PROFESSOR: There is an effective potential, as it's described here. Yes, there is a potential, the central force, but if there is angular momentum, it's like a centrifugal barrier. With angular momentum it becomes very difficult to reach the origin. Because if you want to reach the origin, you have to spin faster and faster. So it becomes very hard to reach the origin.

And this is like a repulsive potential here. So you have an effective potential. You started with just a potential, the central potential, but that will become an effective potential. This whole thing in brackets over there is sometimes called the effective radial potential. And it's this V of r plus h squared over 2mr squared l times l plus 1.

So if you have a Coulomb potential, that would try to make the electrons go all the way to the proton. This is attractive. Well, even that is not problematic for l equals 0. But for l different from 0, you have to add here a potential that diverges. And it's positive, say, for some l positive and falls off very fast because forms of like r squared, this, the Coulomb potential falls off at 1/r. So when you combine the two, so this is just the centrifugal barrier, but you combine the two, this diverges faster as well, so the total potential is something like that in between.

And then you can have bound states. And all the theorems we've learned about bound states and eigenstates of one-dimensional potential, now you discover that they're very useful in three dimensions. Many things just carry through.

So this is effective potential. And it's going to recap and remember that the solution that you've written is Re. But Re is u of r over r Y lm of theta and phi.

And one more thing that I want you to realize, does the function u depend on l? Yes. The differential equation depends on l. Does the function u depend on m? Yes or no?

STUDENT: [INAUDIBLE]

PROFESSOR: No. There's no m in the differential equation, so that should be good enough. So what's happened is the m dependence is kind of very simple, always very simple. It is the e to the im phi and nothing else. The u doesn't know about it.

On the other hand, the u depends on l because it shows in the differential equation. So I could write here of e and l because it depends on l, and it is the solutions of the Schrodinger equation in one potential. So there will be quantization of energy, or there might be stationary states that depend on the energy. So this is the function that knows about l, knows about the energy. And we've been totally successful with the angular dependence. Yes.

STUDENT: That should be theta and phi, dy?

PROFESSOR: Yes, theta and phi. Thank you. Good.

Normalization, the last thing that has to work out nicely. Let's try to see what does the normalization say about this function.

Well, we should find that the integral of psi squared d cubed x is equal to 1. But that integral, as you now know, it's the integral of r squared dr from 0 to infinity. And how do you integrate over volume? You integrate over r. That has the right units. And you integrate over solid angle, d omega.

Of psi squared, so let's do the arithmetic here. We have a Y lm star of theta and phi, a Y lm of theta and phi. We have a u squared and then r squared. Poor graph. But anyway, you can read it.

Now, what happens is just good stuff. r squared cancels. And this solid angle angle integral is a perfect integral for our normalization. So this gives you 1. And therefore the end result for all this integral is just the integral from 0 to infinity of dr u of r squared. And that must be equal to 1.

So not only is little u a nice variable that satisfies a one-dimensional Schrodinger equation, but you can remember that your more complicated wave function will be normalized if u is normalized in the one-dimensional sense. If u squared integrated over x is equal to 1, yes, you're normalized.

So the set-up to convert the three-dimensional differential equation into a one-dimensional differential equation has been very successful. We've reduced it to a one-dimensional problem. We have to solve those. Each time somebody gives you a spherical potential, you look at that equation, the radial equation, and try to solve for u's. If you solve for u's, then you can append the angular dependents that correspond to angular momentum eigenstates. We call this angular moment eigenstates. They are the most you can ask from angular momentum.

And then you have many solutions. So I want to conclude that part of the analysis by mentioning something about solutions with the appropriate boundary conditions. We have x, in the one-dimensional problem we call it x. And you'd run from minus infinity to infinity.

The one difference here is that you have r, and r runs from 0 to infinity. So you may wonder if you have some issues with r going to 0. What should the wave function do when r goes to 0?

OK. Then the way we think about it is not completely general but is good enough. We think of the differential equation as we have here and imagine a potential that when r goes to 0, the centrifugal barrier dominates. So our potential of the form 1 over r to the fourth would be even more singular than the barrier. And I don't know what happens in that case. Most likely it's no good, not interesting. You cannot find solutions.

But if the potential, like the Coulomb potential, is weaker than the centrifugal barrier as r goes to 0, the centrifugal barrier dominates when r goes to 0. And this differential equation, as r goes to 0, has a potential infinite term which corresponds to the centrifugal barrier. So we think of this as r goes to 0, the differential equation roughly becomes minus h squared over 2m d second u dr squared plus h squared l times l plus 1 over r squared of u, roughly 0. At least the leading behavior of these things should work out correctly.

So the h squareds-- here is a 2m, as well, I'm sorry. The h squared over 2m's cancel, and you get d second u dr squared plus-- no, is equal to l times l plus 1 over r squared u. And we're only interested for this as r goes to 0.

And it's not an exact statement. It's a discovery. We're trying to discover what's happening with the wave function near r equals 0.

Well, this has two kinds of solutions. You can try a polynomial. So this could go like r to the l plus 1. If you take two derivatives, that works out. You get the l plus 1 and l. And it solves the equation. Or r to the minus l also solves the equation.

As you can imagine, this is going to be problematic in general. It's too divergent. It will not be possible to normalize it, in general, for arbitrary values of l.

So this is a very brief analysis. I'm not going into all the detail that probably this deserves at this moment. But this is ruled out, and this is ruled in.

The only thing-- so it's true that this one is ruled out, and it has problems for normalization. It is too divergent as l. But for l equals 0, it's not divergent. But for l equals 0, there's another reason why this is not good. It turns out that for l equals 0 this doesn't quite solve the Schrodinger equation, the exact Schrodinger equation.

So the bottom line of this analysis is that we will have u of r behave like r to the l plus 1 as r goes to 0. And in particular, when l is equal to 0, u of r will behave like r. So it will vanish as r goes to 0.

So the only question is how fast it vanishes. It vanishes as r goes to 0 for l equals 0. It vanishes even faster for higher l. So it always vanishes, the wave function at r equals 0. And that's why we can usually think of it as having an infinite barrier.

The wave function could not exist for r less than 0. That physically doesn't exist. And the boundary conditions are such that that cannot happen.

All right. We've finished the discussion of the radial equation. Now, our main interest with the radial equation, of course, is, at this moment, the hydrogen atom. So we're going to turn to that.