8.321 Quantum Theory-I Fall 2017

Prob Set 6

1. **Aharanov-Bohm effect I**

   For a charged particle in an electromagnetic field the Lagrangian is

   $$L = \frac{1}{2}m\dot{\vec{x}}^2 - q\phi(\vec{x}) + \frac{q}{c}\vec{v}\cdot\vec{A} \tag{1}$$

   (a) Is the classical action $S = \int dtL(x, \dot{\vec{x}})$ gauge invariant? How does it change under gauge transformation?

   (b) How does the propagator $K(\vec{x}, t; \vec{x}'t') = \int \mathcal{D}xe^{\frac{iS}{\hbar}}$ change under gauge transformation?

   (c) The essence of the Aharanon-Bohm effect can be illustrated by considering a particle moving on a ring of circusference l with a non-zero flux Φ threading the ring. The vector potential may be taken to be tangential to the ring and has value $A = \frac{\Phi}{l}$. The propagator for this system may be found as a sum over trajectories with different windings m around the ring. For $0 < x < x' < l$,

   $$\begin{aligned} K(x, t; x', t')_{ring} &= \sum_{m=-\infty}^{\infty} K(x, t; x', t')_{m,ring}, & (2) \\ K(x, t; x', t')_{m,ring} &= K(x + ml, t; x', t')_{line} & (3) \end{aligned}$$

   In the last equation the contribution from the paths winding m times around the ring is expressed as that of a straight path on a line $-\infty < x, x' < \infty$ representing an unwound line.

   Find the propagator $K(x, t; x', t')$ for $x = x'$. Use the result for the free particle combined with the gauge transformation discussed above.

(d) Discuss the behavior of the propagator as a function of applied flux Φ. Is it periodic? What is the period?

2. **Aharanov-Bohm effect II**

Consider the same system (particle in a ring threaded by magnetic flux) using a Hamiltonian approach. The Hamiltonian is

$$H = \frac{(p - \frac{q}{c}A)^2}{2m} \tag{4}$$

with $A = \frac{\Phi}{l}$ and $p = -i\hbar\frac{d}{dx}$ is the momentum component tangential to the ring. The wave function, which is single valued, obeys the periodic boundary condition $\psi(x + l) = \psi(x)$.

(a) Find the spectrum and eigenstates.

(b) Plot the energy of the ground state as a function of Φ.

(c) For a particle in the ground state find the expectation value of the electric current along the ring, $j = \frac{q}{m}\langle\psi|p - \frac{q}{c}A|\psi\rangle$ and evaluate the magnetic moment as a function of Φ.

3. In 1984, 25 years after the prediction of the Aharonov-Bohm (AB) effect, Aharonov and Casher [Phys. Rev. Lett. 53, 319 (1984)] predicted a "dual" effect. . In both effects, a particle is excluded from a tubular region of space, but otherwise no force acts on it. Yet it acquires a measurable quantum phase that depends on what is inside the tube of space from which it is excluded. In the AB effect, the particle is charged and the tube contains a magnetic flux. In the Aharonov-Casher (AC) effect, the particle is neutral, but has a magnetic moment, and the tube contains a line of charge. To derive their effect, Aharonov and Casher first obtained the nonrelativistic Lagrangian for a neutral particle of magnetic moment μ interacting with a particle of charge e. In Gaussian units

$$L(\vec{r}, \vec{v}, \vec{R}, \vec{V}) = \frac{1}{2}m\vec{v}^2 + \frac{1}{2}M\vec{V}^2 + \frac{e}{c}(\vec{v} - \vec{V}).\vec{A}(\vec{r} - \vec{R}) \tag{5}$$

$$A(\vec{r} - \vec{R}) = \frac{\vec{\mu} \times (\vec{r} - \vec{R})}{|\vec{r} - \vec{R}|^3} \tag{6}$$

where $M, \vec{R}, \vec{V}$ and $m, \vec{r}, \vec{v}$ are the mass, position and velocity of the neutral and charged particle, respectively, and the vector potential

$\vec{A}(\vec{r}-\vec{R})$ is due to the magnetic moment. In the AB effect, the electron does not cross through a magnetic field; in the AC effect, the neutral particle does cross through an electric field. However, there is no force on either particle.

(a) Show that L is invariant under respective interchange of $\vec{r}, \vec{v}$ and $\vec{R}, \vec{V}$. Thus L is the same whether an electron interacts with a line of magnetic moments (AB effect) or a magnetic moment interacts with a line of electrons (AC effect).

(b) Using the Lagrangian above find the phase difference between wave functions of magnetic dipoles (say, neutrons) going different ways around a line of electric charges. Show that the AC effect is geometric: the AC phase does not depend on particle velocity, it depends only on particle path.