MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — September 5, 2012
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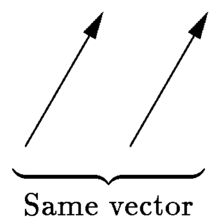
LECTURE NOTES 1
VECTOR ANALYSIS

DEFINITION: A *vector* is a quantity that has magnitude and direction.

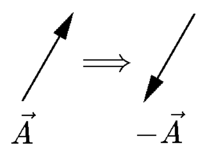
Examples: displacement, velocity, acceleration, force, momentum, electric and magnetic fields.

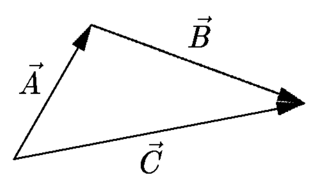
Vectors do not have position:



OPERATIONS:

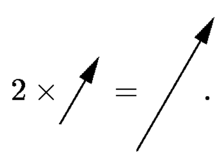
Magnitude: $\|\vec{A}\| \equiv$ magnitude of $\vec{A}$. (Here $\equiv$ means "is defined to be".) Often A is used to denote $\|\vec{A}\|$.

Negation:


Addition:


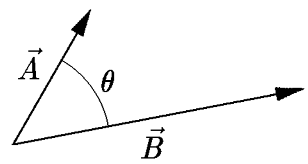
$\vec{A} + \vec{B} = \vec{C}$

Subtraction: $\vec{A} - \vec{B} \equiv \vec{A} + (-\vec{B})$.

Multiplication by a scalar:


Property— Distributive: $a(\vec{A} + \vec{B}) = a\vec{A} + a\vec{B}$.

Dot product of two vectors: $\vec{A}\cdot\vec{B} \equiv |\vec{A}||\vec{B}|\cos\theta$, where θ is the angle between $\vec{A}$ and $\vec{B}$:



always tail to tail!

Properties:

Rotational invariance. The value of the dot product does not change if both of the vectors are rotated together.

Cummutative: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$.

Distributive: $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$.

Scalar multiplication: $(a\vec{A})\cdot\vec{B} = a(\vec{A}\cdot\vec{B})$.

Query: Why $\cos\theta$??? If I defined a Guth-dot product by

$$\vec{A}\cdot\vec{B}\Big|_{\text{Guth}} \equiv |\vec{A}||\vec{B}|\sin\theta\ ,$$

and hired a really good advertising agency, could my product (note the pun!) compete?

Tentative answer: Maybe a really good advertising agency can do anything, but I would have a serious marketing problem. My dot product would not be distributive. In fact, one can show that if $\vec{A}\cdot\vec{B}$ obeys rotational invariance and the distributive law, then

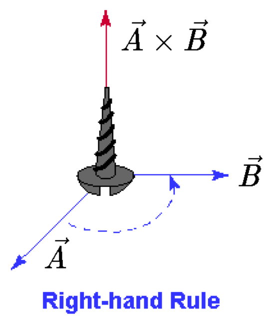
$$\vec{A}\cdot\vec{B} = \text{const}|\vec{A}||\vec{B}|\cos\theta\ .$$

We'll come back to this later.

Cross product of two vectors:

$$\vec{A} \times \vec{B} \equiv |\vec{A}||\vec{B}| \sin\theta \, \hat{n} \ ,$$

where $\hat{n}$ is a unit vector perpendicular to $\vec{A}$ and perpendicular to $\vec{B}$. The choice of the two (opposite) directions that are perpendicular to both $\vec{A}$ and $\vec{B}$ is determined by the right-hand rule:



(Source: Modified from chortle.ccsu.edu/vectorlessons/vch12/rightHandRule.gif.)

Properties:

Rotational invariance: If both vectors are rotated by the same rotation R, then the result of the cross product is also rotated by R.

Anticommutative: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

Distributive: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$.

Scalar multiplication: $(a\vec{A}) \times \vec{B} = a(\vec{A} \times \vec{B})$.

Query: Why $\sin\theta$???

Answer: Again, the function of θ is required for rotational invariance and distributivity. If these two properties hold, then one can show that

$$\vec{A} \times \vec{B} = \text{const}|\vec{A}||\vec{B}| \sin\theta \, \hat{n} \ .$$

COMPONENT FORM:

$$\vec{A} = A_x \hat{e}_x + A_y \hat{e}_y + A_z \hat{e}_z \ ,$$

where $\hat{e}_x$ is a unit vector in the direction of the positive x-axis. (Various notations are in use. Griffiths uses $\hat{x}$, $\hat{y}$, and $\hat{z}$, and many other books use $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.)

Operations:

Vector addition:

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{e}_x + (A_y + B_y)\hat{e}_y + (A_z + B_z)\hat{e}_z \ .$$

Vector dot product:

$$\vec{A} \cdot \vec{B} = (A_x \hat{e}_x + A_y \hat{e}_y + A_z \hat{e}_z) \cdot (B_x \hat{e}_x + B_y \hat{e}_y + B_z \hat{e}_z) \; .$$

$$\hat{e}_x \cdot \hat{e}_x = \hat{e}_y \cdot \hat{e}_y = \hat{e}_z \cdot \hat{e}_z = 1 \; ,$$

and $\hat{e}_x \cdot \hat{e}_y = 0$, as does the dot product of any two basis vectors.

$$\therefore \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \; .$$

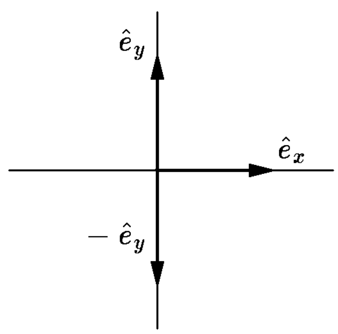
Return to query: why $\cos\theta$???

Rotational invariance $\implies$ $\hat{e}_x \cdot \hat{e}_x = \hat{e}_y \cdot \hat{e}_y = \hat{e}_z \cdot \hat{e}_z = \text{const.}$

What about $\hat{e}_x \cdot \hat{e}_y$?: Rotational invariance also implies that

$$\hat{e}_x \cdot \hat{e}_y = \hat{e}_x \cdot (-\hat{e}_y) \; ,$$

since the pair $(\hat{e}_x, -\hat{e}_y)$ can be obtained from the pair $(\hat{e}_x, \hat{e}_y)$ by rotating both vectors 180° about the x-axis:



Thus $\hat{e}_x \cdot \hat{e}_y + \hat{e}_x \cdot (-\hat{e}_y) = 2\hat{e}_x \cdot \hat{e}_y$. But then the distributive law implies that

$$\hat{e}_x \cdot \hat{e}_y + \hat{e}_x \cdot (-\hat{e}_y) = \hat{e}_x \cdot (\hat{e}_y + (-\hat{e}_y)) = \hat{e}_x \cdot \vec{0} = 0 \; .$$

Similarly the dot product of any two distinct basis vectors must vanish, so

$$\vec{A} \cdot \vec{B} = \text{const}\,(A_x B_x + A_y B_y + A_z B_z) \; .$$

This had better be equivalent to $\text{const}|\vec{A}||\vec{B}|\cos\theta$, but we can see it explicitly by using the rotational invariance to orient $\vec{A}$ along the positive x-axis, so $A_x = |\vec{A}|$, and $A_y = A_z = 0$. Then the above formula gives $\vec{A} \cdot \vec{B} = \text{const}\, A_x B_x$, but $B_x = |\vec{B}|\cos\theta$, where θ is the angle between $\vec{B}$ and the x-axis, but that is also the angle between $\vec{B}$ and $\vec{A}$.

Vector cross product:

$$\vec{A} \times \vec{B} = (A_x \hat{e}_x + A_y \hat{e}_y + A_z \hat{e}_z) \times (B_x \hat{e}_x + B_y \hat{e}_y + B_z \hat{e}_z) \ ,$$

where

$$\hat{e}_x \times \hat{e}_x = \hat{e}_y \times \hat{e}_y = \hat{e}_z \times \hat{e}_z = 0 \ ,$$

and

$$\hat{e}_x \times \hat{e}_y = \hat{e}_z \ , \text{and cyclic permutations,}$$

$$\hat{e}_y \times \hat{e}_z = -\hat{e}_z \ , \text{and cyclic permutations.}$$

Expanding,

$$\vec{A} \times \vec{B} = (A_x B_y - A_y B_x)\hat{e}_z + (A_y B_z - A_z B_y)\hat{e}_x + (A_z B_x - A_x B_z)\hat{e}_y \ .$$

Equivalently, one can write

$$\vec{A} \times \vec{B} = \det \begin{pmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix} \ .$$

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