Quantum Physics III (8.06) — Spring 2018

Assignment 9

PRACTICE PROBLEMS

Posted: Thursday, May 15, 2018

Readings

- Griffiths, Chapter 5.
- Optional: Shankar, Ch. 10; Cohen-Tannoudji, Ch. XIV

Problem Set 9

1. **The "Exchange Force" (15 points)**

   (a) *Spatial wavefunctions:* Let $|\alpha\rangle, |\beta\rangle$ be two orthogonal single-particle states for particles in 1-d. with $\psi_\alpha(x) = \langle x|\alpha\rangle$ and $\psi_\beta(x) = \langle x|\beta\rangle$. Define the distinguishable, symmetric and antisymmetric states to be

$$|\Psi_D\rangle \equiv |\alpha\rangle \otimes |\beta\rangle \tag{1}$$

$$|\Psi_S\rangle \equiv \frac{|\alpha\rangle \otimes |\beta\rangle + |\beta\rangle \otimes |\alpha\rangle}{\sqrt{2}} \tag{2}$$

$$|\Psi_A\rangle \equiv \frac{|\alpha\rangle \otimes |\beta\rangle - |\beta\rangle \otimes |\alpha\rangle}{\sqrt{2}} \tag{3}$$

   Using tensor product notation, the position of the first (resp. second) particle is given by the operator $\hat{x}_1 \equiv \hat{x} \otimes I$ (resp. $\hat{x}_2 \equiv I \otimes \hat{x}$). Define $D_D = \langle\Psi_D|(\hat{x}_1 - \hat{x}_2)^2|\Psi_D\rangle$, $D_s = \langle\Psi_S|(\hat{x}_1 - \hat{x}_2)^2|\Psi_S\rangle$ and $D_A = \langle\Psi_A|(\hat{x}_1 - \hat{x}_2)^2|\Psi_A\rangle$. Calculate $D_S - D_D$ and $D_A - D_D$. Give a brief intuitive explanation of the signs of your answers.

   (b) *Spins:* Now consider two spin-s particles. For a single particle, let $|s, m_a\rangle$ be the $\hbar m_a$ eigenstate of the S_z operator. Suppose $m_a \neq m_b$. Define

$$|\Psi_D\rangle \equiv |s, m_a\rangle \otimes |s, m_b\rangle \qquad D_D \equiv \langle\Psi_D|(\vec{S}_1 - \vec{S}_2)^2|\Psi_D\rangle \tag{4}$$

$$|\Psi_S\rangle \equiv \frac{|s, m_a\rangle \otimes |s, m_b\rangle + |s, m_b\rangle \otimes |s, m_a\rangle}{\sqrt{2}} \qquad D_S \equiv \langle\Psi_S|(\vec{S}_1 - \vec{S}_2)^2|\Psi_S\rangle \tag{5}$$

$$|\Psi_A\rangle \equiv \frac{|s, m_a\rangle \otimes |s, m_b\rangle - |s, m_b\rangle \otimes |s, m_a\rangle}{\sqrt{2}} \qquad D_A \equiv \langle\Psi_A|(\vec{S}_1 - \vec{S}_2)^2|\Psi_A\rangle \tag{6}$$

Calculate D_D, D_S, D_A in terms of m_a, m_b. Order them from smallest to largest; i.e. write down an expression of the from $D_X \leq D_Y \leq D_Z$ with $\{X, Y, Z\}$ some permutation of $\{D, S, A\}$.

[*Hint:* You will find it useful to express $\vec{S}_1 \cdot \vec{S}_2$ in terms of the appropriate raising and lowering operators.]

2. **Two Electrons: Spin-dependent Interaction and Heisenberg Hamiltonian (20 points)**

Consider two electrons with the spatial wave function of one of them given by ψ_1 and that of the other one by ψ_2. We will first ignore the interactions between the electrons. That is, the Hamiltonian of the system can be written as

$$\mathcal{H} = H_0(\vec{r}_1) + H_0(\vec{r}_2) \tag{7}$$

where H_0 denotes the Hamiltonian for a one-electron system (which is spin-independent). Assume for simplicity that ψ_1 and ψ_2 are distinct (i.e. orthogonal) eigenstates of H_0 of the same energy.

The spatial wave function of the full system may be either symmetric or antisymmetric under the interchange of the electrons' coordinates. Since the electrons are spin-$\frac{1}{2}$ fermions, the overall wave function must be antisymmetric under the simultaneous interchange of both space coordinates and spin.

(a) Suppose the spatial wave function is antisymmetric, write down the full wave functions for the system. For these states, what are the eigenvectors and eigenvalues of the square and the z-component of the total spin operator $\vec{S}_{\text{tot}} = \vec{S}_1 + \vec{S}_2$? ($\vec{S}_{1,2}$ are the spin operator for each electron respectively.)

(b) Repeat part (a) in the case where the spatial wave function is symmetric.

(c) So far all the states enumerated in parts (a) and (b) have the same energy. Now add the following term to the Hamiltonian (7):

$$\mathcal{H}' = C\vec{S}_1 \cdot \vec{S}_2 \ . \tag{8}$$

That is, the electrons interact by a spin-spin force due to the interaction of the magnetic moment of each with the magnetic field generated by the other. What are the eigenstates of the system including the interaction $\mathcal{H}'$? What are the energies of the states in parts (a) and (b)?

(d) Now suppose we *ignore* the interaction (8) and consider the following Coulomb repulsion between the two electrons

$$\mathcal{H}'' = \frac{e^2}{|\vec{r}_1 - \vec{r}_2|} \tag{9}$$

Using perturbation theory, compute the first-order contribution of $\mathcal{H}''$ to the energy difference ϵ between the states in (a) and (b). You may leave your answer in integral form since the explicit spatial wave functions are not given.

(e) Suppose that the system has no spin-spin interaction $\mathcal{H}'$ but does have the Coulomb repulsion $\mathcal{H}''$. Argue that such a two-electron system can be described by an *effective* Hamiltonian of the form

$$\mathcal{H}_{\text{eff}} = -J\vec{S}_1 \cdot \vec{S}_2 \tag{10}$$

and express J in terms of ϵ. From your answer to 1(a), what do you think the sign of J will be? [*Note:* You do *not* need to do a detailed calculation or a rigorous proof here. Instead it is enough to give a hand-wavy argument: from 1(a), (anti)symmetric wavefunctions yield particles that are (closer or farther) to each other, which should (increase or decrease) the Coulomb repulsion.]

[*Another note:* This exercise tells us that the Hamiltonian for a two-electron (or more generally many-electron) system which depends only on space and not on spin variables can in fact be mimicked by an effective spin-spin interaction. This is a direct consequence of the Pauli exclusion principle: spin space and real space are interconnected quantum mechanically. Equation (10) was first realized by Heisenberg who used it to understand the origin of ferromagnetism. Note that in a solid, direct spin-spin interactions (8) are also present, but are much weaker (about one hundred times smaller) than (10) which arises from electrostatic interactions.]

3. **Three particles on a three-level system (10 points)** (Inspired by Cohen-Tannoudji, Problem 1, p.1447.)

Consider a Hamiltonian H_0 for the orbital degrees of freedom of a particle, with normalized eigenstates $\psi_0(x)$ of energy zero, $\psi_1(x)$ of energy $\hbar\omega$, and $\psi_2(x)$ with energy $2\hbar\omega$. If the particle has spin s, each of the above eigenstates is degenerate with multiplicity $2s+1$.

Now consider three identical particles with Hamiltonian

$$H = H_0^{(1)} + H_0^{(2)} + H_0^{(3)}\,,$$

where the superscript denotes particle label.

(a) Assume the three particles are electrons. Find the energy levels of the Hamiltonian and the degeneracies. Write out the wavefunction for the state (or states) in which the three electrons have their spin up.

(b) Assume the three particles are spin zero bosons. Find the energy levels of the Hamiltonian and the degeneracies.

(c) Assume the three particles have Maxwell-Boltzman statistics (no spin). Find the energy levels of the Hamiltonian and the degeneracies.