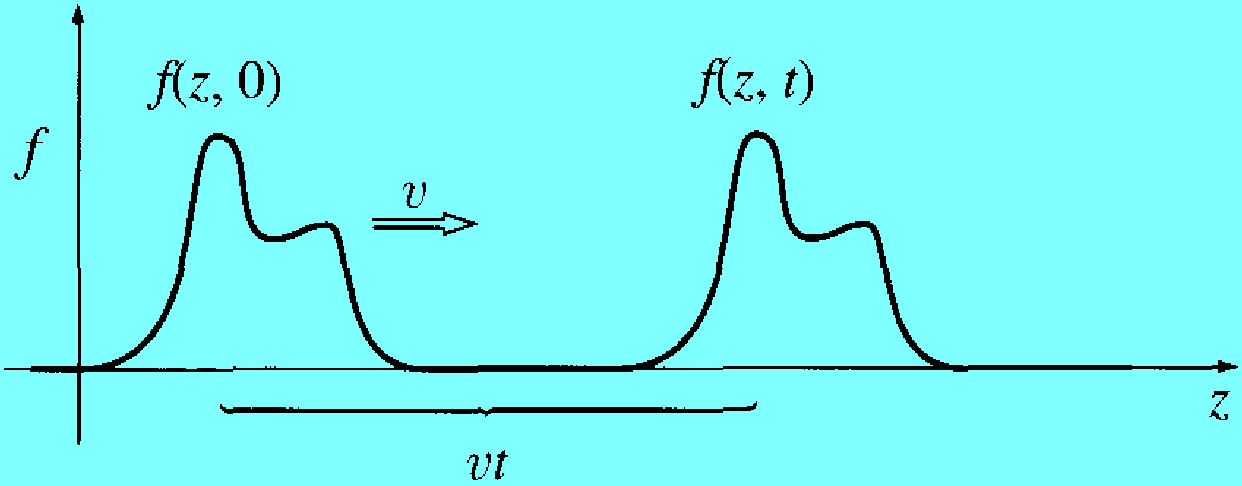
8.07 Lecture 35: December 7, 2012
ELECTROMAGNETIC WAVES

The Wave Equation in 1 Dimension:

The travelling wave:



$$f(z,t) = f(z - vt, 0) \equiv g(z - vt) \ . \tag{1}$$

But waves can move in both directions:

$$f(z,t) = g_1(z - vt) + g_2(z + vt) \ . \tag{2}$$

Differential equation for $f(z,t)$:

$$\begin{aligned} \frac{\partial f}{\partial z} &= g_1'(z - vt) + g_2'(z + vt) \\ \frac{\partial^2 f}{\partial z^2} &= g_1''(z - vt) + g_2''(z + vt) \\ \frac{\partial f}{\partial t} &= -v g_1'(z - vt) + v g_2'(z + vt) \\ \frac{\partial^2 f}{\partial t^2} &= v^2 g_1'(z - vt) + v^2 g_2'(z + vt) \ . \end{aligned} \tag{3}$$

Wave equation:

$$\boxed{\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0 \ .} \tag{4}$$

Sinusoidal Waves:

$$
\begin{aligned}
f(z,t) &= A\cos\left[k(z-vt)+\delta\right] \\
&= A\cos\left[kz-\omega t+\delta\right] \ ,
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
v &= \frac{\omega}{k} = \text{phase velocity} \\
\omega &= \text{angular frequency} = 2\pi\nu \\
\nu &= \text{frequency} \\
\delta &= \text{phase (or phase constant)} \\
k &= \text{wave number} \\
\lambda &= 2\pi/k = \text{wavelength} \\
T &= 2\pi/\omega = \text{period} \\
A &= \text{amplitude.}
\end{aligned}
\tag{6}
$$

Any wave can be constructed by superimposing sinusoidal waves (Fourier's Theorem, aka Dirichlet's Theorem).

Complex Notation:

Let $\tilde{A} = Ae^{i\delta}$. Then

$$f(z,t) = \mathrm{Re}\,[\tilde{A}e^{i(kz-\omega t)}], \tag{7}$$

where we used

$$e^{i\theta} = \cos\theta + i\sin\theta\ . \tag{8}$$

Conventions: drop "Re", and drop $\tilde{}$ on $\tilde{A}$.

$$f(z,t) = Ae^{i(kz-\omega t)}\ . \tag{9}$$

General solution to wave equation:

$$f(z,t) = \int_{-\infty}^{\infty} A(k)\, e^{i(kz-\omega t)}\, \mathrm{d}k\ , \tag{10}$$

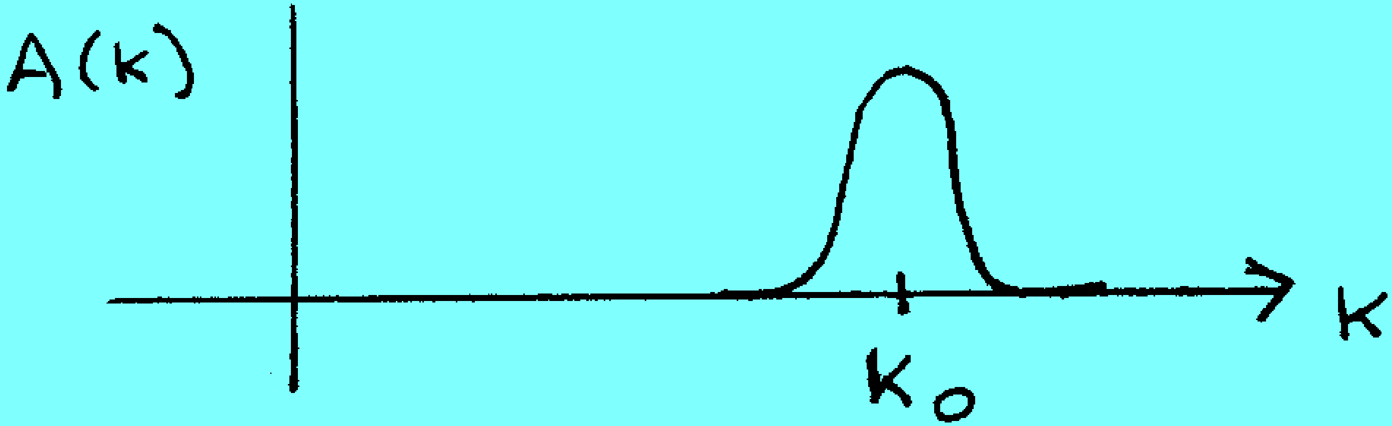
where $\omega/k = v$, v = wave speed = phase velocity.

Group Velocity and Phase Velocity:

Not in Griffiths. In Jackson, pp. 324-325.

ω can sometimes depend on k: dispersion.

Consider a wave packet centered on k_0:



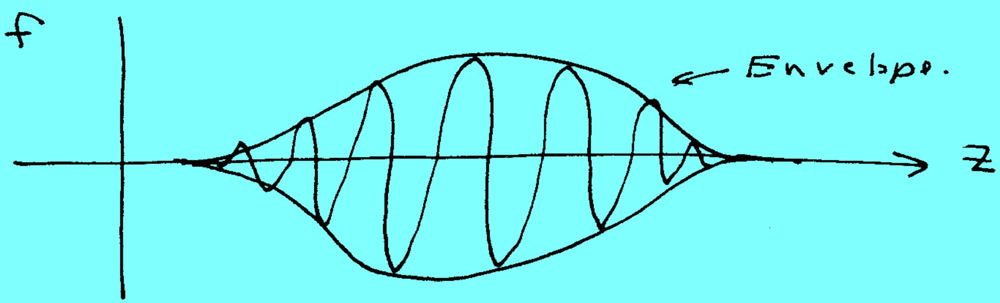
$$\begin{aligned} \omega(k) &= \omega(k_0) + \frac{d\omega}{dk}(k_0)(k - k_0) + \ldots \\ &= \omega(k_0) - k_0 \frac{d\omega}{dk} + k \frac{d\omega}{dk} + \ldots \; . \end{aligned} \tag{11}$$

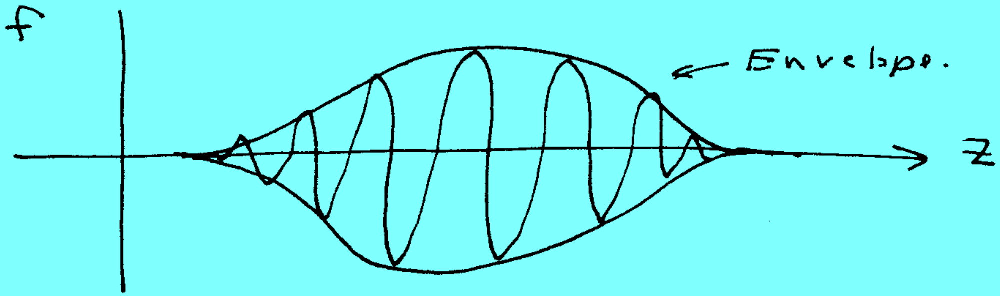
$$\begin{aligned} \omega(k) &= \omega(k_0) + \frac{\mathrm{d}\omega}{\mathrm{d}k}(k_0)(k - k_0) + \ldots \\ &= \omega(k_0) - k_0\frac{\mathrm{d}\omega}{\mathrm{d}k} + k\frac{\mathrm{d}\omega}{\mathrm{d}k} + \ldots\ . \end{aligned} \tag{11}$$

$$f(z,t) = e^{i\left[\omega(k_0) - k_0\frac{\mathrm{d}\omega}{\mathrm{d}k}\right]t}\int_{-\infty}^{\infty}\mathrm{d}k\,A(k)e^{ik\left(z - \frac{\mathrm{d}\omega}{\mathrm{d}k}t\right)}\ . \tag{12}$$

The integral describes a wave which moves with

$$\boxed{v_{\text{group}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}(k_0)\ .} \tag{13}$$





Envelope moves with $v = v_{\rm group}$.

Waves inside envelope move with $v_{\rm phase} = v = \omega(k)/k$.

If $v_{\rm phase} > v_{\rm group}$, then waves appear at the left of the envelope and move forward through the envelope, disappearing at the right.

Electromagnetic Plane Waves

Maxwell Equations in Empty Space:

$$\begin{aligned} \nabla \cdot \vec{E} &= 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \, , \\ \nabla \cdot \vec{B} &= 0 \qquad \nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \, , \end{aligned} \tag{14}$$

where $1/c^2 \equiv \mu_0 \epsilon_0$. Manipulating,

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= \vec{\nabla} \underbrace{(\vec{\nabla} \cdot \vec{E})}_{=0} - \nabla^2 \vec{E} \\ &= \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \, , \end{aligned} \tag{15}$$

$$
\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= \vec{\nabla} \underbrace{(\vec{\nabla} \cdot \vec{E})}_{=0} - \nabla^2 \vec{E} \\
&= \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \; ,
\end{aligned} \tag{15}
$$

so

$$
\boxed{\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \; .} \tag{16}
$$

This is the wave equation in 3 dimensions. An identical equation holds for $\vec{B}$:

$$
\boxed{\nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \; .} \tag{17}
$$

Each component of $\vec{E}$ and $\vec{B}$ satisfies the wave equation. This implies that waves travel at speed c! But the wave equation is not all: $\vec{E}$ and $\vec{B}$ are still related by Maxwell's equations.

Try

$$\vec{E}(\vec{r},t) = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\hat{n} \ , \tag{18}$$

where $\tilde{E}_0$ is a complex amplitude, $\hat{n}$ is a unit vector, and $\omega/|\vec{k}| = v_{\text{phase}} = c$. Then

$$\vec{\nabla}\cdot\vec{E} = i\hat{n}\cdot\vec{k}E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \ , \tag{19}$$

so we require

$$\hat{n}\cdot\vec{k} = 0 \qquad \text{(transverse wave)} \ . \tag{20}$$

The magnetic field satisfies

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} = -i\vec{k} \times \vec{E} = -i\vec{k} \times \hat{n}\,\tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \;. \tag{21}$$

Integrating,

$$\vec{B} = \frac{\vec{k}}{\omega} \times \hat{n}\,\tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \;, \tag{22}$$

so, remembering that $|\vec{k}| = \omega c$,

$$\boxed{\vec{B} = \frac{1}{c}\,\hat{k} \times \vec{E} \;.} \tag{23}$$

Energy and Momentum:

Energy density:

$$u = \frac{1}{2}\left[\epsilon_0|\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right] \ . \tag{24}$$

The $\vec{E}$ and $\vec{B}$ contributions are equal.

$$u = \epsilon_0 E_0^2 \underbrace{\cos^2(kz - \omega t + \delta)}_{\text{averages to } 1/2} \ , \qquad (\vec{k} = k\,\hat{z})$$

$$\begin{aligned} \vec{S} &= \frac{1}{\mu_0}\vec{E}\times\vec{B} = uc\ \hat{z} \\ \mathcal{P}_{\text{EM}} &= \frac{1}{c^2}\vec{S} = \frac{u}{c}\,\hat{z} \\ I\ (\text{intensity}) &= \left\langle |\vec{S}| \right\rangle = \frac{1}{2}\epsilon_0 E_0^2 \ . \end{aligned} \tag{25}$$

Electromagnetic Waves in Matter

For linear, homogeneous materials, Maxwell's equations are unchanged except for the replacement $\mu_0\epsilon_0 \to \mu\epsilon$. Define

$$n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \text{index of refraction.} \tag{26}$$

Then

$$v = \text{phase velocity} = \frac{c}{n} \ . \tag{27}$$

When expressed in terms of $\vec{E}$ and $\vec{B}$, everything carries over, with these substitutions:

$$
\begin{aligned}
u &= \frac{1}{2}\left[\epsilon|\vec{E}|^2 + \frac{1}{\mu}|\vec{B}|^2\right] \\
\vec{B} &= \frac{n}{c}\,\hat{k}\times\vec{E} \\
\vec{S} &= \frac{1}{\mu}\vec{E}\times\vec{B} = \frac{uc}{n}\ \hat{z}\ .
\end{aligned}
\tag{28}
$$

Boundary Conditions, Transmission and Reflection

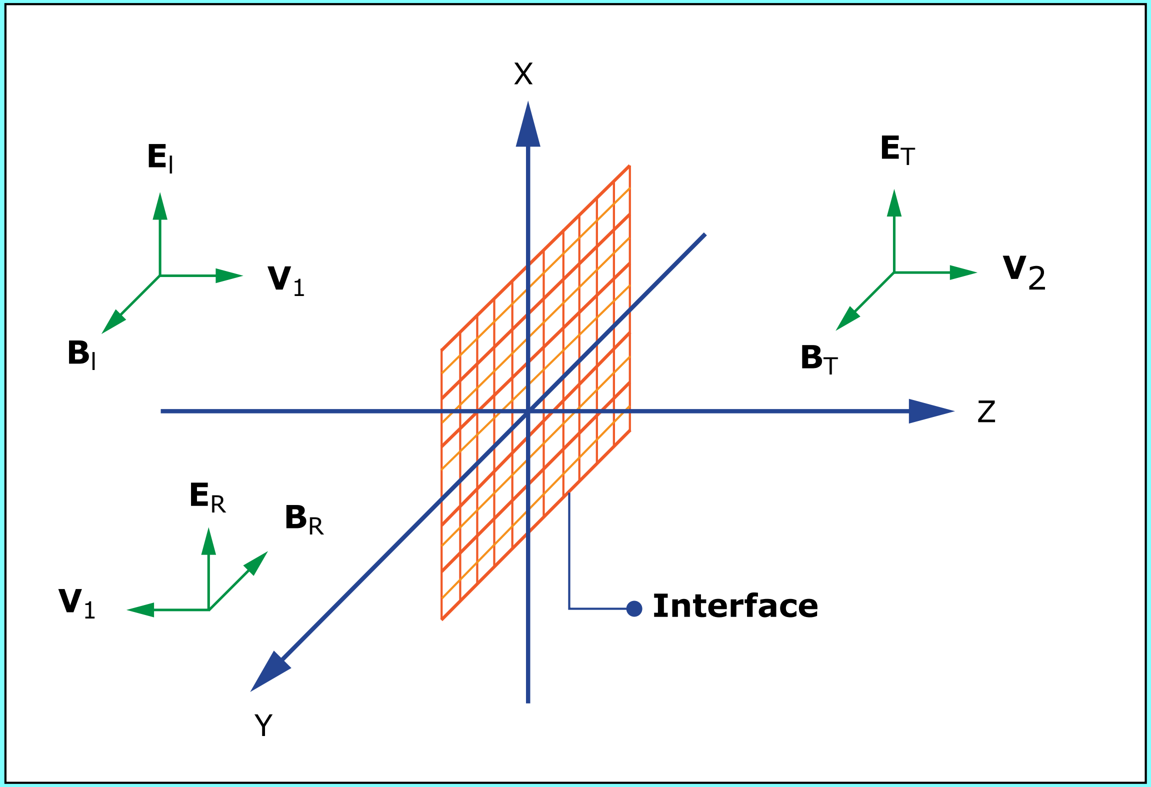


Image by MIT OpenCourseWare.

Boundary Conditions:

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \qquad \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \ ,$$
$$B_1^{\perp} = B_2^{\perp} \qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel} \ . \tag{29}$$

Incident wave ($z < 0$):

$$\vec{E}_I(z,t) = \tilde{E}_{0,I}\, e^{i(k_1 z - \omega t)}\, \hat{x}$$
$$\vec{B}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0,I}\, e^{i(k_1 z - \omega t)}\, \hat{y} \ . \qquad (30)$$

Transmitted wave ($z > 0$):

$$\vec{E}_T(z,t) = \tilde{E}_{0,T}\, e^{i(k_2 z - \omega t)}\, \hat{x}$$
$$\vec{B}_T(z,t) = \frac{1}{v_2}\tilde{E}_{0,T}\, e^{i(k_2 z - \omega t)}\, \hat{y} \ . \qquad (31)$$

ω must be the same on both sides, so

$$\frac{\omega}{k_1} = v_1 = \frac{c}{n_1} \ , \qquad \frac{\omega}{k_2} = v_2 = \frac{c}{n_2} \ . \qquad (32)$$

Reflected wave ($z < 0$):

$$\vec{E}_R(z,t) = \tilde{E}_{0,R}\, e^{i(-k_1 z - \omega t)}\, \hat{x}$$
$$\vec{B}_R(z,t) = -\frac{1}{v_1}\tilde{E}_{0,R}\, e^{i(-k_1 z - \omega t)}\, \hat{y}\ . \tag{33}$$

Boundary conditions:

$$\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \quad \Longrightarrow \quad \tilde{E}_{0,I} + \tilde{E}_{0,R} = \tilde{E}_{0,T}\ , \tag{34}$$

$$\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel} \quad \Longrightarrow \quad \frac{1}{\mu_1}\left(\frac{1}{v_1}\tilde{E}_{0,I} - \frac{1}{v_1}\tilde{E}_{0,R}\right) = \frac{1}{\mu_2}\frac{1}{v_2}\tilde{E}_{0,T}\ . \tag{35}$$

Two equations in two unknowns: $\tilde{E}_{0,R}$ and $\tilde{E}_{0,T}$.

Solution:

$$\tilde{E}_{0,R} = \left| \frac{n_1 - n_2}{n_1 + n_2} \right| \tilde{E}_{0,I} \qquad E_{0,T} = \left(\frac{2n_1}{n_1 + n_2} \right) \tilde{E}_{0,I} \ . \qquad (36)$$