MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — September 5, 2012

Prof. Alan Guth

PROBLEM SET 1

DUE DATE: Friday, September 14, 2012. Either hand it in at the lecture, or by 5:00 pm in the 8.07 homework box.

READING ASSIGNMENT: Chapter 1 of Griffiths: *Vector Analysis.*

PROBLEM 1: VECTOR IDENTITIES INVOLVING CROSS PRODUCTS

(20 points)

In manipulating cross products, it is useful to define ε_{ijk} (the Levi-Civita antisymmetric symbol) to be:

$$\varepsilon_{ijk} = \begin{cases} +1 \text{ if } ijk = (123, 231, 312) \\ -1 \text{ if } ijk = (213, 321, 132) \\ \quad 0 \text{ otherwise} . \end{cases} \tag{1.1.1}$$

That is, ε_{ijk} is nonzero only when all three indices are different; it is then equal to +1 if ijk is a cyclic permutation of 123, and -1 if ijk is an anti-cyclic permutation. Note that ε_{ijk} is totally antisymmetric, in the sense that it changes sign if any two indices are interchanged:

$$\varepsilon_{ijk} = -\varepsilon_{ikj} = \varepsilon_{kij} \ . \tag{1.1.2}$$

With this definition, the i^{th} component of the cross product of two vectors $\vec{A}$ and $\vec{B}$ can be written as

$$\left(\vec{A} \times \vec{B}\right)_i = \varepsilon_{ijk} A_j B_k \ , \tag{1.1.3}$$

where we have used the summation convention that repeated indices are summed over (that is, $\varepsilon_{ijk} A_{jl} B_{km} = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} A_{jl} B_{km}$). For the rest of this problem set, we will always assume that this summation convention is implied, unless explicitly stated otherwise.

(a) From the definition in Eq. (1.1.1), show that

$$\varepsilon_{ijk} \varepsilon_{inm} = \delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn} \ , \tag{1.1.4}$$

where of course there is an implied sum over the i index in Eq. (1.1.4), but the indices j, k, n, and m are free.

(b) Using Eqs. (1.1.3) and (1.1.4), show that for any vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$,

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A} \cdot \vec{C}) - \vec{C}\,(\vec{A} \cdot \vec{B}) \tag{1.1.5}$$

(c) Using Eqs. (1.1.3) and (1.1.4), show that for any vectors $\vec{A}$ and $\vec{B}$,

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \ . \tag{1.1.6}$$

(d) Using Eqs. (1.1.3) and (1.1.4), show that for any vector $\vec{A}$,

$$\vec{A} \times \vec{\nabla} \times \vec{A} = \frac{1}{2}\vec{\nabla} A^2 - \left(\vec{A} \cdot \vec{\nabla}\right) \vec{A} \ . \tag{1.1.7}$$

(e) Using Eqs. (1.1.3) and (1.1.4), show that for any vectors $\vec{A}$ and $\vec{B}$,

$$\vec{\nabla} \times \left(\vec{A} \times \vec{B}\right) = \left(\vec{B} \cdot \vec{\nabla}\right) \vec{A} - \left(\vec{A} \cdot \vec{\nabla}\right) \vec{B} + \vec{A} \left(\vec{\nabla} \cdot \vec{B}\right) - \vec{B} \left(\vec{\nabla} \cdot \vec{A}\right) \ . \tag{1.1.8}$$

PROBLEM 2: TRIPLE CROSS PRODUCTS *(10 points)*

Griffiths Problem 1.2 (p. 4), Griffiths Problem 1.6 (p. 8).

PROBLEM 3: PROPERTIES OF THE ROTATION MATRIX *R* *(15 points)*

Griffiths Eq. (1.31), p. 11, is

$$\bar{A}_i = \sum_{j=1}^{3} R_{ij} A_j \ .$$

If we use the convention that repeated indices are summed over, then this can be written as

$$\bar{A}_i = R_{ij} A_j \ . \tag{1.2.1}$$

(a) Show that the elements (R_{ij}) of the three-dimensional rotation matrix must satisfy the constraint

$$R_{ij} R_{ik} = \delta_{jk} \tag{1.2.2}$$

in order to preserve the length of $\vec{A}$ for all $\vec{A}$. Matrices satisfying Eq. (1.2.2) are called *orthogonal*. Here δ_{jk} is the Kronecker delta (δ_{jk} is 1 if $j = k$ and 0 otherwise), and we use the summation convention above.

(b) Using the orthogonality constraint (1.2.2), show that

$$A_i = R_{ji} \bar{A}_j \ . \tag{1.2.3}$$

Note that we can now show that $R_{ji} R_{ki} = \delta_{jk}$ using this relation, in a manner similar to the procedure in (a) (you do not have to show this).

(c) Using the chain rule for partial differentiation and the results of (b), show that if f is scalar function of $\vec{r} \equiv (x_1, x_2, x_3)$, then $\vec{\nabla} f(\vec{r})$ transforms as a vector; i.e., show that if

$$\bar{f}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = f(x_1, x_2, x_3) \ , \tag{1.2.4}$$

where $\bar{x}_i = R_{ij} x_j$, then

$$\frac{\partial \bar{f}}{\partial \bar{x}_i} = R_{ij} \frac{\partial f}{\partial x_j} \ . \tag{1.2.5}$$

PROBLEM 4: USE OF THE GRADIENT *(10 points)*

Griffiths Problem 1.12 (p.15), Griffiths Problem 1.13 part (a) only (p.15).

PROBLEM 5: THE DIRAC DELTA FUNCTION AND $\nabla^2(1/4\pi r)$ *(20 points)*

One of the most used identities in this course is be the relation

$$-\nabla^2 \frac{1}{4\pi r} = -\vec{\nabla} \cdot \left[\vec{\nabla} \frac{1}{4\pi r}\right] = \vec{\nabla} \cdot \left[\frac{\hat{r}}{4\pi r^2}\right] = \delta^3(r) = \delta(x)\,\delta(y)\,\delta(z) \; . \tag{1.5.1}$$

It turns out of course (see Griffiths 1.5.1, p. 45) that

$$-\nabla^2 \frac{1}{4\pi r}$$

is zero everywhere except at the origin, and ill-defined there. To get a better feel for the fact that

$$-\nabla^2 \frac{1}{4\pi r}$$

is a delta function, let's look at a different function which approaches $-(1/4\pi r)$ in some limit, but which is well-behaved everywhere. The function is

$$f_a(r) = -\frac{1}{4\pi} \frac{1}{\sqrt{r^2 + a^2}} \; . \tag{1.5.2}$$

For a nonzero, $f_a(r)$ is well-behaved everywhere, and

$$\lim_{a \to 0} f_a(r) = -\frac{1}{4\pi r} \tag{1.5.3}$$

(a) Calculate $g_a(r) = \nabla^2 f_a(r)$ and show that it is also well behaved for all r. Sketch $g_a(r)$ for some value of a as a function of r/a.

(b) Show that

$$\int_{all\ space} g_a(r)\, d^3x = 1 \; . \tag{1.5.4}$$

(c) Show that

$$\lim_{a \to 0} g_a(r) = 0 \;\text{ if }\; r \neq 0 \; . \tag{1.5.5}$$

Thus in the limit that a goes to zero, our well-behaved function $g_a(r)$ exhibits the properties we expect of a three-dimensional delta function.

PROBLEM 6: EXERCISES WITH δ-FUNCTIONS *(10 points)*

(a) A charge Q is spread uniformly over a spherical shell of radius R. Express the volume charge density using a delta function in spherical coordinates. Repeat for a ring of radius R with charge Q lying in the xy plane.

(b) In cartesian coordinates, we can write $\delta^3(\vec{r} - \vec{r}') = \delta(x - x')\delta(y - y')\delta(z - z')$. How would one express $\delta^3(\vec{r} - \vec{r}')$ in cylindrical coordinates $(s, \phi, andz)$.

(c) A charge λ per unit length is distributed uniformly over a cylindrical surface of radius b. Give the volume charge density using a delta function in cylindrical coordinates

(d) What is $\nabla^2 \ln r$ in two dimensions? (Here r is the radial coordinate, $r = \sqrt{x^2 + y^2}$.)

PROBLEM 7: COROLLARIES OF THE FUNDAMENTAL INTEGRAL THEOREMS *(15 points)*

This problem is closely related to Problem 1.60, p. 56 of Griffiths. You will find useful hints there– but try without hints first!!. Show that:

(a) $\int_V \vec{\nabla}\psi \, d^3x = \int_S \psi \, d\vec{a}$, where S is the surface bounding the volume V. Show that as a consequence of this, $\int_S d\vec{a} = 0$ for a closed surface S.

(b) $\int_V \vec{\nabla} \times \vec{A} \, d^3x = -\int_S \vec{A} \times d\vec{a}$, where S is the surface bounding the volume V.

(c) $\int_S \vec{\nabla}\psi \times d\vec{a} = -\oint_\Gamma \psi \, d\vec{l}$, where Γ is the boundary of the surface S.

(d) For a closed surface S, one has $\int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = 0$.