Quantum Physics II (8.05) Fall 2013
Assignment 5

Massachusetts Institute of Technology
Physics Department
October 5, 2013

Due October 11, 2013
3:00 pm

This week in lecture we will study uncertainty relations.

Reading Assignment for Week Five

Uncertainty relations: Griffiths, section 3.5, Shankar Chapter 9.

Problem Set 5

1. **Translation operators** [5 points]

   Consider the coordinate-space and momentum-space translation operators

   $$T_x = \exp\left(-\frac{i\hat{p}x}{\hbar}\right), \quad \tilde{T}_p = \exp\left(\frac{ip\hat{x}}{\hbar}\right).$$

   (a) Verify that the above are translation operators by calculation of

   $$T_x^\dagger\, \hat{x}\, T_x \quad \text{and} \quad \tilde{T}_p^\dagger\, \hat{p}\, T_p\,.$$

   (b) Since $\hat{x}$ and $\hat{p}$ do not commute, the translation operators T_x and $\tilde{T}_p$ do not generally commute. But they sometimes do! Compute the commutator

   $$\left[\, T_x\,, \tilde{T}_p\,\right] = \ldots$$

   You should find the CBH formula useful. What is the condition satisfied by x and p that guarantees that T_x and $\tilde{T}_p$ commute?

2. **Position and momentum operators** [5 points]

   In lecture we showed that

   $$\langle x|\hat{p}|\psi\rangle = \frac{\hbar}{i}\frac{d}{dx}\langle x|\psi\rangle\,.$$

   (a) Show that

   $$\langle x|\hat{p}^n|\psi\rangle = \left(\frac{\hbar}{i}\frac{d}{dx}\right)^n \psi(x)\,.$$

(b) Show that $\hat{x}$ is represented as $i\hbar\frac{d}{dp}$ in the *momentum* representation, namely

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{d}{dp}\langle p|\psi\rangle \ .$$

(c) Use the result in (b) to calculate the action of $[\hat{x}, \hat{p}]$ on the state ket $|\psi\rangle$ in the *momentum* representation. Verify that you obtain the expected result.

3. **Elaborations on a theorem** [10 points]

We have shown in lecture that $\langle v, Tv\rangle = 0$ for all $v \in V$ implies that $T = 0$ if V is a complex vector space. If V is a real vector space one can't prove $T = 0$. To distinguish the two cases let

$$\text{Real case: } \langle u, Su\rangle = 0\,, \text{ for all } u\,, \quad \text{Complex case: } \langle v, Tv\rangle = 0\,, \text{ for all } v$$

We first want to examine the case of dimension two to see in a simple example why the theorem is true and why it fails for real vector space. So we will consider two-by-two matrices and two-component vectors.

(a) Let S be represented by a real matrix S_{ij} and u by the two real components u_i with $i, j = 1, 2$. For the complex case let T be represented by a complex coefficient matrix T_{ij} and v by the two complex components v_i with $i, j = 1, 2$. Write out the quadratic forms and then apply the conditions that they vanish for all u and v, respectively. Show that you find $T_{ij} = 0$. For what kind of matrices S does the vanishing of $\langle u, Su\rangle$ imply the vanishing of S.

(b) Extend your argument to arbitrary size matrices, showing that $T_{ij} = 0$ and again stating for what kind of matrices S the theorem holds.

(c) Consider a complex vector space and an arbitrary linear operator. It can be shown that there is a basis for which the matrix representing the operator has an upper triangular form (the elements below the diagonal vanish). On the light of the above analysis explain why the same does not hold for arbitrary linear operators on real vector spaces.

4. **Projectors and the $P^2 = P$ condition** [10 points]

Consider a vector space V and a linear operator P that satisfies the equation $P^2 = P$.

(a) Show that $V = \text{null}\, P \oplus \text{range}\, P$.

The condition $P^2 = P$, however, is not enough to show that P is an orthogonal projector. One must additionally prove that any vector in the first summand is orthogonal to any vector in the second summand.

(b) Show that any of the two conditions below guarantees that orthogonality:

(1) P is Hermitian.

(2) $|Pv| \leq |v|$ for any $v \in V$.

Case (2) is harder than case (1). You may find it useful to prove first the following result: Let $u, v \in V$. Then $\langle u, v\rangle = 0$ if and only if $|u| \leq |u + av|$ for any $a \in \mathbb{F}$.

(c) Invent a two-by-two matrix P that satisfies $P^2 = P$ but fails to be a projector because (as you will demonstrate) violates both conditions (1) and (2) of part (b).

5. **Exercise with matrices.** [5 points]

Consider two hermitian matrices A_1 and A_2 that commute:

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

The matrix A_1 has eigenvalues and orthonormal eigenvectors

$$\lambda_1 = 2, \big|u_1\big\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; \quad \lambda_2 = 0, \big|u_2\big\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}; \quad \lambda_3 = 0, \big|u_3\big\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

In the basis $\{\big|u_1\big\rangle, \big|u_2\big\rangle, \big|u_3\big\rangle\}$ the matrix A_2 takes the form

$$\begin{pmatrix} 3 & * & * \\ 0 & * & -\sqrt{2} \\ 0 & * & * \end{pmatrix}. \tag{1}$$

Determine the missing entries (denoted by $*$) in the above matrix. Use your result to find the eigenvalues of A_2.

6. **Minimum uncertainty** [5 points]

We showed in class that for two hermitian operators A and B the uncertainty inequality

$$(\Delta A)^2(\Delta B)^2 \geq \Big(\langle\Psi|\,\frac{1}{2i}\big[\,A,\,B\,\big]\,|\Psi\rangle\Big)^2$$

is saturated on a state $|\Psi\rangle$ that satisfies

$$\big(B - \langle B\rangle\big)|\Psi\rangle = i\gamma\big(A - \langle A\rangle\big)|\Psi\rangle\,, \quad \text{with} \;\; \gamma = \pm\frac{\Delta B}{\Delta A}\,.$$

Verify explicitly this claim for the Gaussian states

$$\psi(x) = N e^{i\langle p\rangle x/\hbar} e^{-x^2/(2\Delta^2)}$$

that saturate the uncertainty inequality for the product of $\hat{x}$ and $\hat{p}$ uncertainties.

7. **Griffiths 3.32, p.126. Testing a version of the time-energy uncertainty relation** [7 points]

8. **Upper and lower bounds for ground state energy.** [8 points]

   Consider the harmonic oscillator Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2\,.$$

   Use a gaussian trial function and the variational principle to find an upper bound for the ground state energy. Use the uncertainty principle (as explained in lecture) to derive a lower bound for that same ground state energy. Use those two bounds to determine the ground state energy.

9. **Simultaneous diagonalization of two hermitian matrices** [10 points]

   Consider the hermitian matrices A_1 and A_2:

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & -1 \\ -1 & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} & 1 \end{pmatrix}.$$

   These matrices commute so they can be simultaneously diagonalized: there is a unitary matrix U such that

$$U^{-1}A_1U = D_1\,, \quad U^{-1}A_2U = D_2\,,$$

   where D_1 and D_2 are two diagonal matrices. Determine the matrices U, D_1 and D_2. Find the common eigenvectors of the two matrices and label them as u_{a_1,a_2} where a_1 and a_2 are the eigenvalues of A_1 and A_2 respectively. (You are urged to use a mathematical manipulator to avoid tedious arithmetic!).