what happens if I let go? What happens if I look at the system at time t? It's going to spread out. It's going to disperse. We're going to talk about that in more detail later.

So it's going to spread out. And eventually, it will get out arbitrarily far away with whatever probably you like. So the probability distribution is not time invariant. That is to say it's not a stationary state. It's not an energy eigenstate. Saying something is bound means that it never gets away.

So bound states are specifically energy eigenstates that are strictly localized, that

fall off at least exponentially as we go away from the origin. Cool? It's just terminology, what I mean by a bound state. Questions?

OK. So let's talk about the finite well. So I need to give you definitions of the parameters. Let's draw this more precisely. Here's my well. Asymptotically, the potential is zero. The potential depth, I'm going to call minus v naught. OK. And I'm going to center the well around zero. And I'll call the sides minus l and l.

And I want to find bound states of this potential, just like we found bound states of the harmonic oscillator, i.e. states with energy e, which is less than zero. So these are going to give us bound states because we're going to have exponential fall offs far away.

So a couple of things to note. The first is on, I think, problem set three or four you showed that if you have a potential, which is symmetric, which is even, under x goes to minus x, then every energy eigenfunction, or at least every bound state energy eigenfunction, every energy eigenfunction can be written as phi e symmetric or phi e anti-symmetric. So it's either even or odd. It's either even or odd under the exchange of x to minus x.

So when our potential is symmetric, the wave function or the energy eigenfunctions are either symmetric or anti-symmetric. OK. So we want to solve for the actual eigenfunctions. So we want to solve that equation.

And we have this nice simple fact that we know the solutions in this region. We know the general solution in this region. We know the general solution in this region. So I'm going to call these regions one, two, and three, just to give them a name.

So in region one--

That's actually sort of stupid. Let's call this inside, left, and right. Good. OK. So let's look at this equation.

We have two cases. If the energy is greater than the potential in some region, then this is of the form phi prime prime is equal to energy greater than potential. This is a

negative number. And so this is a minus k squared phi. And we get exponentials. And if, on the other hand, e is less than v of x, then phi prime prime is equal to plus alpha squared phi. I should say oscillator.

And in particular here, I want the k squared is equal to 2m upon h bar squared. It's just the coefficient v minus e. And alpha squared is equal to 2m over h bar squared e minus v.

OK. So let's apply that here. So in this region, we're going to get oscillations because we're in a classically allowed region, where the energy is greater than the potential. So we'll get oscillatory solutions. And the salient value of k inside, is k is equal to the square root of 2m over h bar squared times v minus e. So that's minus v naught.

What did I do? I did. It's e minus v. I thank you. Yes. And I want the other one to be l so it'd be minus v. Good. Good. Excellent.

So root 2 over h bar squared. And now we have e minus v naught, which is the actual value of e, which is negative. Right? But minus v naught or plus v naught is positive and greater in magnitude. So this is a nice positive number, and k is the square root of it. This is controlling how rapidly the wave function oscillates in this region.

Similarly, out here we have alpha is equal to--

Well, here the potential is zero. So it's particularly easy. Alpha is equal to the square root of 2m upon h bar squared of--

Now z minus e is zero minus e, which is a negative number. So we can just write e absolute value. So we can write the general solution of this eigenfunction, of this eigenvalue equation, as phi e of x is equal to--

Let's break it up into inside and outside. Well, inside we know, since it's constant with this value of k, we get superpositions of oscillatory solutions. It' a second-order difference equation. There are two solutions and two integration constants.

So first we have a cosine of kx plus b sine kx. This is inside. And then on the left we have a combination of exponentially growing and exponentially decreasing. So the exponentially growing is e to the alpha x plus de the minus alpha x. And on the right, similarly by symmetry, we have some combination of e--

But I don't want to call it the energy, so I'll call it the curly e.

--to the alpha x plus fe to the minus alpha x. OK. So that's the general solution. We solve the problem as a superposition of the two oscillatory solutions or a superposition of the two exponentially growing and damped solutions or exponentially growing and collapsing functions on the left and right. Questions?

So a couple of things to note at this point. So the first is we have boundary conditions to impose. We have boundary conditions at these two interfaces. But we also have boundary conditions off in infinity. What are the boundary conditions at infinity?

Yeah, exactly. It should vanish. So we want the system to be normalizable. So normalizable is going to say that phi goes to zero at minus infinity. Phi of x goes to minus infinity goes to zero, which it equals. And phi of x goes to plus infinity should also be zero. OK. And then we're also going to have the conditions at the left boundary, and we're going to have conditions at the right boundary.

[LAUGHTER]

All right. So what are the boundary condition at the left boundary condition? So first off, what are the boundary conditions we want to impose at the left and right boundaries?

AUDIENCE: Continuous.

PROFESSOR: Continuous, and the derivative should be continuous. Exactly. So we have that phi is continuous, and phi prime--

Good god.

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--phi prime is continuous. Similarly, phi continuous, phi prime continuous. OK. Do we have enough boundary conditions to specify our function? So we have now for our solution, we have six undetermined coefficients. And we have six boundary conditions. So that looks good.

Are they all independent? Ponder that one. So in particular, let's start with the normalizable. So in order for phi to go to zero at minus infinity deep out on the left, what should be true? Yeah, d goes to zero. Oops, equals zero. And on the right?

Yeah, that curly e equals 0, which is nice so I don't ever have to write it again. So that's zero. And that's zero. OK. That's good.

We can take advantage though of something nice. We know that the wave function has to be either symmetric or anti-symmetric. Right? So we can exploit that and say, look, the wave function is going to be different from our boundary conditions, but it's a true fact, and we can take advantage of it. We can use the parody of the well. I can never--

So we can use the parody of the potential to say that the system is either symmetric or anti-symmetric. And these are often said as even or odd because the function will be, as a function of x, either even or odd. You either pick up a minus sign or a plus sign under taking x to minus x.

So if the system is even, what can we say about these coefficients? What must be true of b, for example?

AUDIENCE: It's zero.

PROFESSOR: Yeah. So b equals 0. And what else?

AUDIENCE: It equals f.

PROFESSOR: This equals f. Yeah, good. OK. This equals f. And if the system is anti-symmetric, then a equal to 0. And we see that c is equal to minus f. Yeah? So that's a useful simplification.

So it's easy to see that we could do this either way. We could do either symmetric or anti-symmetric. I'm going to, for simplicity in lecture, focus on the even case. b is equal to 0, and c is equal to minus f. Sorry, c is equal to f. So plus c. So now we're specifically working with the even solutions. And on your problem set, you'll repeat this calculation for the odd functions.

So we're going to focus on the even solutions. And now what we have to do is we have to impose the boundary conditions for phi and phi prime. So that's easy enough.

We have the function. All we have to do is impose that the values are the same. So for example, let's focus on the left boundary conditions. Sorry. Let's focus on the right because I don't want to deal with that minus sign. So let's focus on the right boundary conditions.

So this is x is equal to plus l. So when x is equal to plus l, what must be true? Phi and phi prime must be continuous. So what's phi? So phi is equal to--

Well, from inside, it's a cosine of kl. Yeah. And b is gone because we're only looking at the even functions. On the right, however, it's equal to c e to the minus alpha l because we're evaluating at the right boundary. Yeah?

OK. So this is cool. It allows us to determine c in terms of a. And if we solve that equation for c in terms of a, we'll get an eigenfunction, phi even, with one overall normalization coefficient, a. And then we can fix that to whatever it has to be so that everything integrates to one. Yeah? So that seems fine. It seems like we can solve for c in terms of a. c is equal to--

This is weight. Well, OK. So c is equal to a cosine e to the plus alpha l. On the other hand, we also have a condition on the derivative. And the condition on the derivative is that phi prime is continuous. And the derivative of this, well that's easy. It's the derivative of cosine.

So this is going to be minus sine. But we pull out a factor of k, because we're taking derivative with respect to x. Minus k is sine of kx. Evaluate it out, sine of kl, l. And

this is going to be equal to minus alpha c e to the minus alpha l.

AUDIENCE: You forgot an a.

PROFESSOR: Oh, yes. There should be an a. Thank you. OK. So but now we've got a problem because this says that c is equal to minus a times k over alpha sine of kl times e to the alpha l. And that's bad because c can't be equal to two different numbers at the same time. There's a certain monogamy of mathematical equations. It just doesn't work.

So how do we deal with this? Well, let's think about what these equations would have meant. Forget this one for the moment, and just focus on that first expression. I'm going to rewrite this slightly. a cosine of kl. OK. So what does this expression say?

Well, it seems like it's just saying if we fix c to be equal to this value, for fixed value of kappa l and alpha, then there's a solution. However, what is k? What are k and alpha. k and alpha are functions of the energy. So it would seem from this point of view, like for any value of the energy, we get a solution to this equation. Everyone see that?

But we know that can't possibly be right because we expect the solutions to be discrete. We don't expect any value of energy to lead to a solution of the energy eigenvalue equation. There should be only discrete set of energies. Yeah?

AUDIENCE: Did you pick up an extra minus sign in the expression for c?

PROFESSOR: You do. Thank you. The sign's cancel. Yes, excellent. I've never written this equation in my life. So thank you. Yes, extra minus sign.

So what's going on here? Well, what we see is that we've written down the general form of the solution. Here were imposing that we've already imposed the condition that we're normalizable at infinity. Here, we're imposing the continuity condition on the right. And if we impose just the continuity condition for the wave function, we can find a solution.

Similarly, if we impose only the continuity condition for the derivative, we can find a solution for arbitrary values of the energy. But in order to find a solution where the wave function and its derivative are both continuous, it can't be true that the energy takes just any value because it would tell you that c takes two different values. Right? So there's a consistency condition.

For what values of energy or equivalently, for what values of k and alpha are these two expressions equal to the same thing? Cool? So we can get that by saying, look we want both of these equations to be true. And this is easy. I can take this equation and divide it by this equation. And I will lose my coefficients c. I will lose my coefficients a. What do we get?

On the right hand side, we get--

And I'm going to put a minus sign on everything. So minus, minus, minus. So if we take this equation and we divide it by this equation, on the right hand side, we get alpha, because the c exponential drops off. And on this side, we lose the a. We get a k. And then we get sine over cosine of kl, also known as tangent of kl. Here we have a kl. Here we have a k.

These are all dimensionful things. Let's multiply everything by an l. And this is nice and dimensionless. Both sides are dimensionless.

So we get this condition. This is the consistency condition, such that both the wave function and its derivative can be continuous at the right boundary. OK? And this is a pretty nontrivial condition. It says, given a value of k, you can always determine the value of alpha, such as this equation as true. But remember that k and alpha are both known functions of the energy. So this is really an equation, a complicated, nonlinear equation for the energy.

So this is equal to a horrible expression, a condition, badly nonlinear, in fact, transcendental condition on the energy. And where's it coming from? It's coming from normalizability and continuity everywhere. And a useful thing to check, and I invite you to do this on your own, is to check that the boundary conditions at the left

wall give the same expression. Yeah.

AUDIENCE: For our final form of that equation, is there a reason that we prefer to multiply both sides by l than divide both sides by k?

PROFESSOR: Yeah. And it'll be little more obvious in a second. But here's the reason. So let's divide through by l. This is the form that we got. What are the units? What are the dimensions of this expression?

k is a wave number, so it has units of 1 upon length. Right? And that's good because that's 1 upon length times the length, and you'd better have something dimensionless inside a tangent. But it seems there are two things to say about this. The first is it seems like l is playing an independent role from k in this equation. But this is dimensionless. These are both dimensionful units of 1 over length.

So we can make the entire expression dimensionless and make it clear that k and l don't have an independent life. The dimensionless quantity, kl, times the tangent of that dimensionless quantity is equal to this dimensionless quantity. So the reason that this is preferable is twofold.

First off, it makes it sort of obvious that k and l, you can't vary them independently in this sense. But the second is that it makes it nice and dimensionless. And you'll always, whenever possible, want to put things in dimensionless form.

I mean it's just multiplying by l. So it's obviously not all that deep. But it's a convenient bit of multiplication by l. Other questions?

OK. So where are we? So I'd like to find the solutions of this equation. So again, just to--

Let me write this slightly differently where k squared is equal to 2m upon h bar squared v0 plus e. And alpha squared is equal to 2m upon h bar squared e, the positive value of e. So this is a really complicated expression as a function of e. So I'd like to solve for the actual energy eigenvalues. I want to know what are the energy eigenvalues of the bound states of the finite potential well, as a function of l,

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for example.

Sadly, I can't solve this equation. It's a transcendental equation. It's a sort of canonically hard problem to solve. You can't write down a closed from expression for it. However, there are a bunch of ways to easily solve it.

One is take your convenient nearby laptop. Open up Mathematica, and ask it to numerically find solutions to this. And you can do this. It's a good exercise. I will encourage you to do so on your problem set. And in fact, on the problem set, it asks you to do a calculation. And it encourages you do it using Mathematica.

Let me rephrase the statement in the problem set. It would be crazy for you to try to do it only by hand. You should do it by hand and on computer because they're both easy. And you can check against each other. They make different things obvious. This should be your default is to also check on Mathematica.

The second thing we can do is we can get a qualitative solution of this equation just graphically. And since this is such a useful technique, not just here, but throughout physics to graphically solve transcendental equations, I'm going to walk through it a little bit. So this is going to be the graphical solution. And we can extract, it turns out, an awful lot of the physics of these energy eigenstates and their energies through this graphical technique.

So the first thing is I write this in nice, dimensionless form. And let me give those dimensionless variables a name. Let me call kl is equal to z, just define a parameter z. And alpha l is a parameter y. And I want to note that z squared plus y squared is equal to a constant, which is, I will call if you just plug these guys out, kl squared plus al squared. That's easy. Kl squared is this guy times l squared. Al squared is this guy times l squared. And so the e and the minus e cancel when we add them together.

So we just get 2mv0 over h bar squared times l squared. So 2m upon h bar squared l squared v0. And I'm going to call this r naught in something of a pathological abusive notation. OK. So this is our expression. And I actually want to call this r0

squared.

I know. I know. It's awful. But the reason I want to do this is that this is the equation for a circle. Yeah? And a circle has a radius. The thing that goes over here is r squared.

OK. So at this point, you're thinking like, come on, circle. So let's plot it. So how are we going to solve this equation? Here's what I want to solve. I have now two equations relating z and y.

We have that from this equation z tangent z is equal to y. And from this equation we have that z squared plus y squared is a constant r0 squared. Where r0 squared depends on the potential and the width in a very specific way, on the depth of the potential and the width in a very specific way. So we want to find--

Bless you.

--simultaneously, we want to find simultaneous solutions of these two equations. Yeah? So that's relatively easy. So here's y, and here's z.

So this equation has solutions. Any time that y plus z squared is equal to r0 squared, that means any time we have a circle. So solutions for fixed values of r0 lie on circles. Oh, I really should have drawn this under here. Sorry. y and z. So those are the circles.

Notice that I'm only focusing on y and z, both positive. Why? Not yz, but W-H-Y . Why am I focusing on the variables y and z being positive? Because we started out defining them in terms of k and l, which were both positive, and alpha and l, which were both positive.

Can we find solutions to this equation that have x and y negative? Sure. But they don't mean anything in terms of our original problem. So to map onto solutions of our original problem, we want to focus on the positive values of y and z. Cool?

OK. So that's this one. The solutions lie on circles. So given a value of y, you can find a solution of z. But we want to also find a solution of this equation. And this

equation is a little more entertaining to plot. Here's y. Here's z. So what does z tangent z do?

Oh, shoot. I want to plot y vertical. Otherwise, it's going to a giant pain. Happily, this plot can be left identical. Let's plot y vertically.

So the reason I want to plot y vertically is that this is z tangent z. So first off, what does tangent z look like? Yeah. This is awesome. Yeah, it looks like this. Yes, exactly.

So tangent is sine over cosine. Sine is zero, and cosine is one. So it does this, as you go to a value where the argument, let's call the argument z. So if we just plot tangent--

OK. So when z is equal to pi over 2, then the denominator cosine vanishes, and that diverges. Oops. OK. So here's pi over 2. Here's pi. Whoops. Pi, pi over 2, and here's 3pi over 2, and so on.

Now, we're only interested in the first quadrant. So I'm just ignore down here. OK. So this is pi over 2. This is pi, 3pi over 2. OK.

But this is not what we're interested in. We're not interested in tangent of z. We're interested in z tangent z. And what does z tangent z look like? Well, it's basically the same. Right?

z tangent z, it has an extra factor of zero here and remains extra small at the beginning. But it still curves off roughly like this. And z is just nice and linear, nice and regular throughout this. So it doesn't change the fact that we have a divergence at pi over 2. And it doesn't change the fact that it vanishes again at pi and becomes positive again. It just changes the shape of the curve.

And in fact, the way it changes the shape of the curve is this becomes a little fatter around the bottom. It's just a little more round. And when we get out to large values of z, it's going to have a more pronounced effect because that slope is, at every example where it crosses z, that slope is getting larger because the coefficient of z

is getting larger. OK. So it's just going to get more and more sharp.

But anyway, with all that said, here's 0. Here's pi. Here's pi over 2. Here's pi. Here's 3pi over 2.

The second plot we want to plot, y is z tangent z. We know how to plot this. Cool? And what we want to find are simultaneous solutions of this, values of y and z, for which this equation is solved and this equation is solved for the same value of y and z. This is a graphical solution. So let's combine them together. And the combined plots look like this. First we have pi over 2. So let's plot the tangents.

And then we have these circles for various values of r. So for a particular value of r, for example, suppose this is the value of r. This is r0. So how many solutions do we have?

One. One set of common points where at y and z solve both equations. So we immediately learn something really lovely. What happens to the radius of that circle as I make the well deeper?

Yeah, as I make the well deeper, that means v0 gets larger and larger magnitude, the radius gets larger. So does the circle. So if I make the well deeper, I make this the circle larger.

Will I still have a solution? Yeah, I'll still have a solution. But check this out. Now, I'll have a new solution.

And you can even see the critical value of the depth and the width of the well. In order to have exactly a new bound state appearing, what must the value of r0 be? Well, it's got to be that value, such that r0 squared is pi. Yeah?

And similarly, let me ask you the following question. As I make the well deeper and deeper and deeper, holding the width, and make it deeper and deeper and deeper, does the number of states increase or decrease? It increases. If you make it deeper and deeper, the radius of that circle is getting bigger and bigger. There are more points where this circle intersects this point.

So here's another one. We've got one here, one here, one here, three solutions. And the number of solutions just goes. Every time we click over a new point by increasing the radius of the circle, we get a new solution. We get another bound state.

But here's the thing that I really want to focus on. Let's make the well less and less deep. Let's make it shallower and shallower. At what depth do we lose that first bound state?

We never do. Right? There is no circle so small that it doesn't intersect this curve. In a 1D, finite well potential, there is always at least one bound state. There are never zero bound states. This will turn out not to be true in three dimensions, which is kind of interesting. But it's true in one dimension that we always have at least one bound state.

And in fact, you can decorate this. You can use this and fancy it up a bit to argue that in any potential in 1D, there's always at least one bound state unless the potential is constant, I mean any potential that varies and goes to zero infinity. Yeah?

And so we still don't have any numbers. But we know an awful lot about the qualitative structure of the set of energy eigenvalues of the spectrum of the energy. Questions? Yeah?

AUDIENCE: So what happens if r is bigger than pi or y is bigger than pi and you get two solutions?

PROFESSOR: Great. So when we have two solutions, what does that mean? Well, you've got to bound states, two different energies. Right? It's a good question. Every solution here corresponds to some particular value of y and some particular value of z. But those values of y and z are just telling you what k and alpha are. And so that's determining the energy.

So a different value of k is going to give you a different value of the energy. So we

can just eyeball this in particular. Let's look at alpha. Alpha is just e. Alpha squared is just e. Yes?

So here's a quick question. If alpha is just e, and alpha l is y--

So this is our y value. y is roughly alpha, the width, which means it's roughly the energy square root. So this value, the vertical value of each of these intersection points on a given circle corresponds to the square root of the energy times some coefficients. So which state has the largest value of the energy? Absolute value, which state is most deeply bound on this circle?

Yeah, the first one. Right? Because it's got the largest value of alpha. So this is nice. We see that the first state always has a higher value of alpha than the second state, which always has a higher value of alpha than the third state. And every time we add a new state, we make the depth of these guys the binding energy of the already existing states. We make it just a little bit deeper. We make them a little more tightly bound. And only eventually then do we get a new bound state appearing.

And what's the energy of that new bound state when it appears? Zero energy. It's appearing just at threshold. OK. So we'll explore that in more detail in the problem set. But for now, let me leave it at that. Questions? Other questions? Yeah.

AUDIENCE: You said that this can be generalized to any nonconstant function that you'd like. So there's always going to be at least one bound state. What about, like with delta function?

PROFESSOR: Excellent question. What about the delta function? We're going to come back to that in just a few minutes. It's a very good question.

So the question is, look, if any potential that goes to zero infinity and wiggles inside, if any potential like that in 1D has a bound state, what about the delta function? We briefly talked about that. So we're going to come back to that in just a few minutes. But it's a pressing question. OK. Other questions? Yes.

AUDIENCE: So the energy is zero, but that's not possible.

PROFESSOR: Thank you. OK. Good. So let me talk about that in a little more detail. So I wasn't going to go into this, but--

So when new bound states appear, so let's consider a point where our r0 is, let's say, it's just the right value so that r0 is equal to pi. OK. And we see that we're just about to develop a new bound state. So let's think about what that bound state looks like. So this is the new bound state.

And I'm going to put this in parentheses because it's got bound state. And we say at threshold. OK. At threshold, i.e. at the energy is roughly zero, and r0 is equal to pi. So this is really what we mean. This new state, when r0 is pi and we have a solution on that second branch. Cool?

So what does this wave function look like? What does it look like when you have a wave function that just appeared? It's just barely bound.

Well, first off, what does it mean to be bound? Let's just step back and remember for now. What does it mean to be a bound state? It means you're an energy eigenfunction and you're localized. Your wave function falls off at infinity.

Now, if it falls off at infinity, do these guys fall off at infinity? These wave functions, sure, they fall of with an exponential damping. And in particular, let's look at the right hand side of the well.

This new bound state is appearing just at zero energy. So out here, what is the wave function? It's e to the minus alpha x. But what's alpha? Zero, right? There it is, zero.

So this is e to the minus alpha x where alpha is equal to 0. This is constant. So what does the wave function look like? Well, the wave function, again over the same domain--

Here's 0, l. And here's the value zero. We know that in this domain it's oscillatory, and in this domain, it's constant. And actually, since we know that it's the first excited state, we know that it does this.

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So if we make the well ever so slightly deeper, ever so slightly deeper, which means making the radius of the circle ever so slightly larger, we will get a nonzero value for the alpha of this solution. Right? It'll be just tiny. But it'll be nonzero. So we make the well just a little tiny bit deeper. We get something. OK, good.

So what's going to happen to the wave function? Well, instead of going flat. This is going to curve. It's got a little more energy. There's a little more curvature. So it curves just a little tiny bit more. And then it matches onto a very gradually decaying exponential.

OK. So what's happening as we take this bound state, the second bound state, and we make the well a little more shallow? We make the well a little more shallow. It's a little less curved inside. And the evanescent tails, the exponential tails become longer and longer and longer and broader until they go off to infinity, until they're infinitely wide.

And is that normalizable anymore? No, that's not normalizable. So the state really isn't strictly localized at that point. It's not really a normalizable state. And just when the state ceases to be normalizable, it disappears. We make the well just a little deeper, and there's no state there at all.

OK. So this tells you a very nice thing. It's a good bit of intuition that when states are appearing or disappearing, when states are at threshold as you vary the depth, those threshold bound states have exceedingly long evanescent tails, and they're just barely bound. OK? This turns out to have all sorts of useful consequences, but let me move on. Did that answer your question? Good. Yeah?

AUDIENCE: So in this case, the radius, as you say, is proportional to the length. Right? But also we have intuition that as we increase the length of the well, the energy's going to keep increasing--

PROFESSOR: Fantastic. OK. So what's up with that? Right. So the question is this. We already have intuition that is if we take a finite well and we make it a little bit wider, the ground state energy should decrease. The energy of the ground state should get

deeper and deeper, or the magnitude should increase, another way to say it. Right? That was our intuition.

So let's check if that's true. What happens if I take the ground state some value of the radius, and then I make the well a little bit wider. Well, if I make the well a little bit wider, what happens to r0, the radius of the circle? Well, if I double the length of the well, the width of the well, then it will double r0, and it will double the radius of the circle.

So if I make it wider, r0 gets bigger, and we go to a bigger circle. And what happened to the energy of this state? Yeah, it got deeper and deeper and deeper. And meanwhile, as we make it wider, as we make the well wider, the circle is getting bigger again. And we're going to get more and more states.

So as we make the well wider, holding the depth fixed, we get more and more states. As we make the well deeper, holding the width fixed, we get more and more states. And so how do you trade off? If I make it twice as wide, how much--

So here's a good question. Suppose I take a well, and it has n states. Suppose I then make it twice as wide. What must I do to the energy so that it still has the same states with the same energies?

AUDIENCE: Divide it by 4.

PROFESSOR: Yep, exactly. I've got to divide it by 4. Because I've doubled the length, that means the radius has gone up by four. But I get exactly the same solutions if I just bring this out. Well, that's almost true. So if I put a factor of 1/4 here, that's almost true, except for the value of y is unchanged, but the alpha hasn't changed. Sorry. The alpha has changed because there's an l.

So y is fixed. But alpha's changed because of the l. So the reason that it's useful to write things in these dimensionless forms is that you can see the play off of the various different dimensionful parameters in changing the answer. Other questions.

OK. So a couple of comments. So the first is let's just check to make sure that this

makes sense. We have already solved this problem. We solved this problem a while ago, but we solved it in a particular limit. We solved it in the limit that the potential one's arbitrarily deep. Right?

So when the potential one's are arbitrarily deep, holding the width fixed, that was the infinite well. That was the first problem we solved. So let's make sure that we recover this in that limit.

So what happens as we make the well arbitrarily deep, holding the length fixed or the width fixed. So if we make this arbitrarily deep, v0 is getting arbitrarily large. That means r0 is getting--

We've got a huge circle. So what do these solutions look like when we have a huge circle. Let me not do that here. Let me do that here.

So if we make the potential nice and deep, Let's think about what that plot looks like. So again, that first plot looks identical with the tangents. So on and so forth.

And what I want to do is I want to dot, dot, dot. OK. So this is way up there. So these guys are basically vertical lines at this point. So for very, very large values of y, and in particular, for very large values that are of order the gigantically deep radius r0, what does the circle look like? So what does the second equation look like, the second curve?

Well, again it's circles. But now it's a gigantic circle. Yeah, exactly. If it's a gigantic circle, it's basically flat. It's not exactly flat, but it's almost flat. It's a circle.

So what are the values? And here's the key question. What are the values of--

Did do that right? Yeah, OK. Good.

So what are the values of the curve, where we get a solution? The values now of z, are just exactly on these vertical lines, on the separatrices. The value of z is at pi over 2. and 3pi over 2. And then another one at 5pi over 2, and so on and so forth. Right?

So what we find is that the allowed values of z--

Sorry. kl, yes, good.

--allowed values of z are equal to 2n plus 1 over pi. Whoops 2n plus 1 over 2 times pi. So let's just check that. So n is 0. That's 0 1/2 pi. n is 1. That's 3/2 pi. Good. So these are the values of z, which says that kl is equal to 2n plus 1 upon 2 pi or k is equal to 2n plus 1 over 2l, which is the width of this well because it's from minus l to l pi.

So is this the correct answer for the infinite square well? Are these the allowed values of k inside the well for the infinite square well? Almost. Instead of 2n plus 1, it should just be n plus 1. We seem to be missing about half of the energy eigenvalues.

AUDIENCE: That's only the even ones.

PROFESSOR: Yeah, thank you. This is only the even ones. We started out saying, oh, look. Let's look only at the even ones. Where do you think the odd ones are going to be? Ah, well the odd ones, so this should be k even.

So what about k odd? Well, we know the answer already--

Whoops. Odd, that's an odd spelling.

--should be equal to 2n over 2l plus 2 over 2l. Whoops. 2 capital L, pi. OK. This is our guess just from matching on to the infinite square well. So what does that mean? Well, that means it should be this one and this one.

So when you go through this exercise on your problem set and you find the solutions for the odd, and you repeat this analysis for the odd ground states. What should you expect? Well, you should expect to find this. And what do you think the curves are that you're going to use int he graphical solution to do your transcendental equation?

Yeah, it's really tempting to say, look. It's just going to be something shifted, like

this. OK. So these are going to be the odd question mark, question mark. OK. So you'll check whether that's correct intuition or not on your problem set. OK. Questions? Yeah?

AUDIENCE: What about the lower end of these states, where it won't have gone off high enough? Is that [INAUDIBLE]?

PROFESSOR: Excellent. So that's a really good question. How to say it? What we've done here to make this an infinite well--

So the question, let me just repeat the question. The question is, look, what about all the other states? OK, it's true that r0 as gigantic. But eventually, we'll go to a large enough z, where it's the circle coming down over here too. So what's up with that? What are those states? Where are they? What do they mean in terms if the infinite square well?

Well, first off, what are the energies of those states?

AUDIENCE: Very low.

PROFESSOR: They're very low in magnitude, which means they're close to what in absolute value?

AUDIENCE: Zero.

PROFESSOR: Zero. They're close to zero. So they're at the top of the finite well. These are the states bound at the top of the finite well. These are the states bound at the bottom the finite well.

But how many states are bound at the top of the finite well when we take the limit that the well goes infinitely deep? Yeah, none of them. Right?

So when we make the well infinitely deep, what we're saying is, pay no attention to the top of the well. Look only at the bottom of the well. And if it's really deep, it's a pretty good approximation.

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So that's what we're doing here. We're saying, look. Pay no attention. There is no top of the well. There's just the bottom. And look at the energy eigenvalues. Does that make sense?

So what we're saying is if you have a preposterously deep well, the energies of a preposterously deep well should be a good approximation to the low-lying energies of an infinitely deep well. Because it's way up there, what difference can it make? And that's what we're seeing work out. Did that answer your question?

AUDIENCE: Wait. I might have this backward. But when it says the high energy instead of looking right at the top of the well--

PROFESSOR: Yeah. OK, good. So this is an important bit of intuition. So when we say this is the energy zero, and the potential has a minimum at minus v0, and we're measuring the energies relative to zero, then the states at the top of the potential well are the states with energy close to zero. And the states at the bottom of the potential well are those states with the energy of order v0. cool?

And what are the energies of all these states? They're order of v0. Right? Because this is--

Are they exactly v0? No, because this isn't linear. It's actually a circle. So there's going to be a correction, and the correction is going to be quadratic. If you work out that correction, it's correct. The depth above the bottom of the potential is correct for the energy of the corresponding infinite well problem. I'll leave that to you as an exercise. Other questions?

OK. So there's another limit of this system that's fun to think about. So this was the infinite well limit. What I want to do is I want to take advantage of the observation we made a second ago that as we make the well deeper, we get more states. As we make the well more narrow, we get fewer states.

To trade that off, consider the following limit. I want to take a potential well, which has a ground state. What does the ground state look like? So the ground state wave function is going to be--

So here's zero. And here's exponentially growing. Here's exponentially decreasing. And if it's a ground state, how many nodes will it have inside? How many nodes will the ground state have?

AUDIENCE: Zero.

PROFESSOR: Zero. Good. OK. So the ground state will look something like this. We more conventionally draw it like this. But just for fun, I'm going to draw it in this fashion.

In particular, it has some slope here. And it has some slope here. Oh, shoot. Did I? Yes, I did. Dammit. I just erased the one thing that I wanted to hold onto.

OK. So there's my wave function. It has some particular slope here. It has some particular slope here. And this is the ground state wave function, with some energy. I don't know. I'll call it this.

Now, what I want to do is we've already shown that as we make the well more and more shallow and more and more narrow, the energy of the ground state gets closer and closer to zero. But there remains always a bound state. There is always at least one bound state. We proved that. Proved, as a physicist would.

So I want to do that. I want to take this seriously. But here's the limit I want to consider. Consider the limit that we make the potential v goes to infinity, v0, while making l go to zero. So what I want to do is I want to take this thing, and I want to make it deeper and deeper, but more and more narrow.

If I do this repeatedly, eventually I will get a delta function. And I will get a delta function if I hold the area of this guy fixed. Yeah? So if I do so, holding the area under this plot fixed, I will get a delta function. Everyone cool with that?

So let's think, though, quickly about what's going to happen to the ground state wave function. So as I make the potential, let's take this wave function, and let's look at this version of the potential. So as I make the potential deeper, what happens to the rate of the oscillation inside or to the curvature inside? It increases. Right?

So the system is oscillating more, it changes more rapidly, because phi double dot or phi double prime is equal to v minus e phi. It oscillates more rapidly. So to make it deeper, the system tends to oscillate more rapidly.

However, as we make it more narrow, the system doesn't have as far to oscillate. So it oscillates more rapidly, but it doesn't oscillate as far. So what's going to happen?

Well, as we make it more and more narrow and deeper and deeper, we again have the wave function coming in. And now it oscillates very rapidly. Let's do it again.

The wave function comes in, and it oscillates very rapidly. And the it evanescent tail out. And now as we have a delta function, exponential damping, it oscillates extremely rapidly over an arbitrarily short distance and gives us the kink that we knew at the very beginning we should expect when the potential is a delta function. Right? From our qualitative structure of the wave function at the very beginning we saw that when we have a delta function potential, we should see a kink in the wave function. Because again, if we have phi prime prime is delta function discontinuous, phi prime is the integral of this. This is a step, and phi is continuous.

And so here we have a step function. We get a discontinuity in the second derivative. Here we have a delta function in the potential, and we get a discontinuity in the first derivative if we get a kink in the wave function. Yeah?

AUDIENCE: Would we get a jump there?

PROFESSOR: Sorry. Say again. For e1?

AUDIENCE: Yeah. Would we get a jump?

PROFESSOR: Very good question. So let me do this more seriously. Let's do this more carefully. So the question is, for the first excited state, do we get a jump? Do we get a discontinuity? What do we get for the first excited state? Right?

So let's talk about that in detail. It's a very good question. Example v is equal to minus v0 delta of x. Now, here I want to just warn you of something. This is totally

standard notation for these problems, but you should be careful about dimensions.

What are the dimensions of v0, the parameter of v0? It's tempting to say energy. That's an energy. That's an energy.

But wait. What are the dimensions of the delta function?

AUDIENCE: Length.

PROFESSOR: Whatever length right? Because we know that delta of alpha x is equal to 1 over norm alpha delta of x. So if I write delta of x, which is always a slightly ballsy thing to do because this should really be dimensionless, but if I write delta of x, then this has units of 1 over length, which means this must have units of length times energy.

OK. Just a little warning, when you check your answers on a problem, you always want to make sure that they're dimensionally consistent. And so it will be important to make sure that you use the energy times the length for the dimensions of that beast.

So my question here is, is there a bound state? So for this example, for this potential, the delta function potential bound state, which again is this guy, is there a bound state? So again, we just ran through the intuition where we made the potential deep and deeper and deeper, v0 divided by epsilon over width epsilon. So v0, in order for this to be an energy has to be an energy times a length because we're going to divide it by length.

So this is going to give us a delta function potential in the limit. We have an intuition that we should get a bound state with a kink. But let's check that intuition. We want to actually solve this problem.

So we'll do the same thing we did before. We now write the general solution in the places where the potential is constant, which is on the left and on the right. And then we want to impose appropriate boundary conditions at the interface and at infinity, where these are going to be normalizable, and this is whatever the right boundary conditions are. So we are going to have to derive the appropriate boundary

conditions. So let's just do that quickly.

So phi with definite e is equal to a--

So in this region in the left, it's either growing or decreasing exponential. So ae to the alpha x plus be to the minus alpha x. And this is x less than 0. And ce to the alpha x plus de to the minus alpha x for x greater than 0.

So first off, let's hit normalizability. What must be true for normalizability? Yeah, they'd better be converging to zero here and converging to zero here, which means that c had better vanish and b had better vanish. OK. So those guys are gone from normalizability.

And meanwhile, if this is symmetric, what is going to be true of the ground state? It's going to be symmetric. So a must be equal to d. Great.

So a is now just some overall normalization constant, which we can fix from normalization. So it looks like this should be the solution. We have an exponential. We have an exponential.

But there's one more boundary condition to fix. We have to satisfy some matching. We have to satisfy the boundary conditions at the delta function. So what are those? What are those matching conditions?

So we can get that from the energy eigenvalue equation, which says that phi prime prime is equal to h bar squared upon 2m. Sorry. Get your dimensions right.

So it's 2m over h bar squared v minus e. In this case, v is equal to minus v0 delta function. That's very strange. So minus 2m over h bar squared v0 delta of x minus e--

I pulled out the minus.

--so plus e phi. So this must be true at every point. This of course, is zero everywhere, except for at the origin. So what we want to do is we want to turn this into a boundary condition. And we know what the boundary condition is.

If v is a delta function, that means that phi prime prime is also a delta function or proportional to a delta function. That means that phi prime is a step function. And how did I get that? I got that by integrating phi prime prime. You integrate a delta function, you get a step function.

Well, that's cool. How do we figure out what step function discontinuity gives us? Let's integrate. Let's integrate right across the delta function and figure out what the discontinuity is. So let's take this equation, integrate it from minus epsilon to epsilon, where epsilon is a very small number. And that's epsilon to epsilon.

So what is this going to give us on the left? Well, integral of a total derivative is just the value of the thing at a value to the point. So this is going to be phi prime at epsilon minus phi prime and minus epsilon. What does that mean?

The difference between the derivative just after the origin and just before the origin. This is the discontinuity for very small epsilon. This is the discontinuity of the derivative at the origin at the delta function. And we already expected it to have a step discontinuity. And there it is.

And how big is it? Well on the right hand side we have 2m minus 2m upon h bar squared. And we're going to get two terms. We get a term from integrating the first term. But over this narrow window, around, let's say epsilon was here, over this narrow window we can treat the wave function as being more or less constant.

But in any case, it's continuous. And this is a delta function. So we know what we get from the integral of the delta function. We just get the value v0 phi at the delta function. Phi of the zero at the delta functions, so phi 0.

We get a second term, which is plus the energy integrated against phi. The energy's a constant. And phi is continuous. So this, whatever else you can say about it, is roughly the constant value of phi at the origin times the energy times the width, which is epsilon. So plus-order epsilon terms. Everyone cool with that?

So now what I'm going to do is I'm going to take the limit as epsilon goes to zero. So

I'm just going to take that the discontinuity just across zero. So this is going to give me, of this gives me the change in the slope at the origin. OK. The derivative just after the origin minus the derivative just before the origin is equal to--

These order epsilon terms go away.

--minus 2m upon h bar squared v0 phi. So that's my continuity. That's the condition for continuity of the derivative and appropriate discontinuity of the first derivative at the origin. And so this, when we plug-in these values of this form for the wave function, when we take a derivative, all we're going to do is we're going to pick up an alpha.

And so when we work all of this out--

I'm not going to go through the algebra. You're going to go through it on the problem set.

--when we take this condition, when we impose this condition with this wave function, it gives us a very specific value for alpha. This is only solvable if alpha is equal to mv0 upon h bar squared. Good. So let's just check the units.

So this is momentum times length, momentum times length. This is mass. This is an energy times a length. So this has overall units of, p squared over m, overall units of 1 upon length, which is what we wan. So that's good.

So we get alpha is equal to mv0 upon h bar. And that gives us the form of the potential. And it also tells us that the energy, plugging this back in, is equal to minus h bar squared alpha squared upon 2m, which we could then plug-in the value of alpha and solve for v0.

So what we found is that there is a single bound state of the delta function potential, which we could have gotten by just taking this limit. It's a fun way to rederive the same result. It's a nice check on your understanding. So we find that there's a single bound state of the delta function potential.

Now, what about an odd bound state? We assumed at an important point that this

was even. What if I assume that it was odd? One node, what if we had assumed that it was odd? What would be true of the wave function?

Well for odd, this would be a, and this would be minus a. So the value of the wave function at the origin is what? Zero. So that tells us the value of the wave function is zero.

What's the discontinuity? Zero. So it's as if there's no potential because it has a zero right at the delta function. Yeah? But that means that this wave function, an odd wave function, doesn't notice the delta function potential. So is there a bound state? No.

So how many bound states are there? Always exactly one for the single isolated delta function. On your problem set, you're going to use the result of the single isolated delta function, and more broadly you're going to derive the results for two delta functions.

So you might say, why two delta functions? And the answer is, the two delta function problem, which involves no math--

Right? It's a totally straightforward, simple problem. You can all do it right now on a piece of paper. The two delta function problem is going to turn out to be an awesome model for the binding of atoms. And we're going to use it as intuition on your problem set to explain how quantum mechanical effects can lead to an attractive force between two atoms. See you next time.

[APPLAUSE]