8.321 Quantum Theory-I Fall 2017

Prob Set 3

1. **Minimum uncertainty states (adapted from Sakurai 1.18)**

   (a) The simplest way to derive the Schwarz inequality goes as follows. First observe

   $$(\langle\alpha| + \lambda^*\langle\beta|) \cdot (|\alpha\rangle + \lambda|\beta\rangle) \geq 0 \tag{1}$$

   for any complex number λ; then choose λ in such a way that the preceding inequality reduces to the Schwarz inequality.

   (b) Show that the equality sign in the generalized uncertainty relation holds if the state in question satisfies

   $$\Delta A|\alpha\rangle = \lambda\Delta B|\alpha\rangle \tag{2}$$

   with λ purely *imaginary*.

   (c) Hence show that the Gaussian wave packet

   $$\langle x'|\alpha\rangle = \left(2\pi d^2\right)^{-1/4} \exp\left(\frac{i\langle p\rangle x'}{\hbar} - \frac{\left(x' - \langle x\rangle\right)^2}{4d^2}\right) \tag{3}$$

   satisfies the minimum uncertainty relation for x and p.

2. **The uncertainty relation for spin (Sakurai 1.19)**

3. **Sakurai 1.22**

4. (a) For a particle in a state described by a real wave function show that the average momentum $\langle p \rangle = 0$.

   (b) Suppose the particle is in a general state with wave function $\psi(x)$ with average momentum p_0. Show that the modified wave function $e^{iPx/\hbar}\psi(x)$ has average momentum $p_0 + P$.

   (c) A different perspective on the result above is obtained by considering the operator $\tilde{T}(P) = e^{\frac{ixP}{\hbar}}$ where x is the position operator and P is real. Find $\tilde{T}(P)p\tilde{T}^{\dagger}(P)$ and $\tilde{T}(P)|p'\rangle$. Argue how the result of (b) follows from this calculation.

   (d) Consider $\tilde{T}(P)$ together with the translation operator $T(a) = e^{-\frac{ipa}{\hbar}}$ defined in class. Is the product $\tilde{T}(P)T(a)$ equal to $e^{\frac{i(Px-pa)}{\hbar}}$?

5. Consider a spin-1/2 particle in the presence of a time independent magnetic field along the z-direction with the Hamiltonian

$$H = -\gamma B S_z \tag{6}$$

(γ is a constant). At time $t = 0$ the spin state is an eigenstate of $\vec{S} \cdot \hat{n}_0$ where $\hat{n}_0$ lies in the xz plane and makes an angle θ with the z axis. Here you will consider the time evolution of this state.

   (a) In the Schrodinger picture find the state at time t in the S_z basis. Hence find the probability that a measurement of S_x at time t will yield the value $+\frac{\hbar}{2}$.

   (b) Recall that any state of a spin-1/2 system can be represented as a point $\hat{n}$ in the Bloch sphere. The evolution of the spin state in this problem can be described as a trajectory $\hat{n}(t)$ on the Bloch sphere with $\hat{n}(t = 0) = \hat{n}_0$. Show explicitly using the result above that $\hat{n}(t)$ precesses about the z-axis at the Larmor frequency γB.

   (c) Now using the Heisenberg picture derive the equation of motion for $\langle \vec{S} \rangle$. Re-derive the same equation within the Schrodinger picture using the calculation above.