MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — November 13, 2012
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FORMULA SHEET FOR QUIZ 2, V. 2

Exam Date: November 15, 2012

*** Some sections below are marked with asterisks, as this section is. The asterisks indicate that you won't need this material for the quiz, and need not understand it. It is included, however, for completeness, and because some people might want to make use of it to solve problems by methods other than the intended ones.

Index Notation:

$$\vec{A}\cdot\vec{B} = A_i B_i\,, \qquad \vec{A}\times\vec{B}_i = \epsilon_{ijk}A_jB_k\,, \qquad \epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$\det A = \epsilon_{i_1 i_2 \cdots i_n} A_{1,i_1} A_{2,i_2} \cdots A_{n,i_n}$$

Rotation of a Vector:

$$A'_i = R_{ij}A_j\,, \qquad \text{Orthogonality: } R_{ij}R_{ik} = \delta_{jk} \qquad (R^T T = I)$$

Rotation about z-axis by ϕ:

$$R_z(\phi)_{ij} \;=\; \begin{array}{c c} & \begin{array}{ccc} j{=}1 & j{=}2 & j{=}3 \end{array} \\ \begin{array}{c} i{=}1 \\ i{=}2 \\ i{=}3 \end{array} & \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Rotation about axis $\hat{n}$ by ϕ:***

$$R(\hat{n},\phi)_{ij} = \delta_{ij}\cos\phi + \hat{n}_i\hat{n}_j(1-\cos\phi) - \epsilon_{ijk}\hat{n}_k\,\sin\phi\,.$$

Vector Calculus:

Gradient: $(\vec{\nabla}\varphi)_i = \partial_i\varphi\,, \qquad \partial_i \equiv \dfrac{\partial}{\partial x_i}$

Divergence: $\vec{\nabla}\cdot\vec{A} \equiv \partial_i A_i$

Curl: $(\vec{\nabla}\times\vec{A})_i = \epsilon_{ijk}\partial_j A_k$

Laplacian: $\nabla^2\varphi = \vec{\nabla}\cdot(\vec{\nabla}\varphi) = \dfrac{\partial^2\varphi}{\partial x_i \partial x_i}$

Fundamental Theorems of Vector Calculus:

Gradient: $\displaystyle\int_{\vec{a}}^{\vec{b}} \vec{\nabla}\varphi\cdot \mathrm{d}\vec{\ell} = \varphi(\vec{b}) - \varphi(\vec{a})$

Divergence: $\displaystyle\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{A}\,\mathrm{d}^3x = \oint_S \vec{A}\cdot\mathrm{d}\vec{a}$

where S is the boundary of $\mathcal{V}$

Curl: $\displaystyle\int_S (\vec{\nabla}\times\vec{A})\cdot\mathrm{d}\vec{a} = \oint_P \vec{A}\cdot\mathrm{d}\vec{\ell}$

where P is the boundary of S

Delta Functions:

$$\int \varphi(x)\delta(x-x')\,\mathrm{d}x = \varphi(x')\,, \qquad \int \varphi(\vec{r})\delta^3(\vec{r}-\vec{r}\,')\,\mathrm{d}^3x = \varphi(\vec{r}\,')$$

$$\int \varphi(x)\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x')\,\mathrm{d}x = -\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=x'}$$

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}\,, \quad g(x_i)=0$$

$$\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right) = -\nabla^2\frac{1}{|\vec{r}-\vec{r}\,'|} = 4\pi\delta^3(\vec{r}-\vec{r}\,')$$

$$\partial_i\left(\frac{\hat{r}_j}{r^2}\right) \equiv \partial_i\left(\frac{x_j}{r^3}\right) = -\partial_i\partial_j\left(\frac{1}{r}\right) = \frac{\delta_{ij}-3\hat{r}_i\hat{r}_j}{r^3} + \frac{4\pi}{3}\,\delta_{ij}\,\delta^3(\vec{r})$$

$$\vec{\nabla}\cdot\frac{3(\vec{d}\cdot\hat{r})\hat{r}-\vec{d}}{r^3} = -\frac{8\pi}{3}(\vec{d}\cdot\vec{\nabla})\delta^3(\vec{r})$$

$$\vec{\nabla}\times\frac{3(\vec{d}\cdot\hat{r})\hat{r}-\vec{d}}{r^3} = -\frac{4\pi}{3}\vec{d}\times\vec{\nabla}\delta^3(\vec{r})$$

Electrostatics:

$\vec{F} = q\vec{E}$, where

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{(\vec{r}-\vec{r}\,')\,q_i}{|\vec{r}-\vec{r}\,'|^3} = \frac{1}{4\pi\epsilon_0}\int \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,\rho(\vec{r}\,')\,\mathrm{d}^3x'$$

ϵ_0 =permittivity of free space $= 8.854\times10^{-12}\ \mathrm{C^2/(N{\cdot}m^2)}$

$$\frac{1}{4\pi\epsilon_0} = 8.988\times10^{9}\ \mathrm{N{\cdot}m^2/C^2}$$

$$V(\vec{r}) = V(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}\,')\cdot\mathrm{d}\vec{\ell}\,' = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\mathrm{d}^3x'$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}\,, \qquad \vec{\nabla}\times\vec{E} = 0\,, \qquad \vec{E} = -\vec{\nabla}V$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}\ \text{(Poisson's Eq.)}\,, \qquad \rho = 0 \implies \nabla^2 V = 0\ \text{(Laplace's Eq.)}$$

Laplacian Mean Value Theorem (no generally accepted name): If $\nabla^2 V = 0$, then the average value of V on a spherical surface equals its value at the center.

Energy:

$$W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{\substack{ij\\ i\neq j}} \frac{q_i q_j}{r_{ij}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x\,\mathrm{d}^3x'\,\frac{\rho(\vec{r})\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$W = \frac{1}{2}\int \mathrm{d}^3x\,\rho(\vec{r})V(\vec{r}) = \frac{1}{2}\epsilon_0\int \left|\vec{E}\right|^2 \mathrm{d}^3x$$

Conductors:

Just outside, $\vec{E} = \dfrac{\sigma}{\epsilon_0}\hat{n}$

Pressure on surface: $\frac{1}{2}\sigma|\vec{E}|_{\text{outside}}$

Two-conductor system with charges Q and $-Q$: $Q = CV$, $W = \frac{1}{2}CV^2$

N isolated conductors:

$$V_i = \sum_j P_{ij}Q_j \ , \qquad P_{ij} = \text{elastance matrix, or reciprocal capacitance matrix}$$

$$Q_i = \sum_j C_{ij}V_j \ , \qquad C_{ij} = \text{capacitance matrix}$$

Image charge in sphere of radius a: Image of Q at R is $q = -\dfrac{a}{R}Q$, $r = \dfrac{a^2}{R}$

Separation of Variables for Laplace's Equation in Cartesian Coordinates:

$$V = \begin{Bmatrix} \cos\alpha x \\ \sin\alpha x \end{Bmatrix} \begin{Bmatrix} \cos\beta y \\ \sin\beta y \end{Bmatrix} \begin{Bmatrix} \cosh\gamma z \\ \sinh\gamma z \end{Bmatrix} \qquad \text{where } \gamma^2 = \alpha^2 + \beta^2$$

Separation of Variables for Laplace's Equation in Spherical Coordinates:

Traceless Symmetric Tensor expansion:

$$\nabla^2\varphi(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\nabla^2_\theta\,\varphi = 0 \ ,$$

where the angular part is given by

$$\nabla^2_\theta\,\varphi \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}$$

$$\nabla^2_\theta\, C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell} = -\ell(\ell+1)C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell} \ ,$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ is a symmetric traceless tensor and

$\hat{n} = \sin\theta\cos\phi\,\hat{e}_1 + \sin\theta\sin\phi\,\hat{e}_2 + \cos\theta\,\hat{e}_3$.

General solution to Laplace's equation:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty}\left(C^{(\ell)}_{i_1 i_2 \ldots i_\ell}r^\ell + \frac{C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}}{r^{\ell+1}}\right)\hat{r}_{i_1}\hat{r}_{i_2}\ldots\hat{r}_{i_\ell} \ , \quad \text{where } \vec{r} = r\hat{r}$$

Azimuthal Symmetry:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) \{\, \hat{z}_{i_1} \ldots \hat{z}_{i_\ell} \,\} \, \hat{r}_{i_1} \ldots \hat{r}_{i_\ell}$$

where $\{\ldots\}$ denotes the traceless symmetric part of $\ldots$.

Special cases:

$$\{\, 1 \,\} = 1$$

$$\{\, \hat{z}_i \,\} = \hat{z}_i$$

$$\{\, \hat{z}_i \hat{z}_j \,\} = \hat{z}_i \hat{z}_j - \tfrac{1}{3}\delta_{ij}$$

$$\{\, \hat{z}_i \hat{z}_j \hat{z}_k \,\} = \hat{z}_i \hat{z}_j \hat{z}_k - \tfrac{1}{5}\big(\hat{z}_i \delta_{jk} + \hat{z}_j \delta_{ik} + \hat{z}_k \delta_{ij}\big)$$

$$\begin{aligned}\{\, \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m \,\} = {} & \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m - \tfrac{1}{7}\big(\hat{z}_i \hat{z}_j \delta_{km} + \hat{z}_i \hat{z}_k \delta_{mj} + \hat{z}_i \hat{z}_m \delta_{jk} + \hat{z}_j \hat{z}_k \delta_{im} \\ & + \hat{z}_j \hat{z}_m \delta_{ik} + \hat{z}_k \hat{z}_m \delta_{ij}\big) + \tfrac{1}{35}\big(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}\big)\end{aligned}$$

Legendre Polynomial / Spherical Harmonic expansion:

General solution to Laplace's equation:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} \, r^\ell + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \phi)$$

Orthonormality: $\displaystyle \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \, Y^*_{\ell' m'}(\theta,\phi) \, Y_{\ell m}(\theta,\phi) = \delta_{\ell' \ell} \delta_{m' m}$

Azimuthal Symmetry:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Electric Multipole Expansion:

First several terms:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\, \frac{Q}{r} + \frac{\vec{p} \cdot \hat{r}}{r^2} + \frac{1}{2} \frac{\hat{r}_i \hat{r}_j}{r^3} \, Q_{ij} + \cdots \right] \quad \text{, where}$$

$$Q = \int \mathrm{d}^3x \, \rho(\vec{r}) \,, \quad p_i = \int d^3x \, \rho(\vec{r}) \, x_i \quad Q_{ij} = \int d^3x \, \rho(\vec{r})(3x_i x_j - \delta_{ij} |\vec{r}|^2) \,,$$

$$\vec{E}_{\rm dip}(\vec{r}) = -\frac{1}{4\pi\epsilon_0} \vec{\nabla} \left(\frac{\vec{p} \cdot \hat{r}}{r^2} \right) = \frac{1}{4\pi\epsilon_0} \frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3} - \frac{1}{3\epsilon_0} p_i \delta^3(\vec{r})$$

$$\vec{\nabla} \times \vec{E}_{\rm dip}(\vec{r}) = 0 \,, \qquad \vec{\nabla} \cdot \vec{E}_{\rm dip}(\vec{r}) = \frac{1}{\epsilon_0} \rho_{\rm dip}(\vec{r}) = -\frac{1}{\epsilon_0} \vec{p} \cdot \vec{\nabla} \delta^3(\vec{r})$$

Traceless Symmetric Tensor version:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C^{(\ell)}_{i_1 \ldots i_\ell} \hat{r}_{i_1} \ldots \hat{r}_{i_\ell} \ ,$$

where

$$C^{(\ell)}_{i_1 \ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{r}) \{ x_{i_1} \ldots x_{i_\ell} \} \, \mathrm{d}^3x \qquad (\vec{r} \equiv r\hat{r} \equiv x_i \hat{e}_i)$$

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{r'^{\ell}}{r^{\ell+1}} \{ \hat{r}_{i_1} \ldots \hat{r}_{i_\ell} \} \hat{r}'_{i_1} \ldots \hat{r}'_{i_\ell} \ , \qquad \text{for } r' < r$$

$$(2\ell-1)!! \equiv (2\ell-1)(2\ell-3)(2\ell-5)\ldots 1 = \frac{(2\ell)!}{2^\ell \ell!} \ , \text{ with } \ (-1)!! \equiv 1 \ .$$

Reminder: $\{\ldots\}$ denotes the traceless symmetric part of $\ldots$.

Griffiths version:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int r'^{\ell} \rho(\vec{r}\,') P_\ell(\cos\theta') \, \mathrm{d}^3x$$

where θ' = angle between $\vec{r}$ and $\vec{r}\,'$.

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta') \ , \qquad \frac{1}{\sqrt{1 - 2\lambda x + \lambda^2}} = \sum_{\ell=0}^{\infty} \lambda^\ell P_\ell(x)$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^{\ell} (x^2 - 1)^\ell \ , \qquad \text{(Rodrigues' formula)}$$

$$P_\ell(1) = 1 \qquad P_\ell(-x) = (-1)^\ell P_\ell(x) \qquad \int_{-1}^{1} \mathrm{d}x \, P_{\ell'}(x) P_\ell(x) = \frac{2}{2\ell+1} \delta_{\ell'\ell}$$

Spherical Harmonic version:***

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{q_{\ell m}}{r^{\ell+1}} Y_{\ell m}(\theta, \phi)$$

where $q_{\ell m} = \int Y^*_{\ell m} r'^{\ell} \rho(\vec{r}\,') \, \mathrm{d}^3x'$

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r'^{\ell}}{r^{\ell+1}} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi) \ , \qquad \text{for } r' < r$$

Electric Fields in Matter:

Electric Dipoles:

$\vec{p} = \int d^3x\, \rho(\vec{r})\, \vec{r}$

$\rho_{\text{dip}}(\vec{r}) = -\vec{p} \cdot \vec{\nabla}_{\vec{r}}\, \delta^3(\vec{r} - \vec{r}_d)$, where $\vec{r}_d$ = position of dipole

$\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E} = \vec{\nabla}(\vec{p} \cdot \vec{E})$ (force on a dipole)

$\vec{\tau} = \vec{p} \times \vec{E}$ (torque on a dipole)

$U = -\vec{p} \cdot \vec{E}$

Electrically Polarizable Materials:

$\vec{P}(\vec{r})$ = polarization = electric dipole moment per unit volume

$\rho_{\text{bound}} = -\nabla \cdot \vec{P}$, $\sigma_{\text{bound}} = \vec{P} \cdot \hat{n}$

$\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}$, $\vec{\nabla} \cdot \vec{D} = \rho_{\text{free}}$, $\vec{\nabla} \times \vec{E} = 0$ (for statics)

Boundary conditions:

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_{\text{free}}$$

$$\vec{E}^{\parallel}_{\text{above}} - \vec{E}^{\parallel}_{\text{below}} = 0 \qquad \vec{D}^{\parallel}_{\text{above}} - \vec{D}^{\parallel}_{\text{below}} = \vec{P}^{\parallel}_{\text{above}} - \vec{P}^{\parallel}_{\text{below}}$$

Linear Dielectrics:

$\vec{P} = \epsilon_0 \chi_e \vec{E}$, χ_e = electric susceptibility

$\epsilon \equiv \epsilon_0(1 + \chi_e)$ = permittivity, $\vec{D} = \epsilon \vec{E}$

$\epsilon_r = \dfrac{\epsilon}{\epsilon_0} = 1 + \chi_e$ = relative permittivity, or dielectric constant

Clausius-Mossotti equation: $\chi_e = \dfrac{N\alpha/\epsilon_0}{1 - \frac{N\alpha}{3\epsilon_0}}$, where N = number density of atoms or (nonpolar) molecules, α = atomic/molecular polarizability ($\vec{P} = \alpha\vec{E}$)

Energy: $W = \dfrac{1}{2}\displaystyle\int \vec{D} \cdot \vec{E}\, \mathrm{d}^3x$ (linear materials only)

Force on a dielectric: $\vec{F} = -\vec{\nabla}W$ (Even if one or more potential differences are held fixed, the force can be found by computing the gradient with the total charge on each conductor fixed.)

Magnetostatics:

Magnetic Force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} , \qquad \text{where } \vec{p} = \gamma m_0 \vec{v} , \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\vec{F} = \int I \mathrm{d}\vec{\ell} \times \vec{B} = \int \vec{J} \times \vec{B} \, \mathrm{d}^3 x$$

Current Density:

Current through a surface S: $I_S = \int_S \vec{J} \cdot \mathrm{d}\vec{a}$

Charge conservation: $\dfrac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$

Moving density of charge: $\vec{J} = \rho \vec{v}$

Biot-Savart Law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\vec{\ell}' \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \, \mathrm{d}a'$$

$$= \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \, \mathrm{d}^3 x$$

where μ_0 = permeability of free space $\equiv 4\pi \times 10^{-7}$ N/A^2

Examples:

Infinitely long straight wire: $\vec{B} = \dfrac{\mu_0 I}{2\pi r} \, \hat{\phi}$

Infintely long tightly wound solenoid: $\vec{B} = \mu_0 n I_0 \, \hat{z}$, where n = turns per unit length

Loop of current on axis: $\vec{B}(0,0,z) = \dfrac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}} \, \hat{z}$

Infinite current sheet: $\vec{B}(\vec{r}) = \dfrac{1}{2} \mu_0 \vec{K} \times \hat{n}$, $\hat{n}$ = unit normal toward $\vec{r}$

Vector Potential:

$$\vec{A}(\vec{r})_{\text{coul}} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \, \mathrm{d}^3 x' \ , \qquad \vec{B} = \vec{\nabla} \times \vec{A} \ , \qquad \vec{\nabla} \cdot \vec{A}_{\text{coul}} = 0$$

$\vec{\nabla} \cdot \vec{B} = 0$ (Subject to modification if magnetic monopoles are discovered)

Gauge Transformations: $\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \vec{\nabla}\Lambda(\vec{r})$ for any $\Lambda(\vec{r})$. $\vec{B} = \vec{\nabla} \times \vec{A}$ is unchanged.

Ampère's Law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \ , \ \text{ or equivalently } \int_P \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{enc}}$$

Magnetic Multipole Expansion:

Traceless Symmetric Tensor version:

$$A_j(\vec{r}) = \frac{\mu_0}{4\pi} \sum_{\ell=0}^{\infty} \mathcal{M}^{(\ell)}_{j;i_1 i_2 \ldots i_\ell} \frac{\{\,\hat{r}_{i_1} \ldots \hat{r}_{i_\ell}\,\}}{r^{\ell+1}}$$

where $\mathcal{M}^{(\ell)}_{j;i_1 i_2 \ldots i_\ell} = \dfrac{(2\ell-1)!!}{\ell!} \displaystyle\int \mathrm{d}^3x J_j(\vec{r})\{\,x_{i_1} \ldots x_{i_\ell}\,\}$

Current conservation restriction: $\displaystyle\int \mathrm{d}^3x \operatorname*{Sym}_{i_1 \ldots i_\ell}(x_{i_1} \ldots x_{i_{\ell-1}} J_{i_\ell}) = 0$

where $\operatorname*{Sym}_{i_1 \ldots i_\ell}$ means to symmetrize — i.e. average over all orderings — in the indices $i_1 \ldots i_\ell$

Special cases:

$\ell = 1$: $\displaystyle\int \mathrm{d}^3x\, J_i = 0$

$\ell = 2$: $\displaystyle\int \mathrm{d}^3x\, (J_i x_j + J_j x_i) = 0$

Leading term (dipole): $\vec{A}(\vec{r}) = \dfrac{\mu_0}{4\pi} \dfrac{\vec{m} \times \hat{r}}{r^2}$,

where

$$m_i = -\frac{1}{2}\epsilon_{ijk}\mathcal{M}^{(1)}_{j;k}$$

$$\vec{m} = \frac{1}{2} I \int_P \vec{r} \times \mathrm{d}\vec{\ell} = \frac{1}{2}\int \mathrm{d}^3x\, \vec{r} \times \vec{J} = I\vec{a} \; ,$$

where $\vec{a} = \displaystyle\int_S \mathrm{d}\vec{a}$ for any surface S spanning P

$$\vec{B}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi}\vec{\nabla} \times \frac{\vec{m} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3} + \frac{2\mu_0}{3}\,\vec{m}\,\delta^3(\vec{r})$$

$$\vec{\nabla} \cdot \vec{B}_{\text{dip}}(\vec{r}) = 0 \; , \qquad \vec{\nabla} \times \vec{B}_{\text{dip}}(\vec{r}) = \mu_0 \vec{J}_{\text{dip}}(\vec{r}) = -\mu_0 \vec{m} \times \vec{\nabla}\delta^3(\vec{r})$$

Griffiths version:

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \oint (r')^\ell P_\ell(\cos\theta')\mathrm{d}\vec{\ell}'$$

Magnetic Fields in Matter:

Magnetic Dipoles:

$$\vec{m} = \frac{1}{2} I \int_P \vec{r} \times \mathrm{d}\vec{\ell} = \frac{1}{2}\int \mathrm{d}^3x\, \vec{r} \times \vec{J} = I\vec{a}$$

$\vec{J}_{\text{dip}}(\vec{r}) = -\vec{m} \times \vec{\nabla}_{\vec{r}} \delta^3(\vec{r} - \vec{r}_d)$, where $\vec{r}_d$ = position of dipole

$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$ (force on a dipole)

$\vec{\tau} = \vec{m} \times \vec{B}$ (torque on a dipole)

$U = -\vec{m} \cdot \vec{B}$

Magnetically Polarizable Materials:

$\vec{M}(\vec{r})$ = magnetization = magnetic dipole moment per unit volume

$\vec{J}_{\text{bound}} = \vec{\nabla} \times \vec{M}$, $\vec{K}_{\text{bound}} = \vec{M} \times \hat{n}$

$\vec{H} \equiv \dfrac{1}{\mu_0}\vec{B} - \vec{M}$, $\vec{\nabla} \times \vec{H} = \vec{J}_{\text{free}}$, $\vec{\nabla} \cdot \vec{B} = 0$

Boundary conditions:

$B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} = 0$ $H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}})$

$\vec{B}^{\parallel}_{\text{above}} - \vec{B}^{\parallel}_{\text{below}} = \mu_0(\vec{K} \times \hat{n})$ $\vec{H}^{\parallel}_{\text{above}} - \vec{H}^{\parallel}_{\text{below}} = \vec{K}_{\text{free}} \times \hat{n}$

Linear Magnetic Materials:

$\vec{M} = \chi_m \vec{H}$, χ_m = magnetic susceptibility

$\mu = \mu_0(1 + \chi_m)$ = permeability, $\vec{B} = \mu \vec{H}$

Magnetic Monopoles:

$\vec{B}(\vec{r}) = \dfrac{\mu_0}{4\pi}\dfrac{q_m}{r^2}\hat{r}$; Force on a static monopole: $\vec{F} = q_m \vec{B}$

Angular momentum of monopole/charge system: $\vec{L} = \dfrac{\mu_0 q_e q_m}{4\pi}\hat{r}$, where $\hat{r}$ points from q_e to q_m

Dirac quantization condition: $\dfrac{\mu_0 q_e q_m}{4\pi} = \dfrac{1}{2}\hbar \times$ integer

Connection Between Traceless Symmetric Tensors and Legendre Polynomials or Spherical Harmonics:

$$P_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell(\ell!)^2}\{\, \hat{z}_{i_1} \ldots \hat{z}_{i_\ell} \,\}\, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell}$$

For $m \geq 0$,

$$Y_{\ell m}(\theta, \phi) = C^{(\ell,m)}_{i_1 \ldots i_\ell}\, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \ ,$$

where $C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell} = d_{\ell m}\{\, \hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell} \,\}$,

with $d_{\ell m} = \dfrac{(-1)^m (2\ell)!}{2^\ell \ell!}\sqrt{\dfrac{2^m\,(2\ell+1)}{4\pi\,(\ell+m)!\,(\ell-m)!}}$,

and $\hat{u}^+ = \dfrac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y)$

Form $m < 0$, $Y_{\ell,-m}(\theta, \phi) = (-1)^m Y^*_{\ell m}(\theta, \phi)$

More Information about Spherical Harmonics:***

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)e^{im\phi}$$

where $P_\ell^m(\cos\theta)$ is the associated Legendre function, which can be defined by

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell$$

Legendre Polynomials:

$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= \tfrac{1}{2}(3x^2 - 1)\\
P_3(x) &= \tfrac{1}{2}(5x^3 - 3x)\\
P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3)
\end{aligned}$$

SPHERICAL HARMONICS $Y_{lm}(\theta,\varphi)$

l = 0

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

l = 1

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

l = 2

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\varphi}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\varphi}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}}\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)$$

l = 3

$$Y_{33} = -\frac{1}{4}\sqrt{\frac{35}{4\pi}}\sin^3\theta e^{3i\varphi}$$

$$Y_{32} = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta e^{2i\varphi}$$

$$Y_{31} = -\frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta(5\cos^2\theta - 1)e^{i\varphi}$$

$$Y_{30} = \sqrt{\frac{7}{4\pi}}\left(\tfrac{5}{2}\cos^3\theta - \tfrac{3}{2}\cos\theta\right)$$

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8.07 Electromagnetism II
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