8.321 Quantum Theory-I Fall 2015

Final Exam

Dec 18, 2015

1. **(25 points)**

   Consider two qubits in the quantum state

   $$|\psi\rangle = N(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \tag{1}$$

   Here the states $|0\rangle$, $|1\rangle$ of a qubit are eigenstate of σ_z with eigenvalues -1, $+1$ respectively.

   (a) **(5 points)** Find N so that the state is normalized.

   (b) **(5 points)** Are the two qubits entangled with each other?

   (c) **(10 points)** Find the density matrix of the first qubit in the Z basis. Suppose we now modify the state by acting unitarily on the second qubit. How does this affect the density matrix of the first qubit?

   (d) **(5 points)** Consider the state $|\psi'\rangle$ obtained by acting with $\sigma_{1z}\sigma_{2z}$ on $|\psi\rangle$. Are the two quits entangled in $|\psi'\rangle$? Calculate the density matrix of the fist qubit in this state.

2. **(25 points)**

   Consider two particles - one with charge $+q$ and the other with charge $-q$ - moving in a uniform magnetic field in two dimensions. The Hamiltonian is

   $$H = \frac{\vec{\Pi}_1^2}{2m} + \frac{\vec{\Pi}_2^2}{2m} + V(\vec{x}_1 - \vec{x}_2) \tag{2}$$

   Here $\vec{\Pi}_1 = \vec{p}_1 - q\vec{A}(\vec{x}_1)$, $\vec{\Pi}_2 = \vec{p}_2 + q\vec{A}(\vec{x}_2)$ are the kinematic momenta of the two particles. $\vec{A}$ is the vector potential corresponding to the uniform magnetic field $\vec{B} = B\hat{z}$, and $\vec{x}_{1,2}$ are the coordinates of the two particles. V is an attractive interaction between the two particles which you can take to have a harmonic form:

$$V(\vec{x}_1 - \vec{x}_2) = \frac{k}{2}|\vec{x}_1 - \vec{x}_2|^2 \tag{3}$$

Below you will consider this problem in center-of-mass $\vec{R} = \frac{\vec{x}_1+\vec{x}_2}{2}$, and relative coordinates $\vec{x} = \vec{x}_1 - \vec{x}_2$.

(a) **(10 points)** Define the two momenta

$$\vec{Q} = \vec{\Pi}_1 + \vec{\Pi}_2 - q\vec{x} \times \vec{B} \tag{4}$$

$$\vec{p} = \frac{\vec{\Pi}_1 - \vec{\Pi}_2}{2} \tag{5}$$

Show that the pairs $(\vec{R}, \vec{Q})$ and $(\vec{x}, \vec{p})$ are canonically conjugate. Check also that $[Q_i, p_j] = [Q_i, x_j] = [Q_i, Q_j] = [p_i, p_j] = [R_i, p_j] = 0$.

(b) **(7 points)** Show that $\vec{Q}$ commutes with the Hamiltonian. Show that this has a classical analog by inspecting the classical equations of motion for the two particles (together with an identification of a classical quantity that corresponds to $\vec{Q}$).

(c) **(5 points)** As $[\vec{Q}, H] = 0$ and the two components of $\vec{Q}$ commute with each other, the Hamiltonian may be diagonalized simultaneously with $\vec{Q}$. For a fixed value of $\vec{Q}$, show that the Hamiltonian may be rewritten as

$$H = \frac{\left(\vec{Q} + q\vec{x} \times \vec{B}\right)^2}{4m} + \frac{\vec{p}^2}{m} + V(\vec{x}) \tag{6}$$

(d) **(3 points)** The energy spectrum will depend on $\vec{Q}$ and can be determined exactly. To keep things simple specialize to $\vec{Q} = 0$, and find the exact spectrum.

3. **(25 points)**

A particle of mass m moves on a ring of fixed radius R. The position of the particle may be parametrized by an angle $\phi \in [0, 2\pi)$. Suppose that that the wave function of the particle changes sign each time it goes around the ring, *i.e* the Hilbert space consists of wave functions that are anti periodic under $\phi \to \phi + 2\pi$:

$$\psi(\phi + 2\pi) = -\psi(\phi) \tag{7}$$

(a) **(5 points)** The angular momentum of the particle is described by the operator $L = -i\hbar \frac{d}{d\phi}$. Determine the allowed eigenvalues of L, and the corresponding eigenfunctions.

(b) **(3 points)** The particle is described by the Hamiltonian

$$H = \frac{L^2}{2mR^2} \tag{8}$$

Obtain the energy eigenvalues, the eigenfunctions, and their degeneracies.

(c) **(3 points)** Time reversal T acts in an unusual way by reversing the direction of this particle's position, *i.e* $T : \phi \to \phi + \pi$. Note that, as always, T is anti-unitary. How does L transform under time reversal? Show that H is time reversal invariant.

(d) **(5 points)** Obtain the action of T on the energy eigenfunctions. Show that $T^2 = -1$ on all the energy eigenfunctions, and hence on any state in the Hilbert space.

(e) **(5 points)** Now consider adding a real potential $V(\phi) = V(\phi + 2\pi)$ to the Hamiltonian. If the Hamiltonian continues to be time reversal symmetric what further restriction does V have to satisfy?

(f) **(4 points)** In the presence of a V respecting time reversal the energy eigenvalues you found above will change. How will the degeneracies of the eigenvalues be affected? (Hint: use Kramers theorem).

4. **(25 points)**

A spin-S moment ($S =$ integer) is described by the Hamiltonian

$$H_0 = DS_z^2 \tag{9}$$

with $D > 0$.

(a) **(5 points)** Obtain the energy spectrum and degeneracies.

(b) **(10 points)** This spin is subjected to a small magnetic field along the x-direction leading to an extra term in the Hamiltonian:

$$H_1 = -\gamma B S_x \tag{10}$$

Using second order perturbation theory, calculate the change in ground state energy. Give a criterion in terms of $D, \gamma B, S$ for the perturbation theory to be accurate.

(c) **(10 points)** What is the ground state wave function to first order in B?