8.04, Quantum Physics I, Fall 2015

FINAL EXAM

Friday December 18, 1:30-4:30 pm

You have 3 hours = 180 minutes.

Answer all problems in the white books provided. Write YOUR NAME and YOUR SECTION on your white book(s).

There are six questions, totalling 105 points.
The first three questions are shorter, the last three questions are longer.

No books, notes, or calculators allowed.

Show your work CLEARLY!

Formula Sheet

- $\hbar c \simeq 197.3 \text{ MeV} \cdot \text{fm}\,, \quad m_e c^2 \simeq 0.511 \text{ MeV}\,, \quad m_p c^2 = 938 \text{ MeV}\,, \quad \frac{e^2}{\hbar c} \simeq \frac{1}{137}$
- Relativity: $p = \gamma\, mv\,, \quad E = \gamma mc^2\,, \quad E^2 = p^2c^2 + m^2c^4\,, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}\,, \quad \beta = \frac{v}{c}$
- Photons: $E = h\nu\,, \; p = \frac{h}{\lambda}\,,$ or $E = \hbar\omega\,, \; p = \hbar k$
- Wavelengths

$$\text{de Broglie: } \lambda = \frac{h}{p}\,, \qquad \text{Compton: } \lambda_C = \frac{h}{mc}\,.$$

- Momentum and position operators

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x}\,, \quad [x, p] = i\hbar\,, \qquad \mathbf{p} = \frac{\hbar}{i}\nabla\,, \quad [x_i, p_j] = i\hbar\,\delta_{ij}\,, \quad [p_i, f(\mathbf{x})] = \frac{\hbar}{i}\frac{\partial f}{\partial x_i}$$

- Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}(\mathbf{x}, t) = \Big(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}, t)\Big)\Psi(\mathbf{x}, t)\,,$$

$$\frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla \cdot \mathbf{J}(\mathbf{x}, t) = 0$$

$$\rho(\mathbf{x}, t) = |\Psi(\mathbf{x}, t)|^2\;; \qquad \mathbf{J}(\mathbf{x}, t) = \frac{\hbar}{m}\text{Im}\,[\Psi^*\nabla\Psi]$$

- Fourier transforms:

$$\Psi(x) = \frac{1}{\sqrt{2\pi}}\int dk\,\Phi(k)e^{ikx}\,, \quad \Phi(k) = \frac{1}{\sqrt{2\pi}}\int dx\,\Psi(x)e^{-ikx}\,, \quad \int dx\,|\Psi(x)|^2 = \int dk\,|\Phi(k)|^2$$

$$\Psi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\,\Phi(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,, \; \Phi(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3x\,\Psi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \; \int d^3x\,|\Psi(\mathbf{x})|^2 = \int d^3k\,|\Phi(\mathbf{k})|^2$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dx = \delta(k)\,, \qquad \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3x = \delta^{(3)}(\mathbf{k})$$

$$\int_{-\infty}^{+\infty} dx\exp\big(-ax^2 + bx\big) = \sqrt{\frac{\pi}{a}}\exp\Big(\frac{b^2}{4a}\Big)\,, \quad \text{when } \text{Re}(a) > 0\,.$$

- Wavepackets

$$v_{group} = \frac{d\omega}{dk}\,, \quad \Delta k\,\Delta x \simeq 1\,, \quad \text{shape preserving : } t\,\Delta v \leq \Delta x$$

- Hermitian conjugation:

$$\int dx\,(K\Psi(x, t))^*\Psi(x, t) = \int dx\,\Psi^*(x, t)\,(K^\dagger\Psi(x, t))$$

If $K^\dagger = K$, then K is Hermitian.

- Expectation values

$$\langle Q\rangle(t) \;=\; \int dx\, \Psi^*(x,t)(Q\Psi(x,t))$$

- Time evolution of expectation value. For Q Hermitian

$$i\hbar\frac{d}{dt}\langle Q\rangle \;=\; \big\langle [Q,H]\big\rangle$$

- Commutator identity

$$[A,BC] = [A,B]C + B[A,C]$$

- Uncertainty ΔQ of a Hermitian operator Q

$$(\Delta Q)^2 \;=\; \langle Q^2\rangle - \langle Q\rangle^2 \;=\; \big\langle (Q-\langle Q\rangle)^2\big\rangle$$

- Uncertainty principle: $\Delta x\,\Delta p \geq \frac{\hbar}{2}$

$$\Delta x = \frac{\Delta}{\sqrt{2}} \quad \text{and} \quad \Delta p = \frac{\hbar}{\sqrt{2}\Delta} \quad \text{for} \quad \psi \sim \exp\Big(-\frac{1}{2}\frac{x^2}{\Delta^2}\Big)$$

- Stationary state:

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar}\,, \qquad -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\,\psi(x)$$

- Infinite square well

$$V(x) = \begin{cases} 0\,, & \text{for} \;\; 0 < x < a, \\ \infty & \text{otherwise} \end{cases}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\,, \qquad E_n = \frac{\hbar^2\pi^2 n^2}{2ma^2}\,, \qquad n = 1,2,\ldots$$

- Finite square well bound states: $E \leq 0$

$$V(x) = \begin{cases} -V_0\,, & \text{for} \;\; |x| < a, \quad V_0 > 0 \\ 0 & \text{for} \;\; |x| > a \end{cases}$$

$$\eta^2 \;\equiv\; \frac{2m(E+V_0)a^2}{\hbar^2}\,, \qquad \xi^2 \;\equiv\; \frac{2m|E|a^2}{\hbar^2}\,, \qquad z_0^2 \;\equiv\; \frac{2mV_0a^2}{\hbar^2}$$

$$\rightarrow \quad \frac{|E|}{V_0} \;=\; \frac{\xi^2}{z_0^2}\,, \qquad \xi^2 + \eta^2 \;=\; z_0^2$$

$$\text{Even solutions:} \quad \xi \;=\; \eta\tan\eta$$

$$\text{Odd solutions:} \quad \xi \;=\; -\eta\cot\eta$$

- Delta function potential:

$$V = -\alpha\,\delta(x), \quad \alpha > 0, \qquad \text{Bound state:} \quad E = -\frac{m\alpha^2}{2\hbar^2}$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\,(\hat{N} + \tfrac{1}{2})\,, \quad \hat{N} = \hat{a}^\dagger\hat{a}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\ , \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\ ,$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)\ , \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})\,,$$

$$[\hat{x}, \hat{p}] = i\hbar\,, \quad [\hat{a}, \hat{a}^\dagger] = 1\ , \qquad [\hat{N}, \hat{a}\,] = -\hat{a}\,, \quad [\hat{N}\,, \hat{a}^\dagger\,] \;=\; \hat{a}^\dagger\,.$$

$$\hat{a}\phi_0 \;=\; 0\,, \quad \phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\ .$$

$$\phi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n\phi_0$$

$$\hat{H}\,\phi_n = E_n\,\phi_n = \hbar\omega\big(n + \tfrac{1}{2}\big)\,\phi_n\,, \quad \hat{N}\,\phi_n = n\,\phi_n\,, \quad (\phi_m\,, \phi_n) \;=\; \delta_{mn}$$

$$\hat{a}^\dagger\phi_n = \sqrt{n+1}\,\phi_{n+1}\ , \quad \hat{a}\,\phi_n = \sqrt{n}\,\phi_{n-1}\ .$$

- Positive energy states

$$\psi(x) \;=\; Ae^{ikx} + Be^{-ikx}\,, \qquad J \;=\; \frac{\hbar k}{m}\big(|A|^2 - |B|^2\big)\,, \qquad E = \frac{\hbar^2k^2}{2m}$$

- Scattering in 1D. $V(x) = \infty$ for $x \leq 0$. Solution $\phi(x) = \sin kx$ when $V = 0$.

$$\psi(x) \;=\; e^{i\delta}\,\sin(kx + \delta)\,, \quad x > R\ (R \text{ is the range})$$

Scattered wave: $\psi = \phi + \psi_s$

$$\psi_s = A_s e^{ikx}\,, \quad A_s = e^{i\delta}\sin\delta$$

$$\text{Time delay:}\quad \Delta t \;=\; 2\hbar\,\frac{d\delta}{dE} \quad\rightarrow\quad \frac{1}{R}\frac{d\delta}{dk} \;=\; \frac{\Delta t}{\text{free transit time}}$$

$$N_{bound} \;=\; \frac{1}{\pi}\big(\,\delta(0) - \delta(\infty)\big) \quad \text{(Levinson's theorem)}$$

Resonances: Rapid growth in δ, large time delay, large amplitude in the inner region.

- Orbital angular momentum

$$\hat{L}_x = \hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y\,, \quad \hat{L}_y \;=\; \hat{z}\,\hat{p}_x - \hat{x}\,\hat{p}_z\,, \quad \hat{L}_z \;=\; \hat{x}\,\hat{p}_y - \hat{y}\,\hat{p}_x\,.$$

$$[\,\hat{L}_x\,,\,\hat{L}_y\,] \;=\; i\hbar\,\hat{L}_z\,, \quad [\,\hat{L}_y\,,\,\hat{L}_z\,] \;=\; i\hbar\,\hat{L}_x\,, \quad [\,\hat{L}_z\,,\,\hat{L}_x\,] \;=\; i\hbar\,\hat{L}_y\,.$$

$$\hat{L}^2 \;\equiv\; \hat{L}_x\hat{L}_x + \hat{L}_y\hat{L}_y + \hat{L}_z\hat{L}_z\,, \quad [\,\hat{L}^2\,,\hat{L}_i\,] \;=\; 0$$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}\,r + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\;; \qquad \hat{L}_\pm = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

- Spherical Harmonics

$$Y_{\ell,m}(\theta,\phi) \;\equiv\; \mathcal{N}_{\ell,m}\,P_\ell^m(\cos\theta)e^{im\phi}$$

$$\begin{aligned}\hat{L}_z\,Y_{\ell m} &= \hbar m\,Y_{\ell m}\\ \hat{L}^2\,Y_{\ell m} &= \hbar^2\,\ell(\ell+1)\,Y_{\ell m}\end{aligned}$$

$$\int d\Omega\,Y^*_{\ell' m'}(\theta,\phi)\,Y_{\ell m}(\theta,\phi) \;=\; \delta_{\ell',\ell}\delta_{m',m}\,, \qquad \int d\Omega \;=\; \int_0^{2\pi} d\phi\int_{-1}^{1} d(\cos\theta)$$

$$Y_{0,0}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}\;; \qquad Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi)\;; \qquad Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

- Central potentials: $V(\mathbf{r}) = V(r)$

$$\psi(r,\theta,\phi) \;=\; \frac{u(r)}{r}\,Y_{\ell m}(\theta,\phi)$$

$$\Big(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V(r) + \frac{\hbar^2\,\ell(\ell+1)}{2mr^2}\Big)u(r) = Eu(r)$$

$$u(r) \sim r^{\ell+1}\,, \quad \text{as } r\to 0\,.$$

- Hydrogen atom:

$$H = \frac{\mathbf{p^2}}{2m} - \frac{Ze^2}{r}$$

$$E_n = -\frac{Z^2e^2}{2a_0}\frac{1}{n^2}\,, \qquad a_0 = \frac{\hbar^2}{me^2} \simeq 0.529\times 10^{-10}\mathrm{m}\,, \qquad \frac{e^2}{2a_0} \simeq 13.6\,\mathrm{eV}$$

$$\psi_{n,\ell,m}(\vec{x}) = A\left(\tfrac{r}{a_0}\right)^{\ell}\Big(\text{Polynomial in } \tfrac{r}{a_0} \text{ of degree } n-(\ell+1)\Big)\, e^{-\frac{Zr}{na_0}}\, Y_{\ell,m}(\theta,\phi)$$

$$n = 1,2,\ldots\,, \qquad \ell = 0,1,\ldots,n-1\,, \qquad m = -\ell,\ldots,\ell$$

$$\psi_{n,\ell,m}(\vec{x}) = \frac{u_{n\ell}(r)}{r}Y_{\ell,m}(\theta,\phi)$$

$$u_{1,0}(r) = \frac{2r}{a_0^{3/2}}\exp(-r/a_0)$$

$$u_{2,0}(r) = \frac{2r}{(2a_0)^{3/2}}\left(1-\frac{r}{2a_0}\right)\exp(-r/2a_0)$$

$$u_{2,1}(r) = \frac{1}{\sqrt{3}}\frac{1}{(2a_0)^{3/2}}\frac{r^2}{a_0}\exp(-r/2a_0)$$

1. **Virial Theorem for one-dimensional potentials.** [15 points]

   (a) Let $\psi(x)$ be an energy eigenstate. Explain why the expectation value $\big\langle [H, \Omega] \big\rangle$ of the commutator of H with an arbitrary operator Ω vanishes on the state ψ.

   (b) Choose $\Omega = xp$, and take

   $$H \;=\; \frac{p^2}{2m} + V(x)\,.$$

   Use the claim from part (a) to find a relation between the expectation value $\langle T \rangle$ of the kinetic energy and the expectation value of a combination of x and the derivative $V'(x)$ of the potential with respect to its argument. Both expectation values are taken on an energy eigenstate.

   (c) What does your result in (b) imply for the relation between $\langle T \rangle$ and $\langle V \rangle$ for the case of the one-dimensional harmonic oscillator?

2. **Electron orbit in the Hydrogen Atom** [15 points]

   Throughout this problem we consider a hydrogen atom with fixed principal quantum number n, with $\ell = n-1$, and $m = n-1$. The value n is arbitrary and possibly large.

   (a) Write the wavefunction $\psi_{n,\ell,m}(r, \theta, \phi)$ in terms of the relevant spherical harmonic and a radial factor fully determined except for an overall unit-free normalization constant N.

   (b) Give, up to normalization, the radial probability density $P(r)$ for which $P(r)dr$ is the probability to find the electron in the interval $(r, r+dr)$. For what value of r is $P(r)$ maximum? For large n this is actually a rather sharp maximum.

   (c) It is known that, up to normalization,

   $$|Y_{\ell,\ell}(\theta, \phi)|^2 \simeq (\sin\theta)^{2\ell}\,.$$

   Sketch $|Y_{\ell,\ell}|^2$ as a function of $\theta \in [0, \pi]$ when ℓ is a large integer. Describe in words and/or with a picture, the locus where the electron is likely to be found for n large and $\ell = m = n-1$.

3. **Finding the outgoing wave-packet** [15 points]

In a one-dimensional scattering problem with a potential of range R we write the solution $\psi(x)$ for $x > R$ as

$$\psi(x) \;=\; e^{i\delta(k)} \sin\bigl(kx + \delta(k)\bigr)\,, \quad x > R\,.$$

(a) Decompose this $\psi(x)$ into the sum of an incident wave $\psi_{inc}(x)$ traveling towards $x = 0$ and an outgoing wave $\psi_{out}(x)$ traveling away from $x = 0$.

(b) We send in a localized wave-packet $\Psi_{inc}(x,t)$ given by

$$\Psi_{inc}(x,t) \;=\; \int_0^{\infty} dk\, f(k)\, e^{-ikx} e^{-iE(k)t/\hbar}\,, \quad x > R\,,$$

with $f(k)$ a function whose magnitude peaks sharply for $k = k_0 > 0$. Write an analogous expression for the associated out-going wave packet $\Psi_{out}(x,t)$.

(c) Use the stationary phase approximation to find the relation between x and t that describes the motion of the outgoing packet $\Psi_{out}(x,t)$.

4. **Towards perfect bomb detection** [20 points]

We modify the Mach-Zehnder setup to increase towards 100% the fraction of Elitzur-Vaidman (EV) bombs that can be vouched to work without detonating them. An EV bomb is triggered by a photon detector: if operational any photon incident on the detector will make the bomb explode, if defective the detector lets all photons through and the bomb does not explode.

To improve detection we use a high reflectivity beam-splitter, henceforth called BS, represented by a two-by-two unitary matrix U of the form

$$U = \begin{pmatrix} \cos\frac{\pi}{2N} & i\sin\frac{\pi}{2N} \\ i\sin\frac{\pi}{2N} & \cos\frac{\pi}{2N} \end{pmatrix},$$

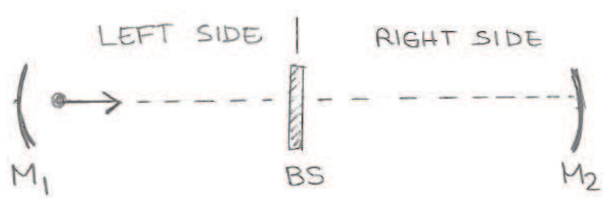
with N some large, fixed, positive integer. Note that BS is a beam splitter with reflectivity R and transmissivity T given by

$$R = \left(\cos\frac{\pi}{2N}\right)^2, \qquad T = \left(\sin\frac{\pi}{2N}\right)^2, \quad R + T = 1\,.$$

We will imagine the beamsplitter BS placed vertically, with a photon to the left of BS represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and a photon to the right of BS represented by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This holds both for photons moving towards or away from BS.

Useful formula: $\begin{pmatrix} \cos\alpha & i\,\sin\alpha \\ i\,\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & i\,\sin\beta \\ i\,\sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} \cos(\alpha+\beta) & i\,\sin(\alpha+\beta) \\ i\,\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}.$

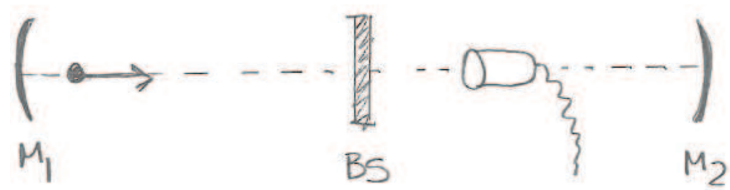
(a) Calculate the k-th power U^k of the matrix U.



(b) We now construct a cavity in which the beam-splitter BS is placed in between perfectly reflecting mirrors M_1 and M_2 at equal distances to the left and to the right. A photon is sent in from the left, as shown in the figure. The photon will hit BS and split, the reflected and transmitted components will bounce off the mirrors and hit BS a second time, and so on and so forth.

After k hits of the BS what is the probability $p_L(k)$ that the photon will be found on the left side of the cavity and what is the probability $p_R(k)$ that it will be found on the right side of the cavity?

What are those probabilities for $k = N$?



(c) A photon detector is inserted on the right side of the cavity, so that any photon reaching the right side will be detected (and absorbed!). As before, a photon is sent in from the left. After waiting for the time needed for N hits on BS, what is the probability $P_L(N)$ that the photon will be found on the left side of the cavity? What is the probability $P_D(N)$ that the photon has been detected?

(d) Estimate $P_L(N)$ and $P_D(N)$ in the limit as N is large. Helpful formulae: $\cos\epsilon \simeq 1 - \frac{1}{2}\epsilon^2$, $(1+\epsilon)^k \simeq 1 + k\epsilon$ for ϵ sufficiently small.

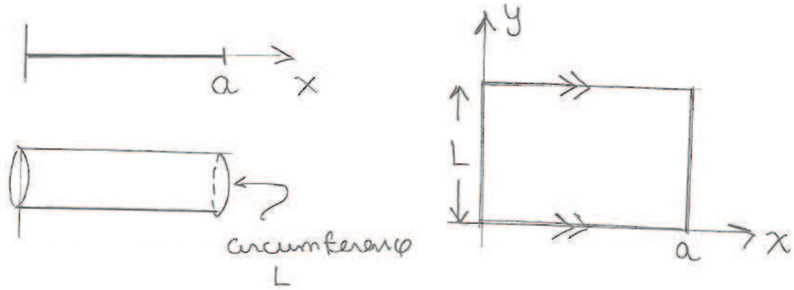
(e) Given an EV bomb, we insert it on the right side of the cavity. We send in a photon from the left and wait for the time needed for N hits of the BS. At that point, if the lab does not blow up, we look for the photon.

  i. What can we conclude if the photon is found on the left side of the cavity?

  ii. What is the probability $P_E(N)$ that an operational EV bomb will explode in this experiment? Give an approximate value for $N = 250$.

5. **Infinite square well with extra dimension = a truncated cylinder** [20 points]

A particle in a one-dimensional infinite square well of width a can be thought as a particle forced to move on a line *segment* of length a. Consider a particle moving on a small *cylinder* of length a. The cylinder has circumference L and it can be represented as a rectangular region in the (x, y) plane, with the y coordinate along the circumference of the cylinder and the horizontal lines with arrows identified:



The system is described by the two-dimensional Schrödinger equation (SE) with a potential that vanishes in the rectangle $\{(x, y) : 0 \leq x \leq a\,,\ 0 \leq y \leq L\}$, and is infinite on the vertical edges at $x = 0$ and $x = a$.

(a) Perform separation of variables in the SE and give the two equations that help determine the energy eigenstates. State the boundary conditions that apply.

(b) Solve for the energy eigenvalues $E_{n\ell}$ and normalized eigenstates $\psi_{n\ell}(x, y)$, where n and ℓ are quantum numbers for the x and y dependence, respectively. State precisely the ranges n and ℓ run over.

(c) What is the ground state energy of the particle?

(d) Assume henceforth that a and L are such that no accidental degeneracies occur (accidental degeneracies are those that require special relations between a and L). What is the list of energy eigenvalues for the particle in the cylinder that coincide with those for the one-dimensional segment $x \in [0, a]$.

(e) What are (or is) the lowest energy levels that exist on the cylinder but do not exist in the segment?

(f) The y dimension that turns the segment into a cylinder may be considered as a yet undetected small extra dimension. Suppose the size L of the extra dimension is about 1000 times smaller than the size a of a small interval where an experimenter has localized a particle. Assume also that the length a and the particle mass m are such that

$$\frac{\hbar^2}{2ma^2} = 1\,\text{eV}.$$

Estimate the minimum energy that the experimenter needs to explore to find evidence for the extra dimension?

6. **Resonant transmission across two delta functions** [20 points]

Consider a potential with two positive-strength delta functions, one at $x = -a$ and another at $x = a$:

$$V(x) = g\delta(x+a) \; + \; g\delta(x-a)\,.$$

Note the unit-free combination λ that represents the effective strength of the potential:

$$\lambda = \frac{mag}{\hbar^2} \geq 0\,.$$

In solving the general scattering problem of a particle incident from the left one sets up a wavefunction

$$\psi(x) \;=\; \begin{cases} Ae^{ikx} + Be^{-ikx}\,, & x < -a\,, \\ Ce^{ikx} + De^{-ikx}\,, & |x| < a\,, \\ Fe^{ikx}\,, & x > a\,. \end{cases}$$

Here A, B, C, D, F are complex constants that must be adjusted for this to be a solution of the time-independent Schrödinger equation. We are interested in finding the energies for which there is resonant transmission, namely, the transmission coefficient is one!

(a) Which of the complex constants in the above ansatz for ψ should vanish for resonant transmission? Explain briefly.

(b) Assume this constant vanishes and find the four equations that implement the boundary conditions. Clean them up and put them in the form:

$$\begin{aligned} C + D\,e^{\cdots} &= \cdots \\ C + D\,e^{\cdots} &= \cdots \\ C - D\,e^{\cdots} &= \cdots \\ C - D\,e^{\cdots} &= \cdots \end{aligned}$$

The expressions indicated by dots should be written in terms of ka, λ, constants in the ansatz for ψ and numerical constants.

(c) We now claim that the existence of a solution for the above equations requires

$$\xi\,\cot\xi \;=\; -2\lambda\,, \quad \text{with} \quad \xi = 2ka\,. \tag{1}$$

You need not prove this! Show a plot of $\xi\cot\xi$ for $\xi \in [0, 3\pi]$. Show the -2λ line in the plot for both very small and very large λ. For $\lambda \ll 1$ what are the approximate values of ka for perfect transmission. For $\lambda \gg 1$ what are the approximate values of the ka for perfect transmission?

(d) Under condition (1) one can prove that

$$\frac{C}{D} \;=\; -\frac{1}{\cos(2ka)}\,, \qquad C \;=\; \Big(1 + \frac{\lambda}{ika}\Big)A\,.$$

Consider the case of $\lambda \gg 1$ and the first resonant transmission. Find an approximate formula for ψ in the region $|x| < a$ and setting $A = 1$ do a rough plot of $|\psi(x)|^2$ for all x. Comment on the features of your plot!