# 8.321 Quantum Theory-I Fall 2017

## Prob Set 5

1. The Lagrangian of a charged particle in an electromagnetic field is $\dot{x}$

$$L(\vec{x}, \dot{\vec{x}}) = \frac{1}{2}m\dot{\vec{x}}^2 - q\phi(\vec{x}) + \frac{q}{c}\vec{v} \cdot \vec{A} \tag{1}$$

and classical action is $S = \int dt L(\vec{x}, \dot{\vec{x}})$. Here $\phi, \vec{A}$ are evaluated on the particle trajectory.

   (a) Show that the least action path satisfies the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{x}}} - \frac{\partial L}{\partial \vec{x}} = 0 \tag{2}$$

   Show explicitly that these lead to the familiar force law for a charged particle in an electromagnetic field.

   (b) Now specialize to the case of one dimension and zero magnetic field. For a free particle find the least action path connecting points $x, t$ and $x', t'$. Repeat for a particle moving in a linear potential $\phi = -xE$.

2. For a free particle, write the propagator $K(x, t; x', t')$ as

$$N(t - t') \exp\left(\frac{im(x - x')^2}{2\hbar(t - t')}\right) \tag{3}$$

From the composition law

$$\int d\tilde{x} K(x, t; \tilde{x}, \tilde{t}) K(\tilde{x}, \tilde{t}; x', t') = K(x, t; x', t'), \qquad t > \tilde{t} > t' \tag{4}$$

obtain a condition for $N(\tau)$. Relate $N(\tau)$ to $N^*(\tau)$ (Hint: use unitarity). Find the most general solutions to those two equations. Is

$N(\tau)$ completely determined? What other information can you use to determine $N(\tau)$ (aside from the Schrodinger equation)?

3. There is a an important class of problems where the stationary phase approximation is a non-approximation, and yields an exact result for the path integral. This happens when the Lagragian is a quadratic polynomial of the position $x$ and velocity $\dot{x}$. To prove this, consider the propagator

$$K(x,t'x',t') = \int [\mathcal{D}x(t)] \exp\left(\frac{iS[x(t)]}{\hbar}\right) \qquad (5)$$

An arbitrary path $x(t)$ from $x', t'$ to $x, t$ can be expressed as a sum of the classical ($i.e$ least action) path $x_{cl}(t)$ with the same end points and a displacement $\delta x(t)$, as follows: $x(t) = x_{cl}(t) + \delta x(t)$. Substituting this expression into the action, and noting that the terms in first order in $\delta x(t)$ cancel (why?), bring Eqn. **??** to the form

$$K(x,t;x',t') = \exp\left(\frac{iS[x_{cl}(t)]}{\hbar}\right) \int [\mathcal{D}\delta x(t)] \exp\left(\frac{iS[\delta x(t)]}{\hbar}\right) \qquad (6)$$

Crucially, only the prefactor $\exp\left(\frac{iS[x_{cl}(t)]}{\hbar}\right)$ depends on the end points $x, x'$, since the path integral is taken over closed paths $\delta x(t') = \delta x(t) = 0$. Thus the full dependence on $x, x'$ is captured by the stationary phase factor, giving the propagator of the form $A(t,t')\exp\left(\frac{iS[x_{cl}(t)]}{\hbar}\right)$.

(a) Use this approach to obtain the propagator for a free particle.

(b) Show that the propagator of a particle moving in a parabolic potential, with Lagrangian $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$, is of the form

$$K(x,t;x',t') = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega(t-t'))}\right)^{1/2} \exp\left(\frac{im\omega}{2\hbar\sin(\omega(t-t'))}[(x^2+x'^2)\cos(\omega(t-t')) - 2xx']\right)$$

The time dependence of this expression is periodic, matching the periodicity of classical motion. Interestingly however the time dependence features two singularities per period, occurring when $\sin(\omega(t-t')) = 0$. Comment on the origin of this behavior.

4. (a) Use the result for the propagator in Prob 3b to determine the energy eigenvalues of the simple harmonic oscillator, and show explicitly that you get the usual result $E = \left(n + \frac{1}{2}\right) \hbar\omega$ with $n = 0, 1, .......$

   (b) Derive the ground state wave function $\psi_0(x)$ of the simple harmonic oscillator from the propagator. You will find it convenient to study the propagator in imaginary time by setting $t = -i\beta$ with $\beta$ real. Specifically consider $K(x, t = -i\beta; x' = x, t' = 0)$. First show that the in limit $\beta \to \infty$ the sum over eigenstates in the expression for the propagator is dominated by the ground state. Apply this to the propagator of the oscillator to extract $|\psi_0(x)|^2$ and show that it agrees with the well known answer.