Lecture 6

Last time: quantized EM radiation field

$$\vec{A}(\vec{x},t) = \frac{1}{\sqrt{V}} \sum_{k,\alpha} c \sqrt{\frac{2\pi\hbar}{\omega}}\, \hat{\varepsilon}^{\alpha} \left[a_{k,\alpha} e^{i\vec{k}\cdot\vec{x} - i\omega t} + a^{+}_{k,\alpha} e^{-i\vec{k}\cdot\vec{x} + i\omega t} \right.$$

$$(\omega = |k|c, \qquad \vec{p} = \hbar\vec{k})$$

$$[a_{k,\alpha}, a^{+}_{k',\alpha'}] = \delta_{kk'}\delta_{\alpha\alpha'}$$

[Fock space: a^{+}'s on $|0\rangle$]

$$H = \sum_{k,\alpha} N_{k,\alpha}\hbar\omega = \sum_{k,\alpha} a^{+}_{k,\alpha} a_{k,\alpha} \hbar\omega$$

$\Downarrow$

From $\langle n'|a|n\rangle = \sqrt{n}\,\delta_{n',n-1}$

Absorbtion matrix element V_{fi} agrees with semiclassical expression where $A_0 \Rightarrow \frac{c}{\sqrt{V}}\sqrt{\frac{2\pi\hbar}{\omega}}\sqrt{n_{k,\alpha}}$.

For emission, use $\langle n'|a^{+}|n\rangle = \sqrt{n+1}\,\delta_{n',n+1}$

$\Downarrow$

Recall SHO matrix elements

$$\langle n' | a^+ | n \rangle = \sqrt{n+1}\ \delta_{n', n+1}$$
$$\langle n' | a | n \rangle = \sqrt{n}\ \delta_{n', n-1}$$

Can now compute matrix element for absorbtion/emission

Absorbtion:

$$V_{fi}^+ = \langle f; n_{k,\alpha} - 1 | -\frac{e}{mc}\, \vec{p}\cdot \varepsilon^{(\alpha)} \underbrace{\left[\frac{c}{\sqrt{V}}\sqrt{\frac{2\pi\hbar}{\omega}}\right]}_{A_0} a_{k,\alpha}\, e^{i\frac{\omega}{c}\hat{n}\cdot\vec{x}} | i; n_{k,\alpha}\rangle$$

agrees with semiclassical expression, where

$$A_0 \Rightarrow \frac{c}{\sqrt{V}}\sqrt{\frac{2\pi\hbar}{\omega}}\sqrt{n_{k,\alpha}}$$

Fits in with

$$\mathcal{U} = \frac{1}{2\pi}\frac{\omega^2}{c^2}|A_0|^2 \Rightarrow \int \mathcal{U} = \sum N_{k,\alpha}\, \hbar\omega = H$$

So: same absorbtion result as semiclassical approach.

Emission:

$$V_{fi} = \langle f; n_{k,\alpha} + 1 | \frac{-e}{mc}\vec{p}\cdot \varepsilon^{*} \left[\frac{c}{\sqrt{V}}\sqrt{\frac{2\pi\hbar}{\omega}}\right] a_{k,\alpha}^{+}\, e^{-i\left(\frac{\omega}{c}\right)\hat{n}\cdot\vec{x}} | i; n_{k,\alpha}\rangle$$

same as before, but

$$A_0 \rightarrow \frac{c}{\sqrt{V}}\sqrt{\frac{2\pi\hbar}{\omega}}\;\underbrace{\sqrt{n_{k,\alpha}+1}}_{\text{for emission formula}}$$

[note: $A \sim \sqrt{n}$ still in $\mathcal{U}$]

Correct emission rate.
agrees at large n, but allows spontaneous emission

6.9 E1 spontaneous emission

Spontaneous emission rate in dipole approximation:

$$n_{k,\alpha} = 0$$

$$\boxed{V_{fi}^{(E1)} = -\frac{e}{mc}\frac{1}{\sqrt{V}}\,c\sqrt{\frac{2\pi\hbar}{\omega}}\,\langle f|\hat{\varepsilon}\cdot\vec{p}\,|i\rangle}$$

generally,

$$\omega_{i,n;f,n+1} = \frac{2\pi}{\hbar}\frac{e^2}{m^2 V}\frac{2\pi\hbar}{\omega}_{(n+1)}\left|\langle f|\hat{\varepsilon}\cdot\vec{p}\,|i\rangle\right|^2 \rho(E)$$

Can compute $\rho(E)$ for photon of energy $E=\hbar\omega$, fixed polarization [HW]

$$\rho(E) = \left(\frac{L}{2\pi c}\right)^3 \frac{\omega^2}{\hbar}\,d\Omega$$

so for spontaneous single photon emission $\left(p = \frac{m}{i\hbar}[x,H]\right)$

$$dw = \frac{e^2\omega^3}{2\pi c^3\hbar}\left|\sum_i \hat{\varepsilon}_i^{(\alpha)}\langle f|x^i|i\rangle\right|^2 d\Omega$$

$$= \boxed{\frac{\alpha}{2\pi}\frac{\omega^3}{c^2}\left|\langle f|\hat{\varepsilon}\cdot\vec{x}\,|i\rangle\right|^2 d\Omega}$$

spontaneous E1 emission rate

selection rules for E1 transition

Since $\vec{X}$ is a first rank tensor,

$$\langle j_f, m_f \mid X^{(1)}_m \mid j_i, m_i \rangle \sim \langle j_f, m_f \mid 1, m; j_i, m_i \rangle \frac{\langle j_f \| X^{(1)} \| j_i \rangle}{\sqrt{2j_i+1}}$$

(Wigner - Eckart)

$$\Rightarrow \quad j_f = j_i \pm 1 \quad \text{or} \quad j_f = j_i,$$

$$j_i = 0 \not\Rightarrow j_f = 0.$$

Also: $P \vec{x} P = -\vec{x}$, so $|i\rangle, |f\rangle$ have opposite parity

$$P_i P_f = -1.$$

Example: Consider $2p \rightarrow 1s$ in Hydrogen

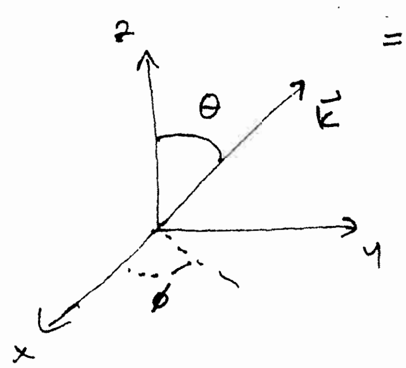
Angular distribution:

in HW did case $m_i = +1 \rightarrow m_f = 0$.

$$\vec{X} \sim \left(\frac{1}{\sqrt{2}} (Y_{1,-1} - \underline{Y_{1 1}}), \frac{1}{\sqrt{2}} (Y_{1 + 1} + \underline{Y_{1-1}}), Y_{10} \right)$$

found $\frac{dW}{d\Omega} \sim \frac{1}{2}\left(1 + \cos^2\theta\right)$

$$= \sum_\alpha \frac{1}{2}\left((\hat{\Sigma}^{(\alpha)}_x)^2 + (\hat{\Sigma}^{(\alpha)}_y)^2 \right)$$



Consider case $m_i = 0 \longrightarrow m_f = 0$

$$\frac{dW}{d\Omega} \sim \sum_{\alpha} \left(\hat{\varepsilon}_z^{(\alpha)}\right)^2 = \sin^2\theta$$

[Note: $\underbrace{\tfrac{1}{2}(1+\cos^2\theta)}_{m_i=1} + \underbrace{\tfrac{1}{2}(1+\cos^2\theta)}_{m_i=-1} + \underbrace{\sin^2\theta}_{m_i=0} = 2$

so isotropic photon distribution if start w/ uniform distributed state]

so

$$dW = \frac{\alpha}{2\pi}\frac{\omega^3}{c^2} |\langle f|z|i\rangle|^2 \sin^2\theta \cdot 2\pi \sin\theta \, d\theta$$

so spontaneous emission rate is

$$A = \int dW = \frac{4}{3}\frac{\alpha\omega^3}{c^2} |\langle f|z|i\rangle|^2$$

Note: same for $m = \pm 1$, since $\int_0^\pi \sin^3\theta = \frac{1}{2}\int_0^\pi \sin\theta(1+\cos^2\theta) = \frac{4}{3}$

Performing explicit calculation [HW]

$$A = 6.25 \times 10^8 \ s^{-1}.$$

Prob. state has decayed at time t is

$$|c_1|^2 = e^{-t/\tau}.$$

$$\tau = \frac{1}{A} = \text{mean lifetime} = 1.6 \times 10^{-9} \ s.$$