MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — October 8, 2012

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LECTURE NOTES 7
A SPHERICAL SHELL OF CHARGE WITH A CIRCULAR HOLE

These notes describe an example that I began to show in lecture on Friday, October 5, 2012, but I was not able to finish. Here I will start it from the beginning, for completeness. The problem comes from J.D. Jackson, *Classical Electrodynamics*, 3rd Edition, Problem 3.2, p. 135.

The problem: *Suppose that a spherical surface of radius R has a surface charge density* $\sigma_0 = \frac{Q}{4\pi R^2}$ *uniformly distributed over its surface except for a spherical cap on the north pole, defined by the cone* $\theta = \alpha$*. Find the potential* $V(\vec{r})$ *everywhere.*

We choose a coordinate system with the center of the sphere at the origin, and the north pole (with its cap) along the positive z-axis. The problem then has azimuthal symmetry, which brings it within the context discussed in Griffiths. If we start with the more general formalism, however, just to show how it works, we can initially write the most general solution to Laplace's equation as

$$V(\vec{r}) = V(r,\theta,\phi) = \sum_{\ell=0}^{\infty} R_\ell(r) C^{(\ell)}_{i_1\ldots i_\ell} \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \ , \tag{7.1}$$

where $C^{(\ell)}_{i_1\ldots i_\ell}$ is a traceless symmetric tensor, $\hat{n}$ is a unit vector in the direction of (θ, ϕ), and i_1 through i_ℓ are repeated Cartesian indices which are summed from 1 to 3. The radial functions $R_\ell(r)$ are constrained by Laplace's equation to take the form

$$R_\ell(r) = A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \ . \tag{7.2}$$

For the case of azimuthal symmetry, we insist that $C^{(\ell)}_{i_1\ldots i_\ell}$ be invariant under rotations about the z-axis. We have not proven it, but the most general such tensor can be constructed by starting with $\hat{z}$, the only unit vector that is invariant under rotations about the z-axis. One can then construct the product $\hat{z}_{i_1} \ldots \hat{z}_{i_\ell}$, and then take the traceless symmetric part of this product. Using curly brackets { } to denote "the traceless symmetric part of," we have

$$C^{(\ell)}_{i_1\ldots i_\ell} \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} = \text{const} \ \{ \, \hat{z}_{i_1} \ldots \hat{z}_{i_\ell} \, \} \, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \propto P_\ell(\cos\theta) \ , \tag{7.3}$$

where $P_\ell(\cos\theta)$ is the standard Legendre polynomial. Absorbing the constants of proportionality into the constants A_ℓ and B_ℓ, we can write

$$V(r,\theta,\phi) = \sum_{\ell=0}^{\infty} R_\ell(r) \, P_\ell(\cos\theta) \ . \tag{7.4}$$

For this problem there is a surface charge density at $r = R$, at least for some angles, so at $r = R$ Laplace's equation does not hold. Thus we expect $V(r, \theta, \phi)$ to have the form of Eq. (7.4) for $r < R$ and $r > R$, but we are likely to need different functions $R_\ell(r)$ for each region. We therefore write

$$\begin{aligned} V_{\text{in}} &= \sum_{\ell=0}^{\infty} \left[A_\ell^{\text{in}}\, r^\ell + \frac{B_\ell^{\text{in}}}{r^{\ell+1}} \right] P_\ell(\cos\theta) \\ V_{\text{out}} &= \sum_{\ell=0}^{\infty} \left[A_\ell^{\text{out}}\, r^\ell + \frac{B_\ell^{\text{out}}}{r^{\ell+1}} \right] P_\ell(\cos\theta) \ , \end{aligned} \tag{7.5}$$

where "in" and "out" refer to the values inside ($r < R$) and outside ($r > R$) the sphere. Since the B_ℓ^{in} terms would be singular at $r = 0$, they are not allowed: $B_\ell^{\text{in}} = 0$. Similarly, the A_ℓ^{out} terms would diverge at infinity, which is not possible for a finite charge distribution, so $A_\ell^{\text{out}} = 0$. At the surface $r = R$ we expect the electric field to be discontinuous, but it will not be infinite — we have calculated the electric field of a surface charge layer, and it is finite. The potential difference between a point just inside the surface and a point just outside the surface is given by a line integral from one point to the other, $\int \vec{E} \cdot \mathrm{d}\vec{\ell}$. Since the integrand is finite and the length of the path is infinitesimal, the integral vanishes. The potential is thus required to be continuous at $r = R$, which implies that

$$A_\ell^{\text{in}}\, R^\ell = \frac{B_\ell^{\text{out}}}{R^{\ell+1}} \tag{7.6}$$

for all ℓ. (The terms have to match for each ℓ, because the Legendre polynomials $P_\ell(\cos\theta)$ are orthogonal, so a mismatch at one value of ℓ cannot be compensated by other values of ℓ.) Thus $B_\ell^{\text{out}} = R^{2\ell+1} A_\ell^{\text{in}}$. The expression for V can be written very compactly by defining

$$\tilde{A}_\ell = R^{\ell+1} A_\ell^{\text{in}} = R^{-\ell} B_\ell^{\text{out}} \ , \tag{7.7}$$

and then

$$V(r, \theta, \phi) = \begin{cases} \displaystyle\sum_{\ell=0}^{\infty} \tilde{A}_\ell \frac{r^\ell}{R^{\ell+1}} P_\ell(\cos\theta) & \text{for } r < R \\ \displaystyle\sum_{\ell=0}^{\infty} \tilde{A}_\ell \frac{R^\ell}{r^{\ell+1}} P_\ell(\cos\theta) & \text{for } r > R \ . \end{cases} \tag{7.8}$$

If we now define

$$r_> \equiv \text{larger}(r, R) \quad \text{and} \quad r_< \equiv \text{smaller}(r, R) \ , \tag{7.9}$$

then Eq. (7.8) can be condensed into the equation

$$V(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \tilde{A}_\ell \, \frac{r_<^\ell}{r_>^{\ell+1}} \, P_\ell(\cos\theta) \ . \tag{7.10}$$

(It is worth mentioning that we could have gotten here faster if we had used cleverer definitions at the start. We could have written $R_\ell(r)$ as

$$R_\ell(r) = A_\ell \frac{r^\ell}{R^{\ell+1}} + B_\ell \frac{R^\ell}{r^{\ell+1}} \ , \tag{7.2$'$}$$

where the extra factors of R do nothing more than redefine the as yet unknown parameters A_ℓ and B_ℓ. Eq. (7.2′) has the virtue that A_ℓ and B_ℓ have the same units. With this starting point, the argument used above leads to the conclusion that $B_\ell^{\rm out} = A_\ell^{\rm in}$, with no factors of R to complicate the equation. It is easy to see that A_ℓ in this approach is equal to $\tilde{A}_\ell$ in the approach that we used.)

To finish the problem, we need to calculate the $\tilde{A}_\ell$. They are clearly determined by the presence of the charge, and we can account for the surface charge density by imposing the discontinuity in the normal component of the electric field that is required by Gauss's law:

$$E_r\big|_{R^+} - E_r\big|_{R^-} = \frac{\sigma(\theta)}{\epsilon_0} \ , \tag{7.11}$$

where

$$\sigma(\theta) = \begin{cases} \sigma_0 & \text{if } \theta > \alpha \\ 0 & \text{if } \theta < \alpha \ . \end{cases} \tag{7.12}$$

Eq. (7.11) can be written in terms of the potential as

$$-\left.\frac{\partial V}{\partial r}\right|_{R^+} + \left.\frac{\partial V}{\partial r}\right|_{R^-} = \frac{\sigma(\theta)}{\epsilon_0} \ . \tag{7.13}$$

Writing out this relation in terms of the expansion (7.10),

$$\sum_{\ell=0}^{\infty} \left[(\ell+1)\tilde{A}_\ell \frac{R^\ell}{R^{\ell+2}}\, P_\ell(\cos\theta) + \ell \tilde{A}_\ell \frac{R^{\ell-1}}{R^{\ell+1}}\, P_\ell(\cos\theta) \right] = \frac{\sigma(\theta)}{\epsilon_0} \ , \tag{7.14}$$

which can be rewritten as

$$\sum_{\ell=0}^{\infty} (2\ell+1)\tilde{A}_\ell \, P_\ell(\cos\theta) = \frac{R^2 \sigma(\theta)}{\epsilon_0} \ . \tag{7.15}$$

Now of course we use the orthogonality of the Legendre polynomials,

$$\int_{-1}^{1} P_{\ell'}(x) P_\ell(x) \, {\rm d}x = \frac{2}{2\ell+1}\, \delta_{\ell'\ell} \ . \tag{7.16}$$

Multiplying both sides of Eq. (7.15) by $P_{\ell'}(x)$ and integrating over $x = \cos\theta$,

$$\sum_{\ell=0}^{\infty}(2\ell+1)\tilde{A}_\ell \int_{-1}^{1} P_{\ell'}(x)P_\ell(x)\,\mathrm{d}x = 2\tilde{A}_{\ell'} = \frac{R^2}{\epsilon_0}\int_{-1}^{1} P_{\ell'}(x)\sigma(x)\,\mathrm{d}x \;, \tag{7.17}$$

so

$$\tilde{A}_\ell = \frac{R^2}{2\epsilon_0}\int_{-1}^{1} P_\ell(x)\,\sigma(x)\,\mathrm{d}x \;. \tag{7.18}$$

In many similar problems one would have to stop at a result such as Eq. (7.18), but in this case one can actually carry out the integration. First use Eq. (7.12) to make the integral explicit:

$$\tilde{A}_\ell = \frac{R^2\sigma_0}{2\epsilon_0}\int_{-1}^{\cos\alpha} P_\ell(x)\,\mathrm{d}x = \frac{Q}{8\pi\epsilon_0}\int_{-1}^{\cos\alpha} P_\ell(x)\,\mathrm{d}x \;. \tag{7.19}$$

Now use the identity

$$\frac{\mathrm{d}P_{\ell+1}(x)}{\mathrm{d}x} - \frac{\mathrm{d}P_{\ell-1}(x)}{\mathrm{d}x} - (2\ell+1)P_\ell(x) = 0 \;, \tag{7.20}$$

which is valid for all $\ell \geq 0$ if one defines

$$P_{-1}(x) \equiv 0 \;. \tag{7.21}$$

Replacing $P_\ell(x)$ in Eq. (7.19) by an expression in terms of derivatives of $P_\ell(x)$, we find

$$\begin{aligned}\tilde{A}_\ell &= \frac{Q}{8\pi\epsilon_0(2\ell+1)}\int_{-1}^{\cos\alpha}\left[\frac{\mathrm{d}P_{\ell+1}(x)}{\mathrm{d}x} - \frac{\mathrm{d}P_{\ell-1}(x)}{\mathrm{d}x}\right]\mathrm{d}x \\ &= \frac{Q}{8\pi\epsilon_0(2\ell+1)}\left\{\big[P_{\ell+1}(\cos\alpha) - P_{\ell+1}(-1)\big] - \big[P_{\ell-1}(\cos\alpha) - P_{\ell-1}(-1)\big]\right\} \;.\end{aligned} \tag{7.22}$$

For $\ell \geq 0$ one has

$$P_\ell(-1) = (-1)^\ell \;, \tag{7.23}$$

which one can verify from Rodriguez's formula, or by knowing that $P_\ell(x)$ is normalized by the condition $P_\ell(1) = 1$, and that $P_\ell(x)$ is an even function for even ℓ, and an odd function for odd ℓ. Thus for $\ell > 0$, the terms $P_{\ell+1}(-1)$ and $P_{\ell-1}(-1)$ cancel, so we have

$$\tilde{A}_\ell = \frac{Q}{8\pi\epsilon_0(2\ell+1)}\big[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\big] \;. \tag{7.24a}$$

For $\ell = 0$ one has from Eqs. (7.21) and (7.22) that

$$\tilde{A}_0 = \frac{Q}{8\pi\epsilon_0}\big[P_1(\cos\alpha) + 1\big] \;. \tag{7.24b}$$

Using Eqs. (7.10) and (7.24), the final answer can be written as

$$V(r,\theta,\phi) = \frac{Q}{8\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\big[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\big]\,\frac{r_<^\ell}{r_>^{\ell+1}}\,P_\ell(\cos\theta) \;, \tag{7.25}$$

where in this formula we intepret $P_{-1}(\cos\alpha)$ as -1, although it conflicts with Eq. (7.21).