MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II October 11, 2012

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LECTURE NOTES 8
THE TRACELESS SYMMETRIC TENSOR EXPANSION AND STANDARD SPHERICAL HARMONICS

These notes are an addendum to Lecture 14, Wednesday October 10, 2012. The notes will describe a topic that I did not have time to include in the lecture: the relation between the traceless symmetric tensor expansion and the standard spherical harmonics.

Using traceless symmetric tensors, we can expand any function of angle as

$$\begin{aligned} F(\hat{n}) &= \sum_{\ell=1}^{\infty} C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\, \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \equiv \sum_{\ell=0}^{\infty} F_\ell(\hat{n}) \\ &= C^{(0)} + C^{(1)}_i\, \hat{n}_i + C^{(2)}_{ij}\, \hat{n}_i \hat{n}_j + C^{(3)}_{ijk}\, \hat{n}_i \hat{n}_j \hat{n}_k + \ldots \ , \end{aligned} \tag{8.1}$$

where the $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ are traceless symmetric tensors, the indices i_1, i_2, $\ldots i_\ell$ are summed from 1 to 3 as Cartesian indices, and

$$\hat{n}(\theta,\phi) = \sin\theta\cos\phi\,\hat{e}_1 + \sin\theta\sin\phi\,\hat{e}_2 + \cos\theta\,\hat{e}_3 \ , \tag{8.2}$$

where $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ can also be written as $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$.

In the more standard approach, an arbitrary function of (θ,ϕ) is expanded in spherical harmonics:

$$F(\hat{n}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta,\phi) \ . \tag{8.3}$$

We have shown that

$$\nabla^2_\theta \left[C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\, \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \right] = -\ell(\ell+1) C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\, \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \ , \tag{8.4}$$

where

$$\nabla^2_\theta = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \ . \tag{8.5}$$

In the standard approach one would show that

$$\nabla^2_\theta\, Y_{\ell m}(\theta,\phi) = -\ell(\ell+1) Y_{\ell m}(\theta,\phi) \ , \tag{8.6}$$

so ℓ apparently has the same meaning in both formalisms. (I am not trying here to derive the standard formalism, but instead I will simply adopt the equations from standard textbooks, and show that we can express these functions in terms of traceless symmetric tensors. A good example of such a standard textbook is J.D. Jackson, **Classical Electrodynamics**, 3rd Edition (John Wiley & Sons, 1999), Sections 3.1, 3.2, 3.5, and 3.6. That means that there must be some particular traceless symmetric tensor, which we will call $C^{(\ell,m)}_{i_1\ldots i_\ell}$ which is equivalent to $Y_{\ell m}(\theta,\phi)$. That is,

$$C^{(\ell,m)}_{i_1\ldots i_\ell}\,\hat n_{i_1}\ldots\hat n_{i_\ell} = Y_{\ell m}(\theta,\phi)\ . \tag{8.7}$$

Our goal is to construct $C^{(\ell,m)}_{i_1\ldots i_\ell}$ explicitly. We have already shown that the number of linearly independent traceless symmetric tensors of rank ℓ (i.e., with ℓ indices) is given by $2\ell+1$, which is not surprisingly equal to the number of $Y_{\ell m}$ functions for a given ℓ. The quantity m is an integer from $-\ell$ to ℓ, so there are $2\ell+1$ possible values.

We consider first the case of azimuthal symmetry, where $F(\hat n)$ is invariant under rotations about the z-axis, and hence independent of ϕ. In that case, within the standard treatment, the most general function can be expanded in Legendre polynomials,

$$F(\hat n) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(\cos\theta)\ . \tag{8.8}$$

The P_ℓ functions are the same as the $Y_{\ell 0}$ functions, except that they are normalized differently:

$$Y_{\ell 0}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\,P_\ell(\cos\theta)\ . \tag{8.9}$$

The Legendre polynomials can be written explicitly using Rodrigues' formula:

$$P_\ell(x) = \frac{1}{2^\ell\,\ell!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell\left[(x^2-1)^\ell\right]\ . \tag{8.10}$$

In the traceless symmetric tensor formalism, the azimuthal symmetry case must be described by traceless symmetric tensors that are invariant under rotations about the z-axis. It is easiest to begin by thinking about $\ell=1$, where we are seeking a tensor $C^{(1)}_i$. Since $C^{(1)}_i$ has one index, it is a vector, which is the same as a tensor of rank 1. It is obvious that the only vector that is invariant under rotations about the z-axis is a vector that points along the z axis. I will let $\hat z$ be a unit vector in the z-direction (which I have also called $\hat e_z$ and $\hat e_3$), and then for azimuthal symmetry we have

$$C^{(1)}_i = \text{const}\,\hat z_i\ , \tag{8.11}$$

where $\hat z_i = \delta_{i3}$ is the i'th component of $\hat z$. The resulting function of $\hat n$ is then

$$F_1(\hat n) = C^{(1)}_i\hat n_i = \text{const}\,\hat z_i\hat n_i = \text{const}\,\hat z\cdot\hat n = \text{const}\,\cos\theta\ , \tag{8.12}$$

which certainly agrees with $P_1(\cos\theta) = \cos\theta$.

To generalize to arbitrary ℓ, we can construct a tensor of rank ℓ that is invariant under rotations about the z-axis by considering the product $\hat{z}_{i_1}\hat{z}_{i_2}\ldots\hat{z}_{i_\ell}$. This is clearly symmetric, but it is not traceless. However, we can make it traceless by taking its traceless part, which I denote by curly brackets.

$$\{\,\hat{z}_{i_1}\hat{z}_{i_2}\ldots\hat{z}_{i_\ell}\,\} \equiv \text{traceless symmetric part of } \hat{z}_{i_1}\hat{z}_{i_2}\ldots\hat{z}_{i_\ell}\ . \tag{8.13}$$

The traceless symmetric part is constructed by starting with the original expression and then subtracting terms proportional to one or more Kronecker δ-functions, where the subtractions are uniquely determined by the requirement that the expression be traceless. For example,

$$\begin{aligned}
\{\,1\,\} &= 1 \\
\{\,\hat{z}_i\,\} &= \hat{z}_i \\
\{\,\hat{z}_i\hat{z}_j\,\} &= \hat{z}_i\hat{z}_j - \tfrac{1}{3}\delta_{ij} \\
\{\,\hat{z}_i\hat{z}_j\hat{z}_k\,\} &= \hat{z}_i\hat{z}_j\hat{z}_k - \tfrac{1}{5}\big(\hat{z}_i\delta_{jk} + \hat{z}_j\delta_{ik} + \hat{z}_k\delta_{ij}\big) \\
\{\,\hat{z}_i\hat{z}_j\hat{z}_k\hat{z}_m\,\} &= \hat{z}_i\hat{z}_j\hat{z}_k\hat{z}_m - \tfrac{1}{7}\big(\hat{z}_i\hat{z}_j\delta_{km} + \hat{z}_i\hat{z}_k\delta_{mj} + \hat{z}_i\hat{z}_m\delta_{jk} + \hat{z}_j\hat{z}_k\delta_{im} \\
&\qquad + \hat{z}_j\hat{z}_m\delta_{ik} + \hat{z}_k\hat{z}_m\delta_{ij}\big) + \tfrac{1}{35}\big(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}\big)\ ,
\end{aligned} \tag{8.14}$$

where the coefficients are all determined by the requirement of tracelessness. We will argue later that Eq. (8.14) gives the only traceless symmetric tensors that are invariant under rotations about the z-axis, and therefore the function

$$F_\ell(\hat{n}) = \{\,\hat{z}_{i_1}\ldots\hat{z}_{i_\ell}\,\}\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell} \tag{8.15}$$

is the only function, up to a multiplicative constant, that is azimuthally symmetric and satisfies $\nabla^2_\theta F_\ell(\hat{n}) = -\ell(\ell+1)F_\ell(\hat{n})$. Since $P_\ell(\cos\theta)$ is azimuthally symmetric and satisfies $\nabla^2_\theta P_\ell(\cos\theta) = -\ell(\ell+1)P_\ell(\cos\theta)$, we must have

$$P_\ell(\cos\theta) = \text{const}\{\,\hat{z}_{i_1}\ldots\hat{z}_{i_\ell}\,\}\,\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\ , \tag{8.16}$$

where the constant is yet to be determined.

Both sides of Eq. (8.16) are polynomials in $\cos\theta$, where the highest power is $\cos^\ell\theta$. If we can find the coefficients of this highest power on each side of the equation, we can determine the constant. On the right-hand side, the highest power comes entirely from the $\hat{z}_{i_1}\ldots\hat{z}_{i_\ell}$ term in $\{\,\hat{z}_{i_1}\ldots\hat{z}_{i_\ell}\,\}$, since all the other terms contain Kronecker δ-functions which result in factors of the form $\hat{n}\cdot\hat{n} = 1$, reducing the number of $\hat{n}$ factors available to give powers of $\cos\theta$. So, the leading term on the right-hand side is simply

$\text{const}(\hat{z}\cdot\hat{n})^\ell = \text{const}\cos^\ell\theta$. For the left-hand side, we can use Rodrigues' formula to extract the highest power:

$$\begin{aligned}
P_\ell(x) &= \frac{1}{2^\ell \ell!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell \left[(x^2-1)^\ell\right] \\
&= \frac{1}{2^\ell \ell!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell \left[x^{2\ell} + (\text{lower powers})\right] \\
&= \frac{1}{2^\ell \ell!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\ell-1} \left[(2\ell)x^{2\ell-1} + (\text{lower powers})\right] \\
&= \frac{1}{2^\ell \ell!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\ell-2} \left[(2\ell)(2\ell-1)x^{2\ell-2} + (\text{lower powers})\right] \\
&= \frac{1}{2^\ell \ell!}\left[(2\ell)(2\ell-1)\dots(\ell+1)x^\ell + (\text{lower powers})\right] \\
&= \frac{(2\ell)!}{2^\ell(\ell!)^2}x^\ell + (\text{lower powers})\ .
\end{aligned} \tag{8.17}$$

Matching these coefficients, we see that

$$P_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell(\ell!)^2}\{\,\hat{z}_{i_1}\dots\hat{z}_{i_\ell}\,\}\,\hat{n}_{i_1}\dots\hat{n}_{i_\ell}\ . \tag{8.18}$$

Now we can return to the general case, in which there is no azimuthal symmetry, and the expansion requires the spherical harmonics, $Y_{\ell m}$. The $Y_{\ell m}$ are chosen to have a very simple dependence on ϕ, namely

$$Y_{\ell m}(\theta,\phi) \propto e^{im\phi}\ . \tag{8.19}$$

This property can be described in terms of how the functions transform under a rotation of the coordinate system about the z-axis. Under a rotation by an angle ψ about the z-axis, the angle ϕ changes by ψ, and $Y_{\ell m}$ changes by a factor $e^{i\psi}$. I have not been careful here about specifying the sign of this rotation, because it will be easy to fix the sign conventions at the end. The important point here is that if we want to match the conventions of the spherical harmonics, we need to construct traceless symmetric tensors that are modified by a rotation only by a multiplicative phase factor. That is, we are looking for tensors that are complex, and that are eigenvectors of the rotation operator.

Naturally we begin by considering a vector (a tensor with one index, or a rank 1 tensor), which under a rotation about the z-axis transforms as

$$\begin{aligned}
v'_x &= v_x\cos\psi - v_y\sin\psi \\
v'_y &= v_x\sin\psi + v_y\cos\psi\ .
\end{aligned} \tag{8.20}$$

We thus seek an eigenvector of the matrix

$$\boldsymbol{R} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \ . \tag{8.21}$$

The eigenvalues λ of the matrix are determined by the characteristic equation

$$\det(\boldsymbol{R} - \lambda \boldsymbol{I}) = 0 \ , \tag{8.22}$$

where $\boldsymbol{I}$ is the identity matrix, which can be expanded as

$$\det \begin{pmatrix} \cos\psi - \lambda & -\sin\psi \\ \sin\psi & \cos\psi - \lambda \end{pmatrix} = 0 \quad \Longrightarrow \quad \lambda^2 - 2\lambda\cos\psi + 1 = 0 \ , \tag{8.23}$$

for which the solutions are

$$\lambda = \cos\psi \pm \sqrt{\cos^2\psi - 1} = \cos\psi \pm i\sin\psi = e^{\pm i\psi} \ . \tag{8.24}$$

The eigenvectors then satisfy

$$\begin{pmatrix} \cos\psi - e^{\pm i\psi} & -\sin\psi \\ \sin\psi & \cos\psi - e^{\pm i\psi} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = 0 \ , \tag{8.25}$$

which simplifies to

$$\begin{pmatrix} \mp i\sin\psi & -\sin\psi \\ \sin\psi & \mp i\sin\psi \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = 0 \ , \tag{8.26}$$

from which we see that $v_y = \mp i v_x$. Constructing normalized eigenvectors, we can define

$$\begin{aligned} \hat{u}^{(1)} \equiv \hat{u}^{+} &= \frac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y) \\ \hat{u}^{(2)} \equiv \hat{u}^{-} &= \frac{1}{\sqrt{2}}(\hat{e}_x - i\hat{e}_y) \ , \end{aligned} \tag{8.27}$$

which are orthonormal in the sense that

$$\hat{u}^{(i)*} \cdot \hat{u}^{(j)} = \delta_{ij} \ . \tag{8.28}$$

We can complete a basis for three-dimensional vectors by adding

$$\hat{u}^{(3)} \equiv \hat{z} = \hat{e}_z \ . \tag{8.29}$$

You might ask how one should visualize a vector with imaginary components. What direction does it point? It certainly points in a definite direction in complex three-dimensional space, which is equivalent to a six-dimensional real-valued space, but for our purposes we do not need to have any geometric picture of these vectors. We are simply going to use them to form dot products to construct (complex-valued) functions of θ and ϕ.

Note that complex conjugation, as used in Eq. (8.28), is essential for defining a positive definite norm for complex vectors. The quantities $\hat{u}^{(+)} \cdot \hat{u}^{(+)}$ and $\hat{u}^{(-)} \cdot \hat{u}^{(-)}$, by contrast, are in fact equal to zero. This leads to the convenient fact that

$$\hat{u}^+_{i_1} \ldots \hat{u}^+_{i_\ell} \tag{8.30}$$

is both traceless and symmetric.

Since

$$\hat{u}^+ \cdot \hat{n} = \frac{1}{\sqrt{2}} \sin\theta e^{i\phi} \ , \tag{8.31}$$

we can construct functions proportional to $e^{im\phi}$, for $m > 0$, by including m factors of $\hat{u}^+$, and arranging for each of them to be dotted into $\hat{n}$. This can be done by considering the function defined by

$$F_{\ell m}(\theta,\phi) \equiv \{\, \hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell} \,\}\, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \ . \tag{8.32}$$

To see that in this expression every $\hat{u}^+$ is dotted into an $\hat{n}$, recall that $\hat{u}^+ \cdot \hat{z} = \hat{u}^+ \cdot \hat{u}^+ = 0$. So, when the right-hand side is expanded and all the indices are summed to give dot products, the only terms that survive are those for which every $\hat{u}^+$ is dotted into $\hat{z}$. Thus, the right-hand side of Eq. (8.32) is proportional to $e^{im\phi}$. From Eq. (8.4), we know that the right-hand side of Eq. (8.32) is an eigenfunction of ∇^2_θ with eigenvalue $-\ell(\ell+1)$. I will argue below that any such eigenvector that is proportional to $e^{im\phi}$ is necessarily proportional to $Y_{\ell m}$. We will return to the question of uniqueness, but let us first assume that uniqueness holds, so that

$$F_{\ell m}(\theta,\phi) \propto Y_{\ell m}(\theta,\phi) \ . \tag{8.33}$$

As in the previous derivation for Legendre polynomials, we can determine the constant of proportionality by matching the leading term in the expansions of both sides of the equation. $F_{\ell m}(\theta,\phi)$ can be written as $(\sin\theta)^m\, e^{im\phi}$ times a polynomial in $\cos\theta$, so we can use the highest power of $\cos\theta$ to determine the matching.

It is easy to extract the leading term from Eq. (8.32), because it comes from the first term in the expansion of

$$\{\, \hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \,\} = \hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell} + \text{terms} \propto \delta_{i_p i_q} \ . \tag{8.34}$$

The first term gives the highest power of $\cos\theta$, because the Kronecker δ-functions that appear in all later terms cause one or more $\hat{n}$'s to dot with other $\hat{n}$'s, reducing the number of $\hat{n}$'s available to appear in the form $\hat{n} \cdot \hat{z} = \cos\theta$. Thus,

$$F_{\ell m}(\theta,\phi) = 2^{-m/2} (\sin\theta)^m e^{im\phi} \left[(\cos\theta)^{\ell-m} + (\text{lower powers of } \cos\theta) \right] \ . \tag{8.35}$$

To compare Eq. (8.35) with the leading term in the expansion for the standard function $Y_{\ell m}$, we need a formula for $Y_{\ell m}(\theta, \phi)$. It is given in Jackson as Eq. (3.53), p. 108, as

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta) e^{im\phi} \ , \tag{8.36}$$

where $P_\ell^m(\cos\theta)$ is the associated Legendre function, which can be defined by Jackson's Eq. (3.50),

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell \ . \tag{8.37}$$

Using the same technique as in Eq. (8.17), we find

$$\begin{aligned} \frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell &= (2\ell)\ldots(\ell+1)\ell(\ell-1)\ldots(\ell-m+1)x^{\ell-m} + (\text{lower powers}) \\ &= \frac{(2\ell)!}{(\ell-m)!}\, x^{\ell-m} + (\text{lower powers}) \ . \end{aligned} \tag{8.38}$$

Matching the coefficients of these leading terms, we find that we can write (for $m \geq 0$)

$$Y_{\ell m}(\theta, \phi) = C^{(\ell,m)}_{i_1 \ldots i_\ell}\, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \ , \tag{8.39}$$

where

$$C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell} = d_{\ell m} \left\{ \hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell} \right\} \ , \tag{8.40}$$

with

$$d_{\ell m} = \frac{(-1)^m (2\ell)!}{2^\ell \ell!}\sqrt{\frac{2^m\,(2\ell+1)}{4\pi\,(\ell+m)!\,(\ell-m)!}} \ . \tag{8.41}$$

For negative values of m, the calculation is identical, except that we use $\hat{u}^-$ instead of $\hat{u}^+$. The result is

$$C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell} = d_{\ell m} \left\{ \hat{u}^-_{i_1} \ldots \hat{u}^-_{i_{|m|}} \hat{z}_{i_{|m|+1}} \ldots \hat{z}_{i_\ell} \right\} = C^{(\ell,|m|)*}_{i_1 i_2 \ldots i_\ell} \ , \tag{8.42}$$

where to allow for negative m we need to write $d_{\ell m}$ as

$$d_{\ell m} = \frac{(-1)^m (2\ell)!}{2^\ell \ell!}\sqrt{\frac{2^{|m|}\,(2\ell+1)}{4\pi\,(\ell+m)!\,(\ell-m)!}} \ . \tag{8.43}$$

It is worth mentioning that the curly brackets indicating "traceless symmetric part" can be put on either factor or both in expressions such as Eq. (8.32). That is,

$$\left\{ \hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell} \right\} \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} = \left\{ \hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell} \right\} \left\{ \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \right\} \tag{8.44a}$$

$$= \hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell} \left\{ \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \right\} \ , \tag{8.44b}$$

where the top line is justified because $\{\, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \,\}$ differs from $\hat{n}_{i_1} \ldots \hat{n}_{i_\ell}$ only by terms proportional to Kronecker δ-functions, which give no contribution when summed with the traceless symmetric tensor $\{\, \hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell} \,\}$. Similarly, once the second factor is written in traceless symmetric form, there is no longer a need to take the traceless symmetric part of the first term, since $\{\, \hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell} \,\}$ differs from $\hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell}$ only by terms proportional to Kronecker δ-functions, which vanish when summed with the traceless symmetric tensor $\{\, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \,\}$.

Finally, we can return to the question of uniqueness. In asserting that $F_{\ell m}(\theta, \phi) \propto Y_{\ell m}(\theta, \phi)$, we knew that both functions are proportional to $e^{im\phi}$, and that both are eigenfunctions of ∇^2_θ with eigenvalue $-\ell(\ell+1)$. We claimed that, up to a multiplicative constant, there is only one function that has these properties. Assuming that the power series representation of Eq. (8.1) always exists, the uniqueness that we need is easy to see. We showed in lecture that the number of linearly independent traceless symmetric tensors of rank ℓ is $2\ell+1$, and now we have constructed $2\ell+1$ such tensors: the $C^{(\ell,m)}_{i_1 \ldots i_\ell}$, for $m = -\ell, \ldots, \ell$. These are clearly linearly independent, since they are each eigenfunctions of rotations about the z-axis with different eigenvalues. Thus, any traceless symmetric tensor of rank ℓ must be a linear sum of the tensors in our basis. When we also specify that the tensor being sought is an eigenvector of rotations about the z-axis, with a specific eigenvalue, then only one of the tensors in our basis can contribute.

The above argument is solid, but one might still wonder what happens if we try, for example, to construct a different tensor by using both $\hat{u}^+$'s and $\hat{u}^-$'s in the same expression. For example, we might consider $\{\, \hat{u}^+_i \hat{u}^-_j \,\}$, which is invariant under rotations about the z-axis. With a little work, however, one can show that

$$\{\, \hat{u}^+_i \hat{u}^-_j \,\} = -\frac{1}{2} \{\, \hat{z}_i \hat{z}_j \,\} \; . \tag{8.45}$$

8.07 Electromagnetism II
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