Lecture 19 (Nov. 15, 2017)

19.1 Rotations

Recall that rotations are transformations of the form $x_i \to R_{ij} x_j$ (using Einstein summation notation), where R is an orthogonal matrix, $R^{\mathrm{T}} R = \mathbb{1}$. This R is called a *rotation matrix.* For now, we will restrict to rotations R with $\det R = +1$ (*orientation-preserving* or *proper* rotations).

Every rotation R of space corresponds to a unitary operator $\mathcal{D}(R)$ on the Hilbert space, which satisfies

$$\mathcal{D}(R_1)\mathcal{D}(R_2) = \mathcal{D}(R_1 R_2)\,. \tag{19.1}$$

We will discuss this composition property more later. A quantum state transforms as $|\alpha\rangle \to |\alpha_R\rangle$ under this rotation, such that

$$|\alpha_R\rangle = \mathcal{D}(R)|\alpha\rangle\,. \tag{19.2}$$

For a vector operator V_i, with $i = 1, \ldots, d$, we require

$$\langle \beta_R | V_i | \alpha_R \rangle = R_{ij} \langle \beta | V_j | \alpha \rangle\,. \tag{19.3}$$

Expanding the transformed bra and ket, this is

$$\langle \beta | \mathcal{D}^\dagger(R) V_i \mathcal{D}(R) | \alpha \rangle = R_{ij} \langle \beta | V_j | \alpha \rangle\,. \tag{19.4}$$

This is true for any states $|\alpha\rangle, |\beta\rangle$, which implies that the operators must be equal:

$$\mathcal{D}^\dagger(R) V_i \mathcal{D}(R) = R_{ij} V_j \tag{19.5}$$

holds as an operator equation.

Consider the infinitesimal rotation $R = 1 - \omega$. The orthogonality condition, $R^{\mathrm{T}} R = \mathbb{1}$, then implies that $\omega^{\mathrm{T}} = -\omega$. Thus, ω is a real, antisymmetric matrix. We can then expand $\mathcal{D}(R)$ in the form

$$\mathcal{D}(R) \approx 1 - \frac{i}{2\hbar} \sum_{ij} \omega_{ij} J_{ij} + O\big(\omega^2\big)\,. \tag{19.6}$$

This expansion identifies the objects $J_{ij} = -J_{ji}$ as the Hermitian generators of rotations. Note that the antisymmetry of J_{ij} follows from the antisymmetry of ω_{ij}.

Let us now specialize to three dimensions, $d = 3$. In this case, we can write

$$\mathcal{D}(R) = 1 - \frac{i}{\hbar}(J_{12}\omega_{12} + J_{23}\omega_{23} + J_{31}\omega_{31}) + O\big(\omega^2\big)\,, \tag{19.7}$$

where we have used the antisymmetry of J_{ij} to group terms. We define

$$J_1 := J_{23}\,, \quad J_2 := J_{31}\,, \quad J_3 := J_{12}\,, \tag{19.8}$$

i.e.,

$$J_i = \frac{1}{2}\epsilon_{ijk} J_{jk}\,, \tag{19.9}$$

where ϵ_{ijk} is the totally antisymmetric symbol with $\epsilon_{123} = +1$, known as the *Levi–Civita symbol.* We can similarly define

$$\theta_1 := \omega_{23}\,, \quad \theta_2 := \omega_{31}\,, \quad \theta_3 := \omega_{12}\,, \tag{19.10}$$

i.e.,

$$\theta_i = \frac{1}{2}\epsilon_{ijk}\omega_{jk}\,, \quad \omega_{ij} = \epsilon_{ijk}\theta_k\,. \tag{19.11}$$

Then, we have

$$\mathcal{D}(R) = 1 - \frac{i}{\hbar}\theta_k J_k + O\big(\theta^2\big)\,. \tag{19.12}$$

Note that

$$R_{ij} = \delta_{ij} - \epsilon_{ijk}\theta_k + O\big(\theta^2\big)\,. \tag{19.13}$$

Thus,

$$x_i \to x_i' = R_{ij}x_j = (\delta_{ij} - \epsilon_{ijk}\theta_k)x_j = x_i - \epsilon_{ijk}x_j\theta_k\,, \tag{19.14}$$

i.e.,

$$\boldsymbol{x} \to \boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{\theta} \times \boldsymbol{x}\,. \tag{19.15}$$

Thus, the meaning of $\boldsymbol{\theta}$ is that $\boldsymbol{x}$ is rotated by an angle $|\boldsymbol{\theta}|$ about the $\boldsymbol{\theta}$-direction.

We now *define* J_k to be the components of the angular momentum. First, we will derive the commutation relations of J_k with any vector operator V_i. We start with the equation

$$\mathcal{D}^\dagger(R)V_i\mathcal{D}(R) = R_{ij}V_j \tag{19.16}$$

and take $R = 1 - \omega$ with ω infinitesimal. The left-hand side then becomes

$$\left(1 + \frac{i\theta_k J_k}{\hbar}\right)V_i\left(1 - \frac{i\theta_\ell J_\ell}{\hbar}\right) = V_i + \frac{i\theta_k}{\hbar}[J_k, V_i]\,, \tag{19.17}$$

while the right-hand side becomes

$$R_{ij}V_j = V_i - \epsilon_{ijk}V_j\theta_k\,. \tag{19.18}$$

Thus, we conclude that

$$[J_k, V_i] = i\hbar\epsilon_{kij}V_j\,. \tag{19.19}$$

We can then use a combination of rotations to deduce the angular momentum algebra, via

$$\mathcal{D}(R_1)\mathcal{D}(R_2) = \mathcal{D}(R_1R_2)\,. \tag{19.20}$$

In particular, this composition rule implies that

$$\mathcal{D}(R_{\boldsymbol{\phi}})\mathcal{D}(R_{\boldsymbol{\theta}})\mathcal{D}\Big(R_{\boldsymbol{\phi}}^{-1}\Big) = \mathcal{D}\Big(R_{\boldsymbol{\phi}}R_{\boldsymbol{\theta}}R_{\boldsymbol{\phi}}^{-1}\Big)\,. \tag{19.21}$$

The rotation $R_{\boldsymbol{\phi}}R_{\boldsymbol{\theta}}R_{\boldsymbol{\phi}}^{-1}$ can be written as a single rotation $R_{\boldsymbol{\theta}'}$ for some $\boldsymbol{\theta}'$. As $\boldsymbol{\theta}$ itself is a vector, for $\boldsymbol{\phi}$ infinitesimal, we have

$$\boldsymbol{\theta}' = \boldsymbol{\theta} + \boldsymbol{\phi} \times \boldsymbol{\theta}\,. \tag{19.22}$$

If we take $\boldsymbol{\theta}$ to be infinitesimal, we have

$$\mathcal{D}(R_{\boldsymbol{\theta}}) = 1 - \frac{i\theta_k J_k}{\hbar} + O\big(\theta^2\big)\,, \tag{19.23}$$

so (19.21) becomes

$$\theta_k\mathcal{D}(R_{\boldsymbol{\phi}})J_k\mathcal{D}\Big(R_{\boldsymbol{\phi}}^{-1}\Big) = \theta_k' J_k \tag{19.24}$$

The left-hand side of this equation, for infinitesimal $\boldsymbol{\phi}$ is

$$\theta_k\left(1 - \frac{i\phi_j J_j}{\hbar}\right)J_k\left(1 + \frac{i\phi_\ell J_\ell}{\hbar}\right) = \theta_k J_k - \frac{i\theta_k\phi_j}{\hbar}[J_j, J_k] + \cdots\,, \tag{19.25}$$

while the right-hand side is

$$\theta_k J_k + \epsilon_{jk\ell}\phi_j\theta_k J_\ell + \cdots\,, \tag{19.26}$$

which leads us to conclude that

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k\,. \tag{19.27}$$

This is the angular momentum commutation algebra. Note that this matches the commutation relation for a vector operator with the angular momentum operator, so this shows that the angular momentum operator is a vector.

In general, we can write the angular momentum as

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}\,, \tag{19.28}$$

with $\boldsymbol{L} = \boldsymbol{x}\times\boldsymbol{p}$ is the orbital angular momentum and $\boldsymbol{S}$ is an internal property that commutes with $\boldsymbol{x}$, $\boldsymbol{p}$, etc. We can check that $\boldsymbol{L}$ on its own satisfies the angular momentum commutation algebra, so the operator $\boldsymbol{J}$ will satisfy the angular momentum algebra if $\boldsymbol{S}$ does. The operator $\boldsymbol{S}$ is the spin operator.

If the Hamiltonian is rotationally invariant, then $[J_i, H] = 0$, which implies that

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = 0\,, \tag{19.29}$$

and so angular momentum is conserved.

19.1.1 Eigensystem of Angular Momentum

Let us now understand the implications of the commutation algebra

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k\,. \tag{19.30}$$

You will show on the homework that

$$\left[\boldsymbol{J}^2, J_i\right] = 0\,. \tag{19.31}$$

This means that we can diagonalize $\boldsymbol{J}^2$ and one component of the angular momentum, say J_z, simultaneously. We can then label the eigenstates of J_z by $|j,m\rangle$, with

$$\boldsymbol{J}^2|j,m\rangle = a|j,m\rangle\,, \quad J_z|j,m\rangle = b|j,m\rangle\,, \tag{19.32}$$

for some eigenvalues a, b. The meanings of the values j and m will become apparent shortly.

It is useful to define the *ladder operators*

$$J_\pm = J_x \pm iJ_y\,, \tag{19.33}$$

which satisfy

$$[J_+, J_-] = 2\hbar J_z\,, \quad [J_z, J_\pm] = \pm\hbar J_\pm\,, \quad \left[\boldsymbol{J}^2, J_\pm\right] = 0\,. \tag{19.34}$$

Using these commutation relations, we see that

$$\begin{aligned} J_z(J_\pm|j,m\rangle) &= (J_\pm J_z \pm \hbar J_\pm)|j,m\rangle \\ &= (b\pm\hbar)(J_\pm|j,m\rangle)\,. \end{aligned} \tag{19.35}$$

Thus, $J_\pm|j,m\rangle$ is also an eigenstates of J_z with eigenvalue $b\pm\hbar$.

We can write

$$\begin{aligned}\boldsymbol{J}^2 &= J_x^2 + J_y^2 + J_z^2 \\ &= J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+) \\ &= J_z^2 + \frac{1}{2}\left(J_+J_+^\dagger + J_-J_-^\dagger\right),\end{aligned} \tag{19.36}$$

which tells us that $\boldsymbol{J}^2 - J_z^2$ is positive semi-definite,

$$\langle j,m|\boldsymbol{J}^2 - J_z^2|j,m\rangle \geq 0\,. \tag{19.37}$$

This implies that $a - b^2 \geq 0$ for all eigenstates. For a fixed a, this means that $|b|$ has a maximum value b_{max}. This seems to be in conflict with the statement that we can use $J_\pm$ to raise or lower the eigenvalue arbitrarily. We conclude that at $b = +b_{\text{max}}$ the state must be annihilated by J_+, and similarly, at $b = -b_{\text{max}}$ the state must be annihilated by J_-.

Call $|\text{max}\rangle$ the state with $b = +b_{\text{max}}$. Then, we have

$$J_+|\text{max}\rangle = 0\,, \tag{19.38}$$

which implies

$$J_-J_+|\text{max}\rangle = 0\,. \tag{19.39}$$

Expanding the ladder operators, this becomes

$$(J_x - iJ_y)(J_x + iJ_y)|\text{max}\rangle = \left(\boldsymbol{J}^2 - J_z^2 - \hbar J_z\right)|\text{max}\rangle = 0\,. \tag{19.40}$$

This gives us

$$a - b_{\text{max}}^2 - \hbar b_{\text{max}} = 0\,, \tag{19.41}$$

i.e.,

$$a = b_{\text{max}}(b_{\text{max}} + \hbar)\,. \tag{19.42}$$

Repeating this argument for the state $|\text{min}\rangle$ with $b = -b_{\text{max}}$ yields the same result.

We now note that, because J_+ increases the eigenvalue b, and this eigenvalue is bounded above by b_{max}, we must be able to reach $|\text{max}\rangle$ from $|\text{min}\rangle$ by repeatedly applying J_+. Say we can reach $|\text{max}\rangle$ from $|\text{min}\rangle$ by n applications of J_+. This implies that

$$b_{\text{max}} = -b_{\text{max}} + n\hbar\,, \tag{19.43}$$

so

$$b_{\text{max}} = \frac{n\hbar}{2} = j\hbar\,, \tag{19.44}$$

with $j \in \frac{1}{2}\mathbb{Z}$. We can then read off the eigenvalues for the state $|j,m\rangle$,

$$a = \hbar^2 j(j+1)\,, \quad b = m\hbar\,, \tag{19.45}$$

and see that m can take any of the $2j+1$ values $-j, -j+1, \ldots, j-1, j$.