6.10 Higher multipole transitions

For some transitions $i \to f$

$$\langle f | \hat{\varepsilon} \cdot \vec{p} | i \rangle = 0 .$$

So E1 transitions not allowed.

Examples:

a) $3d \to 1s$ in hydrogen

$[\cancel{m=2 \to m=0}$ $j_i = 2 \to j_f = 0 \Rightarrow \Delta j = 0$, not E1. also, $P_i P_f = +1]$

b) Hyperfine transition in hydrogen

$[P_i = P_f]$

Need to include higher-order terms in $e^{-i\vec{k}\cdot\vec{x}}$

$$e^{-i\vec{k}\cdot\vec{x}} = 1 - \underbrace{\boxed{i\vec{k}\cdot\vec{x}}}_{E2,\ M1} + \dots$$

For single photon emission

$$V_{fi} = +\frac{e}{m\sqrt{V}}\sqrt{\frac{2\pi\hbar}{\omega}}\, \langle f | e^{i\vec{k}\cdot\vec{x}}\, \vec{p} \cdot \hat{\varepsilon}^{(\alpha)} | i \rangle$$

$$(\vec{k}\cdot\vec{x})(\vec{p}\cdot\hat{\varepsilon}) = \frac{1}{2}\, k_i \hat{\varepsilon}_j \left[(x_i p_j - p_i x_j) + (x_i p_j + p_i x_j)\right]$$

$$= \frac{1}{2}\, k_i \hat{\varepsilon}_j \left[\underbrace{(x_i p_j - x_j p_i)}_{M1} + (x_i p_j + p_i x_j)\right]$$

(since $\vec{k}\cdot\hat{\varepsilon} = 0$)

M1 (magnetic dipole) decay

$$V_{fi}^{(L)} = \frac{ie}{mc\sqrt{V}}\sqrt{\frac{\pi\hbar\omega}{2}}\,\langle f|(\hat{k}\times\hat{\varepsilon})\cdot \vec{L}\,|i\rangle$$

Recall spin $\cdot$ B term in H_{int}

$$H^{(SB)} = -\frac{g\mu_B}{\hbar}\vec{S}\cdot\vec{B}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{i}{\sqrt{V}}(\vec{k}\times\hat{\varepsilon})\,c\sqrt{\frac{2\pi\hbar}{\omega}}\left[a_{k,\alpha}e^{i\vec{k}\cdot\vec{x}-i\omega t} - a^{+}_{k,\alpha}e^{-i\vec{k}\cdot\vec{x}+i\omega t}\right]$$

$$\Downarrow$$

$$V_{fi}^{(S)} = -\frac{ig\mu_B}{\sqrt{V}\,\hbar}\,c\sqrt{\frac{2\pi\hbar}{\omega}}\,\langle f|(\vec{k}\times\hat{\varepsilon})\cdot S|i\rangle$$

using $\mu_B = -e\hbar/2mc$

$$\boxed{V_{fi}^{M1} = \frac{ie}{mc\sqrt{V}}\sqrt{\frac{\pi\hbar\omega}{2}}\,\langle f|(\hat{k}\times\hat{\varepsilon})\cdot(\vec{L}+g\vec{S})|i\rangle}$$

Matrix element for M1 (magnetic dipole) transitions

<u>M1 selection rules:</u>

$\vec{L} + g\vec{S}$ is a vector operator

$$P(\vec{L}+g\vec{S})P = \vec{L}+g\vec{S},$$

so $j_f = j_i \pm 1$, or $j_f = j_i$, no $j_i = 0 \not\rightarrow j_f = 0$.

$$P_i P_f = +1$$

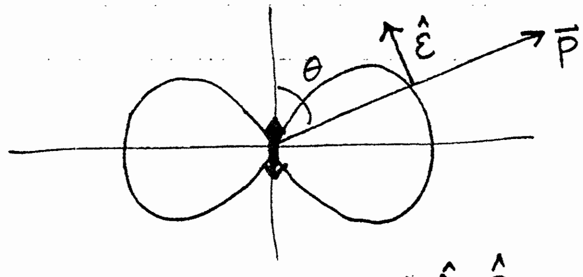
- Can use M1 rule to calculate hyperfine $F=1 \rightarrow F=0$ transition in hydrogen [HW] $(F = I + S)$

- Characteristic strength of M1 interactions

$$\frac{V^{(M1)}}{V^{(E1)}} \sim \frac{\mu_B}{e a_0} \sim \alpha \sim 10^{-2}$$

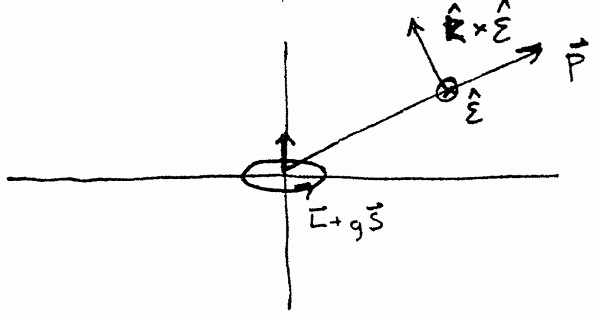
- Compare polarization of E1, M1 radiation

E1 $(\Delta m = 0)$



intensity $\sim \varepsilon_\theta^2 \sim \sin^2\theta$
$\sim (\varepsilon \cdot \langle\vec{x}\rangle)^2$

M1 $(\Delta m = 0)$



Same intensity $\sim \sin^2\theta$

(but down by α)

E2 (electric quadrupole) decay

Return to term $\frac{1}{2} k_i \hat{\varepsilon}_j (x_i p_j + p_i x_j)$

$$= \frac{m}{2i\hbar} k_i \hat{\varepsilon}_j \left(x_i [x_j, H] + [x_i, H] x_j \right)$$

$$= \frac{m}{2i\hbar} k_i \hat{\varepsilon}_j \left\{ x_i x_j H - H x_i x_j \right\}$$

so

$$V_{fi}^{(E2)} = \frac{ie}{m\sqrt{V}} \sqrt{2\pi\hbar\omega} \, \frac{1}{c} \, \frac{m\hbar\omega}{2i\hbar} \hat{k}_i \hat{\varepsilon}_j \langle f | x_i x_j | i \rangle$$

$$\boxed{V_{fi}^{(E2)} = \frac{e\omega}{c\sqrt{V}} \sqrt{\frac{\pi\hbar\omega}{2}} \langle f | \hat{k}_i \hat{\varepsilon}_j \left(x_i x_j - \tfrac{1}{3}\delta_{ij} x^2 \right) | i \rangle}$$

matrix element for E2 (electric quadrupole) transitions

- Can use to calculate $3d \to 1s$ emission [HW]

- Characteristic strength

$$\frac{V^{(E2)}}{V^{(E1)}} \sim \frac{\omega a_0}{c} \sim \alpha \sim 10^{-2}$$

- operator $x_i x_j - \frac{1}{3}\delta_{ij} x^2$ is spin 2 tensor operator

selection rule: $|j_f - j_i| \leq 2 \leq j_i + j_f$

Higher multipoles

Can expand $e^{-i\vec{k}\cdot\vec{x}}$ further ...

Better approach: Vector spherical harmonics

Basic idea:

solve wave equation

$$\nabla^2 \vec{A} - \frac{1}{c^2}\ddot{\vec{A}} = 0$$

in spherical coordinates,

Classify solutions under representations of SO(3) generators

$$\vec{J} = \vec{L} + \vec{S}$$

$$\vec{L} = (\vec{X} \times \vec{p})$$

$$(\vec{S}\cdot\vec{v}) = i\hbar\,\vec{v}\,\times$$

simultaneously rotates vector $\vec{A}$, coordinates.

$S^2\vec{A} = 2\hbar\vec{A}$, since photon has spin 1.

Two types of solutions

$$A^{(e)}_{LM}(r,\theta,\phi): \text{ no radial cpt. to } \vec{B}$$
$$A^{(m)}_{LM}(r,\theta,\phi): \text{ no radial cpt. to } \vec{A}.$$

Can express in terms of Bessel fns, $Y_{LM}(\theta,\phi)$.

Give wavefunction of photons emitted by multipole transitions. [e.g. $A^{(e)}_{11}$ for $2p \to 1s$ [illegible]]
[for more: see nucl. theory texts]