

## Assignment 8

Posted: Thursday, May 3, 2018

### Readings and Announcements

- Scattering Theory: Griffiths, Chapter 11,
- Cohen-Tannoudji, Ch. VIII, and/or Shankar, Chapter 19.

#### 1. Which phase? (10 points)

In the adiabatic theorem we define  $E_n(t)$  and  $|\psi_n(t)\rangle$  according to

$$H(t)|\psi_n(t)\rangle = E_n(t)|\psi_n(t)\rangle. \quad (1)$$

Unfortunately, this equation does not have a unique solution, even if there is no degeneracy. This is clear because multiplying  $|\psi_n(t)\rangle$  by an arbitrary *time-dependent* phase still gives a solution.

Suppose that Alice solves (1) and obtains solutions  $\{|\psi_n^A(t)\rangle\}$  and Bob solves (1) and obtains solutions  $\{|\psi_n^B(t)\rangle\}$ . Assume that they agree at time  $t = 0$ , so that

$$|\psi_n^A(0)\rangle = |\psi_n^B(0)\rangle.$$

At later times their solutions of (1) may be different. As mentioned above, we may have a time-dependent phase  $\alpha_n(t)$  such that

$$|\psi_n^A(t)\rangle = e^{i\alpha_n(t)}|\psi_n^B(t)\rangle, \quad \text{with } \alpha_n(0) = 0.$$

Will this lead Alice and Bob to get different predictions from the adiabatic theorem? More concretely, suppose that at time  $t = 0$  a system is in state

$$|\Psi(t = 0)\rangle = |\psi_n^A(0)\rangle = |\psi_n^B(0)\rangle.$$

Suppose that for times  $0 \leq t \leq T$  the Hamiltonian changes adiabatically. Both Alice and Bob predict the state at time  $T$  using the adiabatic theorem, but each follows their own basis conventions. Write down Alice and Bob's predictions  $|\Psi^A(T)\rangle$  and  $|\Psi^B(T)\rangle$  for the state at time  $T$ . Are these in fact the same state or are they different? Since in the adiabatic theorem we care about phases, equality means equality including phases. Explain your answer.

## 2. Partial Waves (15 points)

Suppose the scattering amplitude for a certain potential is given by

$$f(\theta) = \frac{1}{k} \left( \frac{\Gamma k}{k_0 - k - ik\Gamma} + 3e^{2i\beta k^3} \sin(2\beta k^3) \cos \theta \right),$$

where  $\Gamma$ ,  $k_0$ , and  $\beta$  are constants characteristic of the potential which produces the scattering. As usual,  $\hbar k$  denotes the momentum of the particle.

- What partial waves are active and what are the corresponding phase shifts? Do they have the proper behavior as  $k \rightarrow 0$ ? (The rule of thumb is that  $\delta_l \sim k^{2l+1}$ , which is true for scattering off a hard sphere.)
  - What is the differential cross section  $d\sigma/d$  for general values of  $k$ ?
  - What are the partial wave cross sections  $\sigma_\ell$ ?
  - Assume  $\beta k_0^3 \ll 1$ . What is the total cross section  $\sigma(k)$  for  $k \approx k_0$ .
  - Find the total cross section for arbitrary  $k$  and the imaginary part of the forward scattering amplitude? Do they satisfy the optical theorem?
3. **Scattering from a spherical well (20 points)** For some parameters  $\gamma$  and  $b$ , consider the following spherically symmetrical potential:

$$V(\vec{r}) = V(r) = \begin{cases} -\frac{\hbar^2}{2m}\gamma^2 & r \leq b \\ 0 & r > b \end{cases}$$

We consider  $s$ -wave ( $\ell = 0$ ) scattering of an incoming plane wave with momentum  $\hbar k$ .

- Calculate the phase shift  $\delta_0$ . Leave your answer in terms of  $k$ ,  $b$ , and the real and positive constant  $q$  defined via the relation  $q^2 \equiv k^2 + \gamma^2$ .
- Find the scattering length  $a \equiv -\lim_{k \rightarrow 0} \frac{\tan(\delta_0)}{k}$ , and plot  $a/b$  as a function of  $\gamma b$ . Your plot should have many zeros. For these values of  $\gamma b$  we have  $\sigma_0 = 0$  and there is no  $s$ -wave scattering. This is known as the Ramsauer-Townsend effect. Numerically find the smallest positive value of  $\gamma b$  for which  $a = 0$ . Your plot should also have infinities when  $\gamma b = (n + 1/2)\pi$  for  $n$  a nonnegative integer. What happens to  $\delta_0$  and  $\sigma_0$  at these points? Is this consistent with the bound from partial-wave unitarity?
- Let's try to explore these infinities more. In the above we took the  $E \rightarrow 0$  limit from above, i.e. considering  $E$  to be positive and very small. Now consider  $\lim_{E \rightarrow 0^-}$ ; i.e. suppose  $E < 0$  and take the limit as  $E$  approaches zero. Now solutions to the Schrödinger equation correspond to bound states. We can equivalently think of  $k$  as  $i\kappa$  for some  $\kappa > 0$ . A bound state with  $E$  very close to zero is called a "threshold" bound state. Which values of  $\gamma b$  correspond to threshold bound states? For each such value of  $\gamma b$  how many bound states (i.e. not only including threshold bound states) does the potential support?

[Comment: For partial-wave scattering at fixed  $\ell$ , the  $S$ -matrix element is the extra phase  $e^{2i\delta_\ell}$  in the outgoing wave relative to the incoming wave. When we let  $k = i\kappa$  with  $\kappa > 0$ , the outgoing wave becomes a decaying exponential and the ingoing wave becomes a growing exponential. If for such  $k$  the  $S$ -matrix element  $e^{2i\delta_\ell(k)} = \infty$ , the growing exponential effectively vanishes and we have the description of a bound state.]

- (d) Sketch the radial solution  $u(r)$  as a function of  $r/b$  for  $k = 0$  and  $\gamma b = 0, \pi/4, \pi/2, \pi$ .
- (e) Suppose  $\gamma b$  is slightly larger than  $\pi/2$ , so there is a threshold bound state with energy  $-E_B$ , with  $E_B$  positive. Show that for incoming waves of small positive energy  $E$ ,

$$\sigma_0 \approx \frac{c}{E + E_B}$$

for some constant  $c$ . Find  $c$ . This result shows that low-energy scattering can be used to detect low lying bound states.

#### 4. Scattering from a $\delta$ -shell (15 points)

Consider  $s$ -wave scattering from the potential

$$V(r) = \lambda \frac{\hbar^2}{2mR} \delta(r - R),$$

with  $\lambda$  a large positive constant.

- (a) Let  $u$  denote the radial solution. By comparing  $u'(r)/u(r)$  just inside and just outside  $r = R$ , find a formula to determine  $\delta_0$ .
- (b) Find the scattering length  $a \equiv -\lim_{k \rightarrow 0} \frac{\tan \delta_0}{k}$ .
- (c) Assume  $\lambda \gg 1$ . Sketch  $\delta_0(k)$ . Show that for  $kR$  just below  $n\pi$ , with  $n$  a positive integer,  $\delta_0(k)$  increases very rapidly by  $\pi$  (as  $kR$  increases towards  $n\pi$ ). Sketch the  $s$ -wave cross-section  $\sigma_0$ . Show that the  $s$ -wave scattering from this potential is the same as that from a hard sphere of radius  $R$  for all values of  $kR$  *except* those such that  $kR$  is close to  $n\pi$ . What is the significance of these values?

#### 5. Born Approximation for Scattering From Yukawa and Coulomb Potentials, plus a Practical Example of the Latter (15 points)

Check Griffiths' Examples 11.5 and 11.6 (p.415). He's done some of the work for you.

Consider a Yukawa potential

$$V(r) = \beta \frac{e^{-\mu r}}{r}$$

where  $\beta$  and  $\mu$  are constants.

- (a) Evaluate the scattering amplitude, the differential cross section  $d\sigma/d$ , and the total cross section in the first Born approximation. Express your answer for the total cross section as a function of the energy  $E$ .

- (b) Take  $\beta = Q_1 Q_2$  and  $\mu = 0$ , and show that the differential cross section you obtain for scattering off a Coulomb potential is the same as the classical Rutherford result. Use this differential cross section in part (d) below.
- (c) Differential cross sections  $\frac{d\sigma}{d}$  are what physicists actually use to calculate the rate at which scattered particles will enter their detectors. The number of particles scattered out into solid angle  $d$  per second by a *single scatterer* is given by

$$\frac{d^2 N_*^{out}}{dt d} = \frac{d\sigma}{d} \times \frac{d^2 N^{inc}}{dt d A},$$

where the  $*$  is for single scatterer, and  $\frac{d^2 N^{inc}}{dt d A}$  is the incident flux in units of particles per second per unit area transverse to the beam.

Consider a uniform beam of  $\frac{dN^{inc}}{dt}$  particles per second with a cross sectional area  $A$ . This beam strikes a target with  $n$  scattering sites per unit volume and thickness  $T$ .

Give an expression for the number of particles

$$\frac{d^2 N^{out}}{dt d}$$

scattered into a detector with angular size  $d$  per unit time.

Show that your result is independent of the cross sectional area  $A$  of the beam *even* if the beam is not uniform across this area. This is important because it is typically easy for an experimenter to measure  $\frac{dN^{inc}}{dt}$  but hard to measure the cross sectional area  $A$  or to assess the uniformity of the beam across  $A$ .

- (d) Consider a beam of alpha particles ( $Q_1 = 2e$ ) with kinetic energy 8 MeV scattering from a gold foil. Suppose that the beam corresponds to a current of 1 nA. [It is conventional to use MKS units for beam currents. 1 nA is  $10^{-9}$  Amperes, meaning  $10^{-9}$  Coulombs of charge per second. Each alpha particle has charge  $2e$ , where  $e = 1.6 \times 10^{-19}$  Coulombs.] Suppose the gold foil is 1 micron thick. You may assume the alpha particles scatter only off nuclei, not off electrons. You may also assume that each alpha particle scatters only once. You will need to look up the density of gold and the nuclear charge of gold ( $Q_2$ ). How many alpha particles per second do you expect to be scattered into a detector which occupies a cone of angular extent  $d\theta = d\phi = 10^{-2}$  radians, centered at  $\theta = \pi/2$ ?

**6. Born for 1D problems (15 points) Based on Griffiths's 11.16, 11.17, 11.18).**

Consider the one-dimensional Schrödinger equation for a particle of mass  $m$  moving in a potential  $V(x)$ . For convenience define the rescaled potential function  $U(x)$  from

$$V(x) = \frac{\hbar^2}{2m} U(x)$$

- (a) Find the explicit form of the Green's function  $G(x)$  that will allow you to write the following integral form of the Schrödinger equation for the wavefunction  $\psi(x)$ :

$$\psi(x) = \psi_0(x) + \int_{-\infty}^{\infty} dx' G(x-x') U(x') \psi(x'),$$

where  $\psi_0(x)$  is a solution with zero potential. Use an outgoing type Green's function in analogy to our 3D case.

- (b) Consider one-dimensional scattering on the open line with a potential  $V(x)$  that is non-zero only for  $x \in [-x_0, x_0]$  for some positive  $x_0$ . Consider the above integral equation setting  $\psi_0(x)$  equal to a wave  $Ae^{ikx}$  incident from the left:

$$\psi_0(x) = Ae^{ikx}$$

Show that to first order in the Born approximation the reflection coefficient  $R$  for this potential takes the form

$$R \simeq \left( \frac{m}{\hbar^2 k} \right)^2 \int_{-x_0}^{x_0} dx' e^{2ikx'} V(x')^2.$$

- (c) Evaluate the above expression for  $R$  for the case of the delta function potential

$$V(x) = -\alpha\delta(x),$$

with  $\alpha > 0$ . Write your answer for  $R$  in terms of the particle energy  $E$ ,  $m$ ,  $\alpha$ , and  $\hbar$ . Compare with the exact reflection coefficient given in Griffiths [2.141], p.75.

MIT OpenCourseWare  
<https://ocw.mit.edu>

8.06 Quantum Physics III  
Spring 2018

For information about citing these materials or our Terms of Use, visit: <https://ocw.mit.edu/terms>