MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — October 18, 2012
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FORMULA SHEET FOR QUIZ 1

Exam Date: October 18, 2012

*** Some sections below are marked with asterisks, as this section is. The asterisks indicate that you won't need this material for the quiz, and need not understand it. It is included, however, for completeness, and because some people might want to make use of it to solve problems by methods other than the intended ones.

Index Notation:

$$\vec{A}\cdot\vec{B} = A_i B_i\,, \qquad \vec{A}\times\vec{B}_i = \epsilon_{ijk}A_j B_k\,, \qquad \epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$\det A = \epsilon_{i_1 i_2 \cdots i_n} A_{1,i_1} A_{2,i_2} \cdots A_{n,i_n}$$

Rotation of a Vector:

$$A'_i = R_{ij}A_j\,, \qquad \text{Orthogonality: } R_{ij}R_{ik} = \delta_{jk} \qquad (R^T T = I)$$

Rotation about z-axis by ϕ:

$$R_z(\phi)_{ij} = \begin{array}{c} \\ i{=}1 \\ i{=}2 \\ i{=}3 \end{array}\begin{array}{c} \begin{array}{ccc} j{=}1 & j{=}2 & j{=}3 \end{array} \\ \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Rotation about axis $\hat{n}$ by ϕ:***

$$R(\hat{n},\phi)_{ij} = \delta_{ij}\cos\phi + \hat{n}_i\hat{n}_j(1-\cos\phi) - \epsilon_{ijk}\hat{n}_k \sin\phi\ .$$

Vector Calculus:

Gradient: $(\vec{\nabla}\varphi)_i = \partial_i\varphi\,, \qquad \partial_i \equiv \dfrac{\partial}{\partial x_i}$

Divergence: $\vec{\nabla}\cdot\vec{A} \equiv \partial_i A_i$

Curl: $(\vec{\nabla}\times\vec{A})_i = \epsilon_{ijk}\partial_j A_k$

Laplacian: $\nabla^2\varphi = \vec{\nabla}\cdot(\vec{\nabla}\varphi) = \dfrac{\partial^2\varphi}{\partial x_i \partial x_i}$

Fundamental Theorems of Vector Calculus:

Gradient: $\displaystyle\int_{\vec{a}}^{\vec{b}} \vec{\nabla}\varphi\cdot \mathrm{d}\vec{\ell} = \varphi(\vec{b}) - \varphi(\vec{a})$

Divergence: $\displaystyle\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{A}\,\mathrm{d}^3x = \oint_S \vec{A}\cdot\mathrm{d}\vec{a}$

where S is the boundary of $\mathcal{V}$

Curl: $\displaystyle\int_S (\vec{\nabla}\times\vec{A})\cdot\mathrm{d}\vec{a} = \oint_P \vec{A}\cdot\mathrm{d}\vec{\ell}$

where P is the boundary of S

Delta Functions:

$$\int \varphi(x)\delta(x-x')\,\mathrm{d}x = \varphi(x')\,, \qquad \int \varphi(\vec{r})\delta^3(\vec{r}-\vec{r}\,')\,\mathrm{d}^3x = \varphi(\vec{r}\,')$$

$$\int \varphi(x)\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x')\,\mathrm{d}x = -\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=x'}$$

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}\,, \quad g(x_i)=0$$

$$\nabla^2 \frac{1}{|\vec{r}-\vec{r}\,'|} = -4\pi\delta^3(\vec{r}-\vec{r}\,')$$

Electrostatics:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{(\vec{r}-\vec{r}\,')\,q_i}{|\vec{r}-\vec{r}\,'|^3} = \frac{1}{4\pi\epsilon_0}\int \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,\rho(\vec{r}\,')\,\mathrm{d}^3x'$$

$$V(\vec{r}) = V(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}\,')\cdot \mathrm{d}\vec{\ell}\,' = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\mathrm{d}^3x'$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}\,, \qquad \vec{\nabla}\times\vec{E} = 0\,, \qquad \vec{E} = -\vec{\nabla}V$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \ \ \text{(Poisson's Eq.)}\,, \qquad \rho = 0 \implies \nabla^2 V = 0 \ \ \text{(Laplace's Eq.)}$$

Laplacian Mean Value Theorem (no generally accepted name): If $\nabla^2 V = 0$, then the average value of V on a spherical surface equals its value at the center.

Energy:

$$W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{\substack{ij\\ i\neq j}} \frac{q_i q_j}{r_{ij}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x\,\mathrm{d}^3x'\,\frac{\rho(\vec{r})\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$W = \frac{1}{2}\int \mathrm{d}^3x\rho(\vec{r})V(\vec{r}) = \frac{1}{2}\epsilon_0\int \left|\vec{E}\right|^2\mathrm{d}^3x$$

Conductors:

Just outside, $\vec{E} = \dfrac{\sigma}{\epsilon_0}\hat{n}$

Pressure on surface: $\frac{1}{2}\sigma|\vec{E}|_{\text{outside}}$

Two-conductor system with charges Q and $-Q$: $Q = CV$, $W = \frac{1}{2}CV^2$

N isolated conductors:

$$V_i = \sum_j P_{ij}Q_j\,, \qquad P_{ij} = \text{elastance matrix, or reciprocal capacitance matrix}$$

$$Q_i = \sum_j C_{ij}V_j\,, \qquad C_{ij} = \text{capacitance matrix}$$

Image charge in sphere of radius a: Image of Q at R is $q = -\frac{a}{R}Q$, $r = \frac{a^2}{R}$

Separation of Variables for Laplace's Equation in Cartesian Coordinates:

$$V = \left\{ \begin{matrix} \cos \alpha x \\ \sin \alpha x \end{matrix} \right\} \left\{ \begin{matrix} \cos \beta y \\ \sin \beta y \end{matrix} \right\} \left\{ \begin{matrix} \cosh \gamma z \\ \sinh \gamma z \end{matrix} \right\} \qquad \text{where } \gamma^2 = \alpha^2 + \beta^2$$

Separation of Variables for Laplace's Equation in Spherical Coordinates:

Traceless Symmetric Tensor expansion:

$$\nabla^2 \varphi(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \nabla_\theta^2 \varphi = 0 \ ,$$

where the angular part is given by

$$\nabla_\theta^2 \varphi \equiv \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \varphi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \varphi}{\partial \phi^2}$$

$$\nabla_\theta^2 \, C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} = -\ell(\ell+1) C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \ ,$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ is a symmetric traceless tensor

General solution to Laplace's equation:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) C^{(\ell)}_{i_1 i_2 \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \ , \quad \text{where } \vec{r} = r\hat{n}$$

Azimuthal Symmetry:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) c_\ell \{ \, \hat{z}_{i_1} \ldots \hat{z}_{i_\ell} \, \} \, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell}$$

Special cases:

$$\{ \, 1 \, \} = 1$$

$$\{ \, \hat{z}_i \, \} = \hat{z}_i$$

$$\{ \, \hat{z}_i \hat{z}_j \, \} = \hat{z}_i \hat{z}_j - \tfrac{1}{3} \delta_{ij}$$

$$\{ \, \hat{z}_i \hat{z}_j \hat{z}_k \, \} = \hat{z}_i \hat{z}_j \hat{z}_k - \tfrac{1}{5} \big(\hat{z}_i \delta_{jk} + \hat{z}_j \delta_{ik} + \hat{z}_k \delta_{ij} \big)$$

$$\{ \, \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m \, \} = \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m - \tfrac{1}{7} \big(\hat{z}_i \hat{z}_j \delta_{km} + \hat{z}_i \hat{z}_k \delta_{mj} + \hat{z}_i \hat{z}_m \delta_{jk} + \hat{z}_j \hat{z}_k \delta_{im} + \hat{z}_j \hat{z}_m \delta_{ik} + \hat{z}_k \hat{z}_m \delta_{ij} \big) + \tfrac{1}{35} \big(\delta_{ij} \delta_{km} + \delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk} \big)$$

Legendre Polynomial / Spherical Harmonic expansion:

General solution to Laplace's equation:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \phi)$$

Orthonormality: $\displaystyle\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \, Y^*_{\ell' m'}(\theta, \phi) \, Y_{\ell m}(\theta, \phi) = \delta_{\ell' \ell} \delta_{m' m}$

Azimuthal Symmetry:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Multipole Expansion:

First several terms:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \Big[\, \frac{Q}{r} + \frac{\vec{p} \cdot \hat{r}}{r^2} + \frac{1}{2} \frac{\hat{r}_i \hat{r}_j}{r^3} \, Q_{ij} + \cdots \Big] \quad \text{, where}$$

$$Q = \int \mathrm{d}^3x \, \rho(\vec{r}) \, , \quad p_i = \int d^3x \, \rho(\vec{r}) \, x_i \quad Q_{ij} = \int d^3x \, \rho(\vec{r}) (3 x_i x_j - \delta_{ij} |\vec{r}|^2) \, ,$$

$$\vec{E}_{dip} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \big[3(\vec{p} \cdot \hat{r}) \hat{r} - \vec{p} \big] - \frac{1}{3\epsilon_0} p_i \delta^3(\vec{r})$$

Traceless Symmetric Tensor version:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \, C^{(\ell)}_{i_1 \ldots i_\ell} \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \; ,$$

where

$$C^{(\ell)}_{i_1 \ldots i_\ell} = \frac{(2\ell - 1)!!}{\ell!} \int \rho(\vec{r}\,') \, \{ \, \vec{r}\,'_{i_1} \ldots \vec{r}\,'_{i_\ell} \, \} \, \mathrm{d}^3 x'$$

Griffiths version:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int r'^{\ell} \rho(\vec{r}\,') P_\ell(\cos\theta') \, \mathrm{d}^3 x$$

where θ' = angle between $\vec{r}$ and $\vec{r}\,'$.

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} \, P_\ell(\cos\theta') \, , \qquad \frac{1}{\sqrt{1 - 2\lambda x + \lambda^2}} = \sum_{\ell=0}^{\infty} \lambda^\ell P_\ell(x)$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell (x^2-1)^\ell\,, \qquad \text{(Rodrigues' formula)}$$

$$P_\ell(1) = 1 \qquad P_\ell(-x) = (-1)^\ell P_\ell(x) \qquad \int_{-1}^{1} \mathrm{d}x\, P_{\ell'}(x) P_\ell(x) = \frac{2}{2\ell+1}\delta_{\ell'\ell}$$

Spherical Harmonic version:***

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1} q_{\ell m}\, Y_{\ell m}(\theta,\phi)$$

$$\text{where } q_{\ell m} = \int Y^*_{\ell m} r'^\ell \rho(\vec{r}\,')\, \mathrm{d}^3x'$$

Connection Between Traceless Symmetric Tensors and Legendre Polynomials or Spherical Harmonics:

$$P_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell(\ell!)^2}\{\, \hat{z}_{i_1}\ldots\hat{z}_{i_\ell}\,\}\, \hat{n}_{i_1}\ldots\hat{n}_{i_\ell}$$

For $m \geq 0$,

$$Y_{\ell m}(\theta,\phi) = C^{(\ell,m)}_{i_1\ldots i_\ell}\, \hat{n}_{i_1}\ldots\hat{n}_{i_\ell}\ ,$$

$$\text{where } C^{(\ell,m)}_{i_1 i_2\ldots i_\ell} = d_{\ell m}\,\{\, \hat{u}^+_{i_1}\ldots\hat{u}^+_{i_m}\hat{z}_{i_{m+1}}\ldots\hat{z}_{i_\ell}\,\}\ ,$$

$$\text{with } d_{\ell m} = \frac{(-1)^m(2\ell)!}{2^\ell \ell!}\sqrt{\frac{2^m\,(2\ell+1)}{4\pi\,(\ell+m)!\,(\ell-m)!}}\ ,$$

$$\text{and } \hat{u}^+ = \frac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y)$$

Form $m<0$, $Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi)$

More Information about Spherical Harmonics:***

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P^m_\ell(\cos\theta) e^{im\phi}$$

where $P^m_\ell(\cos\theta)$ is the associated Legendre function, which can be defined by

$$P^m_\ell(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell$$

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