PS 2 due Mon Feb 24

<u>Lecture 5</u>

Last time: [abs & em. by atoms]

$$H = \underbrace{\left[\frac{p^2}{2m} + V\right]}_{H_0} - \underbrace{\frac{e}{mc}\vec{p}\cdot\vec{A}}_{V(t)} + \cancel{H_{RAD}, A^2, S\cdot B}$$

$$\vec{A} = A_0\hat{\varepsilon}\left[\underbrace{e^{i(\frac{\omega}{c})\hat{n}\cdot\vec{x} - i\omega t}}_{abs} + \underbrace{e^{-i(\frac{\omega}{c})\hat{n}\cdot\vec{x} + i\omega t}}_{em.}\right)$$

$$\sigma_{abs} = \frac{\hbar\omega\, W_{i\to n}}{c\mathcal{U}}$$

$$= \frac{4\pi^2\hbar}{m^2\omega}\underbrace{\left(\frac{e^2}{\hbar c}\right)}_{\alpha \simeq 1/137}\left|\langle n|e^{i(\frac{\omega}{c})\hat{n}\cdot\vec{x}}\,\hat{\varepsilon}\cdot\vec{p}\,|i\rangle\right|^2 \delta(E_n - E_i - \hbar\omega)$$

For absorbtion:

$$\omega_{i\to n} = \frac{2\pi}{\hbar} |V^{+}_{ni}|^2 \, \delta(E_n - E_i - \hbar\omega)$$

$$V^{+}_{ni} = \langle n | -\frac{e}{mc} \vec{p}\cdot\vec{A}_{abs} | i \rangle$$
$$= -\frac{eA_0}{mc} \langle n | e^{i(\frac{\omega}{c})(\hat{n}\cdot\vec{x})} \hat{\varepsilon}\cdot\vec{p} | i \rangle$$

Absorbtion cross-section:

$$\sigma_{abs} = \frac{\text{Energy absorbed per unit time}}{\text{Energy flux}}$$

$$= \frac{\hbar\omega \, \omega_{i\to n}}{c\,\mathcal{U}}$$

$$= \frac{4\pi^2\hbar}{m^2\omega} \underbrace{\left(\frac{e^2}{\hbar c}\right)}_{\alpha = 1/137} |\langle n | e^{i(\frac{\omega}{c})\hat{n}\cdot\vec{x}} \hat{\varepsilon}\cdot\vec{p} | i \rangle|^2 \delta(E_n - E_i - \hbar\omega)$$

(units L^2)

Emission probability: same as absorbtion in semiclassical picture (detailed balance)

Dipole approximation:

if $\lambda = \frac{2\pi c}{\omega} \gg R_{atom}$, can

can neglect subleading terms in $e^{i(\frac{\omega}{c})\hat{n}\cdot\vec{x}} = 1 + i\left(\frac{\omega}{c}\right)\hat{n}\cdot\vec{x} + \ldots$

e.g. for Hydrogen:

$$E_I \cong \frac{me^4}{2\hbar^2} \cong 13.6 \text{ eV}$$

$$a_0 \cong \frac{\hbar^2}{me^2} \cong 0.52 \text{ Å} \qquad \text{(Bohr radius)}$$

$$\Delta E \cong \frac{me^4}{2\hbar^2} = \hbar\omega$$

$$\Rightarrow \quad \omega = \frac{me^4}{2\hbar^3} = \alpha \frac{mce^2}{2\hbar^2} = \frac{\alpha}{2}\frac{c}{a_0} \qquad \left(\alpha = \frac{e^2}{\hbar c} \cong \frac{1}{137}\right)$$

$$\Rightarrow \lambda = \frac{2\pi c}{\omega} \cong \frac{4\pi}{\alpha} a_0 \gg a_0$$

so dipole approx. is valid

- generally good for atoms with small Z.
- doesn't work for processes in which E1 (electric dipole) transitions not possible.

So:

$$\langle n| e^{i\left(\frac{\omega}{c}\right)(\hat{n}\cdot\vec{x})} \hat{\varepsilon}\cdot\vec{p}\,|i\rangle \longrightarrow \hat{\varepsilon}\langle n|\vec{p}\,|i\rangle$$

assume wlog $\hat{\varepsilon} = \hat{x}$, $\quad \hat{n} = \hat{z}$

need

$$\langle n|p_x|i\rangle = \frac{m}{i\hbar}\langle n|[x, H_0]|i\rangle$$

$$= i m\,\omega_{ni}\langle n|x|i\rangle$$

$$\boxed{\sigma_{abs} = 4\pi^2\alpha\,\omega_{ni}\,|\langle n|x|i\rangle|^2\,\delta(\omega-\omega_{ni})}$$

For electric dipole [illegible]

6.7 Photoelectric effect

Consider ejection of electron by rad. field (ionization)

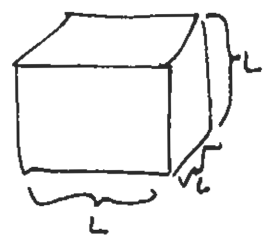
$|i\rangle$ = bound atomic state

$|n\rangle$ = continuum state: plane wave $|p\rangle$

Need to know density of states $\rho(E)$

2 ways to calculate:

a) Box normalization



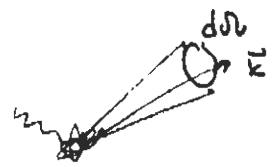
$$\langle \vec{x} | \vec{p} \rangle = \frac{e^{i\vec{k}\cdot\vec{x}}}{L^{3/2}} \qquad \vec{p} = \hbar\vec{k}$$

$$k_i = \frac{2\pi n_i}{L}, \quad n_i \in \mathbb{Z}, \quad i = 1,2,3 \quad (x,y,z)$$

$$E = \frac{\hbar^2 k^2}{2m_e} = \frac{2\pi^2\hbar^2 n^2}{m_e L^2}$$

$$n^2 = n_1^2 + n_2^2 + n_3^2$$

$$dE = \frac{4\pi^2\hbar^2 n\,dn}{m_e L^2}$$



Choose solid angle $d\Omega$

$$dN \cong d\Omega\, n^2 dn = \frac{m_e L^2}{4\pi^2\hbar^2}\, n\, d\Omega\, dE$$

$$= \left(\frac{L}{2\pi}\right)^3 \frac{mp}{\hbar^3}\, d\Omega\, dE$$

$$\rho(E) = \frac{dN}{dE} = \left(\frac{L}{2\pi}\right)^3 \frac{mp}{\hbar^3}\, d\Omega$$

$$|\langle p|V|i\rangle|^2 \rho(E) = \left|\int e^{-ip\cdot x/\hbar} V \psi_i\right|^2 \frac{mp}{(2\pi\hbar)^3}\, d\Omega$$

[note: L dependence cancels]

b) Continuum normalization

$$\langle \vec{x}|\vec{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{p}\cdot\vec{x}/\hbar}$$

$$\sum_n |\langle n|V|i\rangle|^2 \delta(E_n - E)$$

$$\rightarrow \int d^3\vec{p}\, |\langle\vec{p}|V|i\rangle|^2\, \delta\!\left(\frac{p^2}{2m} - E\right)$$

$$= \int d\Omega \int p^2 dp\, |\langle\vec{p}|V|i\rangle|^2 \left(\frac{m}{p}\right) \delta(p - \sqrt{2mE})$$

$$\left[\delta(f(p)) = \frac{\delta(p)}{|f'(p)|},\ f(0) = 0\right]$$

$$= d\Omega\, mp\, |\langle p|V|i\rangle|^2$$

$$= \left|\int d^3x\, e^{-i\vec{p}\cdot\vec{x}} V \psi_i\right|^2 \frac{mp}{(2\pi\hbar)^3}\, d\Omega$$

For photoelectric effect:

$$\frac{d\sigma}{d\Omega} = \frac{4\pi^2 \alpha \hbar}{m^2 \omega} \cdot \frac{mp}{\cancel{(2\pi\hbar)^3}} (\hat{\varepsilon}\cdot\vec{p})^2 |\langle \vec{p}|0\rangle|^2$$

(E1 approximation)

$$= \frac{32\, e^2 p\, (\hat{\varepsilon}\cdot p)^2}{m c \omega \hbar^3 a_0^5} \frac{1}{(1/a_0^2 + p^2/\hbar^2)^4}$$

[Fourier xform: homework]

6.8 Quantization of transverse EM field

Computed absorbtion & emission semiclassically
— results proportional to incoming radiation flux

OK for absorbtion, stimulated emission in strong fields

Clearly fails for spontaneous emission.

For better understanding: quantize EM field

Quantization of radiation field (skipping subtleties)

Write

$$A(\vec{x},t) = \frac{1}{\sqrt{V}} \sum_{k,\alpha} c \sqrt{\frac{2\pi\hbar}{\omega}} [a_{k,\alpha} \varepsilon^\alpha e^{i\vec{k}\cdot\vec{x} - i\omega t} + a^*_{k,\alpha} \varepsilon^\alpha e^{-i\vec{k}\cdot\vec{x} + i\omega t}]$$

$$\omega = |k| c$$

Hamiltonian is

$$H = \frac{1}{8\pi^2}\int (B^2 + \epsilon^2)\, d^3x$$

$$= \frac{1}{2\cancel{8\pi}} \sum_{k,\alpha} \left(a^*_{k,\alpha}\, a_{k,\alpha} + a_{k,\alpha}\, a^*_{k,\alpha} \right) \hbar\omega$$

Hamiltonian of a system of uncoupled oscillators

Quantize: $a, a^* \longrightarrow a, a^+$ operators

$$[a_{k,\alpha}, a^+_{k,\alpha}] = \delta_{kk'}\, \delta_{\alpha\alpha'}$$

Number operator: $N_{k,\alpha} = a^+_{k,\alpha}\, a_{k,\alpha}$

$$H = \sum_{k,\alpha} N_{k,\alpha}\, \hbar\omega$$ (dropping infinite contribution $\Sigma\, \hbar\omega/2$ to vac. energy – relevant for Casimire energy.)

Hilbert space:

Fock space built by acting with a^+'s on vacuum

$|0\rangle$ = vacuum, $a_{k,\alpha}|0\rangle = 0 \quad \forall\, k,\alpha$

$a^+_{k,\alpha}|0\rangle$ = 1-photon state

$a^+_{k,\alpha}\, a^+_{k',\alpha'}|0\rangle = a^+_{k',\alpha'}\, a^+_{k,\alpha}|0\rangle$ = 2 photon state

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