Quantum Physics III (8.06) — Spring 2016

Assignment 10

Readings

The current reading assignments are:

- Griffiths Section 10.2.3 is an excellent treatment of the Aharonov-Bohm effect.
- Quite remarkably, given its length, Cohen-Tannoudji never mentions the Aharonov-Bohm effect. It does have a nice treatment of Landau levels, however, in Ch. VI Complement E.
- Those of you reading Sakurai should read pp. 130-139. Shankar's treatment is very different from the one presented in this course.

Notes

1. In this problem set, the following definitions are used throughout:

$$\omega_L \equiv \frac{qB}{mc}, \qquad l_0 \equiv \sqrt{\frac{\hbar}{m\omega_L}} = \sqrt{\frac{\hbar c}{qB}}, \tag{1}$$

$$v_x = \frac{1}{m}\left(p_x - \frac{q}{c}A_x\right), \qquad v_y = \frac{1}{m}\left(p_y - \frac{q}{c}A_y\right). \tag{2}$$

   Gauge transformations correspond to choosing some function $f(\vec{x}, t)$ and replacing the wavefunction ψ and the vector and scalar potentials $\vec{A}$ and ϕ by the following:

$$\psi'(\vec{x}, t) \equiv \exp\left(-\frac{iq}{\hbar c} f(\vec{x}, t)\right)\psi(\vec{x}, t) \tag{3}$$

$$\vec{A}'(\vec{x}, t) \equiv \vec{A}(\vec{x}, t) - \vec{\nabla} f(\vec{x}, t),$$

$$\phi'(\vec{x}, t) \equiv \phi(\vec{x}, t) + \frac{1}{c}\frac{\partial f}{\partial t}(\vec{x}, t)\ . \tag{4}$$

2. The physics of charged particles in a magnetic field is full of intricacies and subtleties, not all of which we will have time to cover in lecture. The problems in this PSET are exploratory in nature, guiding you through various aspects of the story by working out properties on your own. So instead of viewing it as a check of your understanding, you should view it as a research experience to further explore properties of the system on your own.

Optional Problem Set 10

1. **Classical Motion in a Magnetic Field**

   Consider a particle of mass m and charge q moving along a trajectory $\vec{x}(t)$ through a constant magnetic field along the z-direction, i.e. $B_x = B_y = 0$, $B_z = B = \text{const}$.

   (a) Show that in the $x - y$ plane, the particle travels in a circle around a "center of orbit" with an angular velocity ω_L given by

   $$\omega_L = \frac{qB}{mc}\ . \tag{5}$$

   (b) Suppose that the "center of orbit" has coordinates (X, Y). Show that X, Y can be expressed in terms of the coordinates (x, y) and the velocities (v_x, v_y) of the particle as

   $$X = x + \frac{v_y}{\omega_L}, \qquad Y = y - \frac{v_x}{\omega_L}\ . \tag{6}$$

   (c) Show (by differentiating (6) with respect to time) that X, Y are constants of motion.

2. **General Aspects of Quantum Motion in a Magnetic Field**

   The quantum motion for a particle in a magnetic field shows some resemblances to the classical motion and also many important differences. The differences can be traced to various commutators derived in this problem, in particular equations (7) and (9).

   The questions in this problem should be derived without explicitly choosing a gauge.

   (a) In this part we consider an arbitrary magnetic field (not necessarily constant). Find the commutator

   $$[\hat{v}_x, \hat{v}_y] = ? \tag{7}$$

   where $\hat{v}_{x,y}$ are the velocity operator defined by (2). What can you conclude about the motion of the particle from (7)?

   (b) In this and all parts below, we take $\vec{E} = 0$ and

   $$B_x = B_y = 0, \qquad B_z = B = \text{const},$$

   and look at the motion in $x - y$ plane only. Classically the particle travels in a circle around a "center of orbit" (X, Y), which can be expressed in terms of the coordinates (x, y) and the velocities (v_x, v_y) which you derived in Prob. 2 above. Motivated by the classical expressions (6), we introduce quantum operators

   $$\hat{X} = \hat{x} + \frac{\hat{v}_y}{\omega_L}, \qquad \hat{Y} = \hat{y} - \frac{\hat{v}_x}{\omega_L} \tag{8}$$

   Find the commutators

   $$[\hat{X}, \hat{Y}] = ? \tag{9}$$

   What can you say about the motion in $x - y$ plane from (9)?

(c) Show that $\hat{X}$ and $\hat{Y}$ are gauge invariant (you can use the results from Prob. 1).

(d) Show that

$$[\hat{X}, H] = [\hat{Y}, H] = 0 \ . \tag{10}$$

Equations (9) and (10) imply that one of $\hat{X}$ and $\hat{Y}$ (or an arbitrary linear combination of them, but not both) can be diagonalized together with the Hamiltonian. [Hint: It is convenient to write the Hamiltonian in a form $H = \frac{1}{2}m(\hat{v}_x^2 + \hat{v}_y^2)$ and first find the commutators between $\hat{X}, \hat{Y}$ and $\hat{v}_x, \hat{v}_y$.]

[Note: Equation (10) is the quantum counterpart of the statement you proved in Prob. 2(c), i.e. classically (6) are constants of motion.]

3. **Transformation between Basis Vectors in Different Gauges**

Consider a particle of charge q and mass m moving in a constant magnetic field $\vec{B} = (0, 0, B)$. In the gauge

$$A_x = -By, \qquad A_y = A_z = 0, \tag{11}$$

we found in lecture that a basis of energy eigenvectors in the lowest Landau level is given by (see (1) for notation)

$$\psi_0(x, y; k_x) = e^{ik_x x}\phi_0(y - y_0), \tag{12}$$

where

$$\phi_0(y - y_0) = \frac{1}{(\pi l_0^2)^{\frac{1}{4}}} e^{-\frac{(y-y_0)^2}{2l_0^2}} \tag{13}$$

is the ground state wave function for a harmonic oscillator of frequency ω_L and

$$y_0 = -\frac{c\hbar k_x}{qB} = -l_0^2 k_x \ .$$

(a) Show that in gauge (11), (12) is an eigenvector of $\hat{Y}$ introduced in (8). Find the eigenvalue. Thus (12) diagonalizes H and $\hat{Y}$ simultaneously.

(b) Without doing any calculation, write down energy eigenvectors $\psi_0'(x, y; k_y)$ in the lowest Landau level for gauge choice

$$A_y' = Bx, \qquad A_x' = A_z' = 0 \ . \tag{14}$$

Show that $\psi_0'(x, y; k_y)$ are eigenvectors of $\hat{X}$ introduced in (8) and thus diagonalize H and $\hat{X}$ simultaneously. [Note: in contrast to (12), $\psi_0'(x, y; k_y)$ is a plane wave in the y-direction while a harmonic oscillator in the x-direction.]

(c) Find the gauge transformation between (14) and (11).

(d) You found that $\psi_0(x, y; k_x)$ is an eigenstate of $\hat{Y}$, $\psi_0'(x, y; k_y)$ is an eigenstate of $\hat{X}$, and in (10) you found that $\hat{X}$ and $\hat{Y}$ have a non-trivial commutator. From this, discuss what kind of transformation(s) should relate $\psi_0(x, y; k_x)$ and $\psi_0'(x, y; k_y)$.

(e) Check that $\psi_0'(x,y;k_y)$ can be written in terms of a linear superpositions of (12) as

$$\psi_0'(x,y;k_y) = e^{-\frac{iq}{\hbar c}f}\frac{l_0}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk_x\, e^{-il_0^2 k_x k_y}\,\psi_0(x,y;k_x) \tag{15}$$

where f is the gauge transformation you find from (c). The factor $e^{-\frac{iq}{\hbar c}f}$ in (15) comes from the gauge transformation. [Hint: To prove (15), it is convenient to write equation (13) in terms of its Fourier transform

$$\phi_0(y-y_0) = \frac{l_0}{(\pi l_0^2)^{\frac{1}{4}}}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}\, e^{ik(y-y_0)-\frac{l_0^2k^2}{2}}$$

and use it in the right hand side of (15).]

[**Note:** The fact that energy eigenvectors look so different in different gauges is due to the infinite degeneracy of each Landau level. Different choices of gauges select different sets of bases.]

4. **Electromagnetic Current Density in Quantum Mechanics**

The probability flux in the Schrödinger equation can be identified as the electromagnetic current density, provided the proper attention is paid to the effects of the vector potential. This current density will play an important role in our discussion of the quantum Hall effect.

In 8.04/8.05, you derived the probability flux in quantum mechanics:

$$\vec{S}(\vec{x},t) = \frac{\hbar}{m}\text{Im}\left[\psi^*\vec{\nabla}\psi\right]\ . \tag{16}$$

In the presence of electric and magnetic fields, the probability current is modified to

$$S_i(\vec{x},t) = \frac{\hbar}{m}\text{Im}\left[\psi^*\partial_i\psi\right] - \frac{q}{mc}\psi^*\psi A_i = \text{Re}\left(\psi^*\hat{v}_i\psi\right) \tag{17}$$

This probability flux is conserved and when multiplied by q, the particle's charge, it can be interpreted as the electromagnetic current density, $\vec{j}\equiv q\vec{S}$.

(a) Derive the expression eq. (17) for the probability flux. [Hint: It is convenient to work in a gauge where $\vec{\nabla}\cdot\vec{A}=0$. The derivation of eq. (17) is parallel to that of (16), i.e. you need to derive

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{S}=0$$

with $\rho=\psi^*\psi$ and $\vec{S}$ given by eq. (17).]

(b) Assuming that ψ has units $1/l^{3/2}$ as one would expect from the normalization condition, $\int d^3x\,\psi^*\psi=1$, show that $\vec{j}=q\vec{S}$ has units of charge per unit area per unit time, which are the dimensions of current density.

(c) Now show that $\vec{S}$ has *exactly the same form* in any gauge, that is, show that under gauge transformations (4) and (3), $\vec{S}'$ defined in terms of $\vec{A}'$ and ψ' is identical to $\vec{S}$ defined in terms of $\vec{A}$ and ψ.

5. **The Aharonov-Bohm Effect on Energy Eigenvalues**

You have seen the "standard presentation" of the Aharonov-Bohm effect in lecture. The standard presentation has its advantages, and in particular is more general than the presentation you will work through in this problem. However, students often come away from the standard presentation of the Aharonov-Bohm effect thinking that the only way to detect this effect is to do an interference experiment. This is not true, and the purpose of this problem is to disabuse you of this misimpression before you form it.

As Griffiths explains on pages 385-387 (344-345 in 1st Ed.), the Aharonov-Bohm effect modifies the energy eigenvalues of suitably chosen quantum mechanical systems. In this problem, we work through the same physical example that Griffiths uses.

Imagine a particle constrained to move on a circle of radius b (a bead on a wire ring, if you like.) Along the axis of the circle runs a solenoid of radius $a < b$, carrying a magnetic field $\vec{B} = (0, 0, B_0)$. The field inside the solenoid is uniform and the field outside the solenoid is zero. The setup is depicted in Griffiths' Fig. 10.10. (10.12 in 1st Ed.)

(a) Construct a vector potential $\vec{A}$ which describes the magnetic field (both inside and outside the solenoid) and which has the form $A_r = A_z = 0$ and $A_\phi = \alpha(r)$ for some function $\alpha(r)$. We are using cylindrical coordinates z, r, ϕ.

(b) Now consider the motion of a "bead on a ring": write the Schrödinger equation for the particle constrained to move on the circle $r = b$, using the $\vec{A}$ you found in (a). Hint: the answer is given in Griffiths.

(c) Solve the Schrodinger equation of (b) and find the energy eigenvalues and eigenstates.

(d) Plot the energy eigenvalues as a function of the enclosed flux, Φ. Show that the energy eigenvalues are periodic functions of Φ with period Φ_0, where you must determine Φ_0. For what values of Φ does the enclosed magnetic field have no effect on the spectrum of a particle on a ring? Show that the Aharonov-Bohm effect can only be used to determine the fractional part of Φ/Φ_0.

(e) Suppose we introduce a defect on the ring at $\phi = 0$, which can trap the particle, i.e. in addition to the states you worked out above, there now exist trapped states in which the wave function of the particle is localized around $\phi = 0$. For simplicity, assume the trapped state wave functions vanish outside an interval $(-\phi_0, \phi_0)$ for some $\phi_0 < \pi$. Show that the energy of a trapped state does NOT depend on the existence of the solenoid.

[Hint: find a gauge in which the vector potential vanishes identically in the region where trapped state wave functions are supported. You should also explain why the same argument does not apply to states of part (c).]

[Moral of problem: even though the bead on a ring is in a region in which $\vec{B} = 0$, the presence of a nonzero $\vec{A}$ affects the energy eigenvalues of extended states (those states whose wave functions cover the whole circle). $\vec{A}$, however, does not affect the energies of localized states. This is the counterpart for the energy spectrum of a similar statement for an interference experiment: the interference pattern is shifted *if and only if* the relevant paths enclose the solenoid.]

6. **Phase kickback.** This problem will explain a technique called "phase kickback" in which the control and target of a unitary interaction can be apparently reversed. Part (a) is independent of parts (b), (c) and (d).

(a) Grover's algorithm makes use of reflections about a marked state. This part examines one method of producing these reflections on a quantum computer.

Let f be a function mapping $\{1, 2, \ldots, N\} \to \{0, 1\}$. Suppose that we can implement f using the unitary U_f, defined as

$$U_f(|x\rangle \otimes |y\rangle) = |x\rangle \otimes |y \oplus f(x)\rangle, \tag{18}$$

for $x \in \{1, \ldots, N\}$, $y \in \{0, 1\}$ and $\oplus$ denoting addition mod 2. Define

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad \text{and} \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{19}$$

Show that for any $|\psi\rangle \in \mathbb{C}^N$,

$$U_f(|\psi\rangle \otimes |+\rangle) = (V_+|\psi\rangle) \otimes |+\rangle \qquad \text{and} \qquad U_f(|\psi\rangle \otimes |-\rangle) = (V_-|\psi\rangle) \otimes |-\rangle \tag{20}$$

for some diagonal unitaries V_+, V_-. Find V_+, V_-, and express them in the form $\sum_x g_\pm(x)|x\rangle\langle x|$, for some functions $g_\pm(x)$.

(b) If V is a 2×2 unitary matrix, define the *controlled-V* operation to be the two-qubit unitary

$$|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes V.$$

For U a 2×2 unitary, show that

$$(I \otimes U)C_V(I \otimes U^\dagger)$$

is of the form C_W. Find W.

(c) The C_X gate is also called the "controlled-NOT" because it corresponds to flipping the second bit only if the first bit is equal to one. Calculate $(I \otimes H)C_X(I \otimes H)$, where H is the Hadamard gate:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(d) Calculate $(H \otimes H)C_X(H \otimes H)$. Does your answer have a simple interpretation?

7. **Quantum simulation**. This problem explores Feynman's original 1982 question, which can be paraphrased as "If classical computers have a hard time simulating quantum mechanics, then could a quantum computer do a better job?"

   Consider n spin-1/2 particles on a line subject to the Hamiltonian $H = H_{1,2} + H_{2,3} + \ldots + H_{n-1,n}$, where $H_{i,i+1}$ acts on particles $i, i+1$. Formally we can say that

   $$H_{i,i+1} = I_2^{\otimes i-1} \otimes h_{i,i+1} \otimes I_2^{\otimes n-(i+1)},$$

   where $h_{i,i+1}$ is a two-qubit Hamiltonian. We do not need to assume that the different $h_{i,i+1}$ are the same.

   Our goal in this problem is to approximately implement e^{-iHt} efficiently on a quantum computer (using units where $\hbar = 1$). In other words, we want a sequence of two-qubit unitary operations (called "gates") whose product is approximately e^{-iHt}.

   (a) Describe how to perform $e^{-iH_{i,i+1}t}$ with a single gate. *Hint: don't overthink this one.*

   (b) Assume that n is even, so that $n = 2m$ for some integer m. Define

   $$H_O = H_{1,2} + H_{3,4} + \ldots = \sum_{i=1}^{m} H_{2i-1,2i}$$
   $$H_E = H_{2,3} + H_{4,5} + \ldots = \sum_{i=1}^{m-1} H_{2i,2i+1}.$$

   Note that $H = H_O + H_E$. Describe how to perform $e^{-iH_O t}$ and $e^{-iH_E t}$. How many gates does your construction use?

   (c) Prove that

   $$e^{-iHt} = e^{-iH_O t} e^{-iH_E t} + O(t^2). \tag{21}$$

   (d) Prove that If $A_1, \ldots, A_L, B_1, \ldots, B_L$ are unitaries with $A_i - B_i = O(\delta)$, then

   $$A_1 A_2 \cdots A_L - B_1 B_2 \cdots B_L = O(L\delta). \tag{22}$$

   (e) Since (21) gives us an effective approximation only for very small t, it does not yet give us a good general-purpose algorithm. Assume now that t is arbitrary, but we divide it into L intervals so that t/L is small. In this case, we can approximate e^{-iHt} with

   $$(e^{-\frac{iH_E t}{L}} e^{-\frac{iH_O t}{L}})^L.$$

   Bound the error of this as a function of t and L. How large does L have to be if we want the error to be $O(\epsilon)$? How many gates are in the resulting circuit? Asymptotic answers (i.e. in the form $O(\cdot)$) are ok here.

8. **Wikipedia again.** Look at the wikipedia page that you edited (if you chose this option on pset 9). Have there been more revisions or responses on the talk page? Are your changes still there?