[Lecture 4]

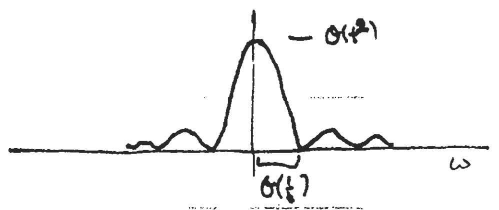
* Problem set 2. due Monday 2/25/02
  Problems 33, 35, 37, 38, 39 From Sakurai Chapter 5

Last time:

1st order TDPT for harmonic / constant $V(t)$.

For constant $V(t) = \hat{V}$, $\quad C_n^{(1)} = \frac{\hat{V}_{ni}}{\hbar\,\omega_{ni}}\left[1 - e^{i\omega_{ni}t}\right]$

$$P^{(1)}_{(i\to n)} = \frac{4\,|\hat{V}_{ni}|^2}{(E_n - E_i)^2}\,\sin^2\left[\frac{(E_n - E_i)\,t}{2\hbar}\right]$$
$$= |C_n^{(1)}(t)|^2$$



$\lim_{t\to\infty} \sim t\pi\,\delta(\omega)$

$\cancel{\lim_{\alpha\to\infty} \frac{\sin^2 \alpha x}{\alpha x^2} = \pi\,\delta(x)}$

Transition rate $\omega_{i\to n} = \frac{d}{dt}|C_n^{(1)}|^2 = \frac{2\pi}{\hbar}|\hat{V}_{ni}|^2\,\delta(E_n - E_i)$

(t big enough for EC, short enough, TDPT ok.)

Introduce density of states $\rho(E)$

$$\omega_{i\to[n]} = \frac{2\pi}{\hbar}\,\overline{|\hat{V}_{ni}|^2}\,\rho(E_n)\Big|_{E_n \simeq E_i}$$

final states w/ $E_n \simeq E_i$

Fermi's Golden Rule

$\left[\ \overline{|\hat{V}|^2} = \lim_{\Delta E\to 0} \frac{1}{\Delta E}\int_{-\Delta E/2}^{\Delta E/2} |\hat{V}_{ni}|^2\,dE_n \quad \text{if } \rho, \hat{V} \text{ smooth on families of final states}\ \right]$

Back to harmonic perturbation

$$V(t) = V e^{i\omega t} + V^{+} e^{-i\omega t}$$

$$C_n^{(1)} = \frac{1}{\hbar}\left[\underbrace{\frac{1-e^{i(\omega_{ni}+\omega)t}}{\omega_{ni}+\omega}}_{\text{peaked near } \omega = -\omega_{ni}} V_{ni} + \underbrace{\frac{1-e^{i(\omega_{ni}-\omega)t}}{\omega_{ni}-\omega}}_{\text{peaked near } \omega = \omega_{ni}} V^{+}_{ni}\right]$$

$\omega \cong -\omega_{ni}$: Stimulated emission

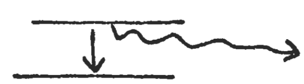
$$|C_n^{(1)}|^2 \approx \frac{4|V_{ni}|^2}{\hbar^2(\omega+\omega_{ni})^2} \sin^2\left[(\omega+\omega_{ni})^2 t/2\right]$$

Transition rate → state w/ energy E_n at large t

$$w_{i\to n} = \frac{2\pi}{\hbar}|V_{ni}|^2 \delta(E_n - E_i + \hbar\omega)$$

$$w_{i\to[n]} = \frac{2\pi}{\hbar}\overline{|V_{ni}|^2}\,\rho(E_n)\Big|_{E_n \cong E_i - \hbar\omega}$$

total emission rate

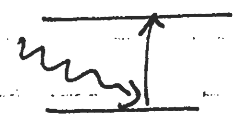


$\omega \cong \omega_{ni}$: absorbtion

$$|C_n^{(1)}|^2 \approx \frac{4|V^{+}_{ni}|^2}{\hbar^2(\omega-\omega_{ni})^2} \sin^2\left[(\omega-\omega_{ni})^2 t/2\right]$$

$$\omega_{i\to n} = \frac{2\pi}{\hbar} |V_{ni}^{+}|^2 \, \delta(E_n - E_i - \hbar\omega)$$

$$\omega_{i\to[n]} = \frac{2\pi}{\hbar} \overline{|V_{ni}^{+}|^2} \, \rho(E_n) \Big|_{E_n \cong E_i + \hbar\omega}$$

total absorbtion rate 

So - harmonic perturbation causes stimulated emission or absorbtion in units of $\hbar\omega$.

- Just what we expect if background made up of quanta w/ $E = \hbar\omega$!

For transitions to occur & satisfy energy conservation, must have

(a) final states exist over continuous energy range, to match $\Delta E = \hbar\omega$ for fixed perturbation frequency ω

-or-

(b) Perturbation must cover sufficiently wide spectrum of ω so that discrete transition with a fixed $\Delta E = \hbar\omega$ is possible.

- Note that spectral lines are not really sharp, due to decay processes.

Note: For two discrete states, $\omega_{i\to n}^{(abs)} = \omega_{n\to i}^{(em.)}$ in semiclassical calc. since $|W_{ni}|^2 = |W_{ni}^{+}|^2$

$\Rightarrow$ Detailed balance

[Really, only true @ $T = \infty$ when rad. field quantized]

Now: Emission & Absorbtion of EM radiation by atoms

6.5 Coupling to radiation field

Recall E & M

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \mu = 0,1,2,3$$

$$A_\mu = (-\phi, \vec{A})$$
$$x^\mu = (ct, \vec{x})$$

$$E_i = F_{i0} = -F_{0i} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \frac{\partial \phi}{\partial x^i}$$
$$B^i = \frac{1}{2}\varepsilon^{ijk} F_{jk} = \varepsilon^{ijk}\partial_j A_k$$

E, B unchanged under gauge xforms

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

For charged particle, spin $\vec{S}$,

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\phi - g_s \mu_B \frac{\vec{S}}{\hbar}\cdot(\vec{\nabla}\times\vec{A})$$

In free space (no sources) Maxwell is

$$\partial_\mu F^{\mu\nu} = 0$$

Choose Coulomb (radiation) gauge

$$A_0 = 0, \qquad \vec{\nabla}\cdot\vec{A} = 0.$$

transversality condition... (Lorentz gauge + get rid of A_0)

Fermi (1930) showed: [see Sakurai: "Advanced QM" for details]

Charged matter + EM fields can be described by [break $A = A_\perp + A_\parallel$]

instantaneous Coulomb interaction

$$H = \underbrace{\left[\frac{p^2}{2m} + V\right]}_{H_0} \underbrace{- \frac{e}{mc}\vec{p}\cdot\vec{A}_\perp}_{V(t)} + H_{RAD}^{(A_\perp)} + \frac{e^2}{2mc^2}A_\perp^2 - \frac{g\mu}{\hbar}S\cdot B$$

(ignore multi-photon processes, spin effects for now)

where $A_\perp$ is purely transverse field. ($\vec{\nabla}\cdot\vec{A}_\perp = 0$, $\vec{\nabla}\times A_\parallel = 0$)

6.6 Absorbtion cross-section

Maxwell eqns for transverse field (drop "$\perp$").

$$\Box A^i = \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)A^i = 0$$

Plane wave solutions

$$\vec{A} = 2A_0\hat{\varepsilon}\cos\left(\frac{\omega}{c}\hat{n}\cdot\vec{x} - \omega t\right)$$

where $\hat{\varepsilon}\cdot\hat{n} = 0$

Energy density

$$\mathcal{U} = \frac{1}{2}\left(\frac{E_{max}^2}{8\pi^2} + \frac{B_{max}^2}{8\pi^2}\right)$$
$$= \frac{1}{2\pi}\frac{\omega^2}{c^2}|A_0|^2$$

$$\vec{A} = A_0\hat{\varepsilon}\left[\underbrace{e^{i\left(\frac{\omega}{c}\right)\hat{n}\cdot\vec{x} - i\omega t}}_{\text{absorbtion}} + \underbrace{e^{-i\left(\frac{\omega}{c}\right)\hat{n}\cdot\vec{x} + i\omega t}}_{\text{emission}}\right]$$