[SQUEAKING]

[RUSTLING]

[CLICKING]

HONG LIU: So last time, we started talking about the quantization of the Dirac theory. And so, if we start with the Lagrangian density of the Dirac theory, then we can isolate the time dependent-- the part with only time directive and has the following form.

And then we interpret this as the canonical momentum for psi. And then we have pi psi. And then this one we just interpret as the Hamiltonian, Hamiltonian density.

And then we can just proceed to quantize the theory. When you write down the canonical quantization condition, and then we expand psi in terms of complete sets of solutions, et cetera.

And then we recognize if we do it in the standard way, say to impose the commutator relation, and then the Hamiltonian actually is unbounded from below. So, actually, the total energy can be as negative as-- yeah, can be x active-- yeah, it can be arbitrarily negative. And so, that theory does not make sense.

And, also, if we use the standard way to quantize it, we would get a theory which, when you exchange two particles, which they commute, which are symmetric. And so, that looks like both a bosonic theory rather than a fermionic theory.

And then, there's a simple fix. So the fix is that instead of imposing the standard quantization-- standard commutation relation, we impose the following one. So we replace everywhere the commutator by anticommutator.

So what we do, say, for the commutation relation, we consider psi alpha px and then pi beta px prime. So the anticommutator is equal to i delta alpha beta and the 3. We impose this condition.

So which is the same if we plug in this pi just given by i psi dagger, and then that just gives you psi alpha px times psi beta dagger px prime, gives you delta alpha beta delta 3 x minus x prime.

So this equation-- so one thing to notice-- so the commutator-- the anticommutator is symmetric between these two because the now is the plus rather than minus.

And so, note, so this commutation relation is self-consistent. So if we call this equation one, so if one is self-consistent at x equal to x prime.

So if you take these two at the same point-- sorry, so this should be prime. So if you take these two at the same point, so the left-hand side, you have psi alpha and psi beta. So you just have psi dagger.

So the left-hand side is a positive definite quantity. You have psi dagger. And the right-hand side is also a positive definite quantity. And you have this delta function evaluated at 0 even though it's a divergent. And so, both sides are positive definite.

So the fact that this is self-consistent, if you trace it back, has to do with the psi-- this pi is actually equal to i psi beta. So if it's equal to minus i psi-- so if pi is equal to i psi dagger.

So if it is equal to minus i psi dagger, and then, of course, you will have a contradiction because the left-hand side will be positive or the right side sign will be negative because you will change the sign. And that sign then trace back to this sign.

So that's the reason actually we want to have this sign here. So the sign here is for a reason. Even though, classically, when you write it down, it doesn't matter what's the-- yeah-- it was not important what the sign is. But quantum mechanically it's actually important. So this comes from-- so the-- yeah, so the sign in L important.

Good. Any questions on this? Yes.

AUDIENCE: Is that the [INAUDIBLE] when you replace the commutator with the anti commutator? Is that the generic feature when we describe things in harmonic series?

HONG LIU: Yeah. I'm going to mention that. Yeah. I'm going to mention that. Yes.

AUDIENCE: So in quantum mechanics, the fundamental commutation relation tells you that x and p don't commute. But in the classical limit, h bar goes to 0, they do.

HONG LIU: Yeah.

AUDIENCE: And that's a very physically intuitive idea.

HONG LIU: Right.

AUDIENCE: But then, here, h bar goes to 0 just tells you something about the anticommutator. So how do we determine the classical?

HONG LIU: Yeah. So the classical limit, yeah, we will actually-- this is a very good question we will discuss later. The classical limit, they become anticommute. Yeah, in the classical limit, essentially they become a anticommute variables.

Yeah. And this is crucial when we later describe path integral. We will discuss this in detail later. Other questions? Yes.

AUDIENCE: So can you say anything about the commutation relation between them, or can you not impose it?

HONG LIU: Oh, yeah. You can talk about the commutation relation between them, just you cannot impose them.

AUDIENCE: But does it-- do you get any physical meaning from the commutation relation between two objects, or--

HONG LIU: Yeah. These are two operators. You can always consider whatever quantities about them.

AUDIENCE: There isn't a general--

HONG LIU: Yeah, but there's no general rule. Yeah. So the general rule, we impose for the anticommutator. And so, and in the second step, we just expand the psi in terms of complete sets of modes.

And so, in the second step, as we do before, we expand the psi x in terms of complete set of modes. So we have already worked out last time, times-- so we have ak, s for the u s. Then we have expansion ikx.

And then, similarly, with the complex scalar convention-- so now let me call this c. Just last time I used b, but it's often conventional to use c. And I put the dagger here. It doesn't matter. It's not so important whether you put dagger here. So but it's a convention.

So this is our complete sets of solutions. And then we expand psi in terms of arbitrary coefficients in terms of them. And since psi is generally complex, so these two don't have to be the same. So these two, yeah, this is a complex scalar case. They don't have to be the same.

And so, quantum mechanically, then a and c, they are operators. A and c, they are operators. So if I call this equation two, and then if we plug two into one, then we can get to the commutators. We can get the anticommutators between a and c.

And so, so we find that, for example, ak r, ak prime s dagger equal to ckr and ck prime s dagger, now it's equal to 2 pi cube delta 3 k minus k prime delta rs.

And with all other anticommutator, so emphasize anticommutators zero. And, yeah, so the story is similar as before, except you just everywhere replace the commutator by anticommutator.

And then, the vacuum, we can still define the vacuum. You can still divide-- later, we will see that this is justified. So we can still define the vacuum as annihilated by ak and ck for any k and s.

So we define the vacuum. The reason this is the vacuum is because, now, if you find the Hamiltonian, so you can plug this in into your expression for the Hamiltonian.

Because they just expressed in terms of psi, then you can just work it out, just straightforward as we did previously for the scalar field. Just now the algebra is slightly more complicated because you have-- yeah.

So now you get omega k. And then you have s 1, 2. And then you have aks dagger aks plus cks dagger cks, then plus some constant. Just as in the scalar case, we always have some zero point energy. We always have some zero point energy.

And so now you see, if you define-- so now you see the Hamiltonian is positive definite. The Hamiltonian is positive definite. And because we have a plus sign here rather than the minus sign if we do the commutation relation.

And then, since this is positive definite, and then justify defining this 0 as the vacuum. Because this state, indeed, has the lowest energy given by this E0. And, yeah, E0 as in the bottom-- in the previous scalar case-- would be divergent. It would be divergent. Any questions on this? Yes.

AUDIENCE: On that line, what is the coefficient of the exponential? It's minus i k x?

HONG LIU: Oh, right. Sorry. Yeah. I did. Good. Thank you. Good.

So let me make some comments on this E0. So, again, the quantization-- yeah, again, the quantized field, as in the scalar case, essentially consists of an infinite number of harmonic oscillators.

So now I need to put the harmonic oscillator by codes. Because now these harmonic oscillators, they defined in terms of the anticommutators rather than commutators. And so, very important feature.

And, normally, we call them fermionic for reasons will be clear very soon. So this is the first note-- for a reason will be very soon and the-- why we call them fermionic oscillator.

And so, we have for each k and the value of the s, and you have an oscillator. And then you have oscillator generated by a and also have oscillator generated by c.

So each fermionic oscillator-- so this is like the harmonic oscillator. They contribute to E0, to this ground state energy. So, remember, previously, for the both-- for the scalar case, each oscillator contributes 1/2 omega k. Remember.

So now, in this case, they contribute by minus 1/2 omega k. It's actually opposite sign for the standard harmonic oscillator. In the standard harmonic oscillator, it's 1/2 omega, and here it's minus 1/2 omega.

So let me just also saying why we call it a-- so a feature of this fermionic harmonic oscillator is that, say, you can act, you can create a state-- yeah.

So let me just make some general remarks. Say if you have a harmonic oscillator satisfy-- yeah-- satisfy this kind of commutation relation. Let's just look at one set of them.

So one side of them, just take any side with any choice of k and s, and then they satisfy that relation. So this defines fermionic harmonic oscillator.

And so, interesting feature of this oscillator is that you can only excite them once. Because if you have a dagger acting on 0, now if you have a dagger square acting on 0, then this is 0. So the only state you have-- you only have two states. One is 0 and the one is a dagger 0.

It's because the anticommutator of a dagger itself is 0. So this just tells you that a dagger squared is equal to 0. And, also, a squared is equal to 0. And so, you act twice. And this is precisely the Pauli principle, and the two particle on the same state, and you get identically 0.

So if you act any of k here, a, k, and s, if they are the same, if you act them twice, you get 0. So if you have two particles in the same state, when k and s is the same, then it means that all their quantum numbers are the same.

And so, this is the Pauli principle. So this is the Pauli principle. So no two particle can be in the same states.

So here, then we conclude that a dagger, aks and cks dagger, they create fermions. They create particles which obey the Pauli principle.

Good? Any questions on this? Good.

So now we can define-- as before, we can define single particle states. So the single particle states, say, you just act them once.

So, again, we define them by-- with this normalization as in the scalar case. And then a s aks dagger on 0. And then we define ks bar to be the one created by c.

So as before, in the scalar case, we can interpret this k as the momentum. You can check explicitly, as we will describe, that the k give you the momentum of the-- yeah, goes back into the momentum eigenvalue.

And then, the s can be interpreted as some kind of polarization. So yeah. And yeah, so this we call particle. So we'll call this particle. And we will call this antiparticle.

And so, each particle has two polarizations. Since s equal to 1, 2. And they are all satisfied, and the k square, of course, all satisfied minus m squared. good? Any questions on this?

AUDIENCE: Is s and s bar different?

HONG LIU: Hmm?

AUDIENCE: Is s and s bar different?

HONG LIU: Is s and s bar different? Yeah, because one is created by a, and one is created by c. So we just use a bar to distinguish them. Yeah, this one corresponding to a polarization of this particle. And one bar corresponding to the polarization of the antiparticle. Good?

Yeah, so you can also find the Noether charge. Say you can find the Noether charge for translation. So we already talked about momentum-- Hamiltonian.

You can also find the Noether charge for spatial translation. And then you just find the P, capital P, is equal to-- yeah, so this is, yeah, you can just similarly work it out.

So, actually, in your pset, you will work out the stress tensor. So you find that the loss of charge for the spatial translation is given by this. And then, if you plug it in, and then you get-- and then, again, you have this a dagger.

Yeah, to save the effort, I just say you have a dagger, a and c, dagger c. So you can see, immediately, these two will be a eigenstate, say, of the momentum operator with eigenvalue k. With eigenvalue k. So that justifies that these are momentum-- these are the momentum eigenstates.

So we can also find we can also define normalization. We define the normalization, yeah, for those-- you can check the normalization for those state. So these are the plane wave. So you can check just by using commutation relation. And this is given by omega k. That's 2 pi cubed.

And, similarly, for the antiparticle, you have the same thing. Same thing. But if you take the overlap between the particle and the antiparticle, you always get the 0, because the commutator between a and c is always 0.

Take k, r, and the k prime. S bar is always 0. They are always orthogonal.

So let us make some further notes here. So since we have two components, since each particle a fermion. So we already said there should be a fermion. And it has two components.

So, again, we guessed they must be spin 1/2 particle. We guess they must be spin 1/2 particle. And so, you can check explicitly. Yeah, so you can guess that this would be spin 1/2 particle. But you can check explicitly.

Indeed, the eigenvalues are-- so you can construct angular momentum operator because this series, Lorentz invariance, so you can construct the Noether charge associated with the Lorentz transformation. And then you can construct the angular momentum operator.

And then you check. And then, indeed, so you find that indeed the ks and k bar-- ks bar has eigenvalue of spin half particles.

So this is you can check explicitly. Good. Any questions on this? And this will be in your pset. This is in your pset.

And this is a little bit non-trivial calculation. But it's an instructive one. So when you see from your hand-- you see it with your own eye and from your own calculation, that this is spin 1/2 is satisfying. Good. Any questions on this?

So, more generally, so we will not be able to-- we will not certainly prove here. So, more generally, you can prove there exist a so-called spin statistical theorem.

So a half integer spin field-- yeah. So half integer spin fields can only be quantized using anticommutation relations-- anticommutators. Not only spin half, say 3/2, 5/2, et cetera.

So Dirac field is the simplest of them. And in the anticommutation relation, and so obey this so-called the Fermi-Dirac statistics. So they obey the Pauli principle.

Yes, in statistical physics, which when you exchange the wave function, you get the minus sign, it's called the Fermi-Dirac statistics. And in contrast, if you have integer spin, say like a scalar or like a photon, they have integer spin, a photon have spin one. And then you can quantize them using commutators.

So for integer spin, and then quantized-- can be consistently quantized using commutators. So, in this case, you get the Bose-Einstein statistics. Means when you exchange the wave function of the particles, the wave function remains-- when you exchange the particles, the wave function remains symmetric.

And so, this is very general. This is very general. Good. Any questions on this? Yes.

AUDIENCE: So are there other relationships we use to quantize a field here?

HONG LIU: Yeah, that's a very good question. So in 2 plus 1 dimension, very special, 2 plus 1 dimension, so there are more general statistics than Bose-Einstein-- than bosons and fermions. There are things in between called anyons, yeah. And they play a very important role in condensed matter physics, yeah.

The anyon, actually, was proposed by our colleague, Frank Wilcock, in early '80s. And, yeah, at the beginning, it sounded like a fantasy. But then later actually find many important applications in condensed matter physics. Yes.

AUDIENCE: What rule we've seen so far tells us that we should interpret the a and the c operators as particles and antiparticles?

HONG LIU: Yeah, it's just convention. It doesn't matter. You can-- as far as they're anti each other.

AUDIENCE: I guess, what indicates they are--

HONG LIU: Good. That's what we are going to talk about. Other questions? Yes.

AUDIENCE: So the spin 1/2 here comes from the possible value of s, which comes from solving the classical Dirac equation?

HONG LIU: Yeah, so spin 1/2, you can guess it from the two components. But for this one, you can just work out the eigenvalues. You will see, it's 1/2. Yeah.

AUDIENCE: So when we generalize it to more-- like more half spin [INAUDIBLE].

HONG LIU: Yeah.

AUDIENCE: We have more components?

HONG LIU: Yeah, you will have more components, yeah.

AUDIENCE: And we will be--

HONG LIU: Yeah, you will have more components, yeah. Other questions? Yes.

AUDIENCE: You say that the energy is positive, positive definite just because you got rid of the negative sign. But how do you know that these new operators, like c is going to behave like you want it to? I guess, because we don't know the spectrum.

HONG LIU: What do you mean?

AUDIENCE: The eigenvalues of c [INAUDIBLE].

HONG LIU: No, we know everything about-- we know everything about them once I specifier the anticommutator, everything is fixed. Then everything is fixed, the spectrum is fixed. The energy spectrum just follow from here.

So each a or c just can create the one particle. Yeah. So the spectrum is completely fixed, the energy is fixed. Yeah, we know everything about them. Yes.

AUDIENCE: Can you have a theory that is spin half particles but no antiparticles?

HONG LIU: Oh, yeah, that's a very good question. And so, that's the Holy Grail of neutrino physics and also in condensed matter physics or quantum computation.

So people have been looking for these Majorana fermions, which we will talk about later. And people have been looking for these Majorana fermions, which is like a real scalar. It's the counterpart of the real scalar.

Here it's more like a complex-- the counterparts. Yeah, here is more like a complex scalar. But you can actually-- we will talk about it. We'll be able to define something like a real fermion and then its own antiparticle. Yeah. Yes.

AUDIENCE: Can you write down something that looks analogous to the Dirac equation for spin 3/2 or 5/2?

HONG LIU: Yeah, you can write it down. Yeah.

AUDIENCE: You can find those?

HONG LIU: Yeah, I think you can write down-- I think people are by now have written down equations for any, yeah. Yeah, in principle, you can write it down for any of them. Yeah, in the end, it boils down to group theory. Yeah. Yes.

AUDIENCE: I don't know if this makes sense. But the particles that are excited, the excitations of this field, will they still obey the uncertainty relation given that we've gotten rid of position momentum and commutation relations?

HONG LIU: So even in the boson case, our commutation relation is not related to position and momentum. There is the field with its conjugate. There's no position operator anymore.

AUDIENCE: How do you get a single particle wave function from a [INAUDIBLE] a single particle wave equation?

HONG LIU: Yeah, in the non-relativistic limit, yeah, you have to take the non-relativistic limit. Yeah. Good. Other questions? Yes.

AUDIENCE: In step two, how do we know that all the solutions that we listed out form a complete set of all the solutions?

HONG LIU: Yeah, because we have solved the Dirac equation. Dirac equation is a linear equation. And, yeah, you solved it, then you know you have found all the solutions.

Yeah, because we do a Fourier transform, convert it into an algebraic equation. The algebraic equation we know when we find all the solutions. Yeah, so those things we can know for full confidence. Good.

So now let's explain in what sense we call one of-- called the particle and antiparticle, in what sense they are anti each other. So this is very similar to in the boson case when you have a complex scalar.

And, remember, in the complex scalar case, the reason we call one particle to be a particle, the other to be antiparticle, is they have the same mass, the same spin, spin 0 in that case, but opposite charge. So here we can also define a charge for them.

Now let's talk about charge. So remember, in your pset, when we first talk about Dirac equation, in your pset you derived that if you treat the Dirac equation as a wave equation, then you can derive a probability current. And then, the zeroth component of that current is actually positive definite. Remember?

So there, when we treat the Dirac equation as a wave equation. And then we find-- from what we did with Schrodinger equation, similarly from what we did with the Schrodinger equation, and we can derive a equation like partial mu, j mu equal to 0.

And the j mu-- yeah, so up to a sign. So let me just choose a minus sign here. So j mu is given by this. Using our current notation. Of course, at the time, we don't know those. Yeah, do we know the bar? I forgot.

Anyway, so you have this j. You can derive this j. And the zeroth component of the j, you just psi dagger psi, which is manifestly positive as a classical function.

And so, I mentioned that this was the Dirac's main motivation for writing down Dirac equation is to look for probability current, which is the positive probability. And so, this is positive definite classically. Yeah, when you treat it as a wave equation.

So now we are-- but now we don't think Dirac equation as a wave equation, we treat it as a field theory. So as a field theory, and then Dirac equation-- so we have the action.

So this action has obvious symmetry similar to the complex boson case. So this, you can rotate psi by a phase, by a constant phase. Then psi bar goes to minus alpha psi.

And so, obviously, this is the invariant if alpha is a constant. And so, this is a u1 symmetry. So when you have a symmetry by rotation by a face, and it's a U1 symmetry.

And so, from the Noether theorem, then there must be a conserved current corresponding to this phase rotation. So without doing any calculation, you should already be able to guess that this is the current.

So this now becomes the Noether current. So it's actually this, if you find the Noether current, you just find this one. Just find this one.

So, now, instead, we interpret J0-- rather than interpret this as a probability current and this as a probability density, if you treat it as a wave equation. And here we just interpret it as the conserved current for this U1 symmetry. And you have some conserved current. And then, this is corresponding to the zeroth component of this conserved current.

And so, this is-- we interpreted as some kind of charge density as we did for the scalar case. Now this is just interpreted as the-- so now, now you can just work it out, what's this, and then the Q.

So the Q defined this way. So Q is the total conserved charge, which is you integrate j0 over all space. And then just equal to that. So, naively, this is a positive definite quantity.

But, actually, now if you plug in the -- the expression for psi, then what you find is the following. You find the Q have the following form.

Now minus cks dagger, cks, and then plus the infinite constant. Plus the infinite constant. So, yeah, so we will define the Q, the quantum operator Q, by forgetting about this infinite constant.

So now you see something very interesting happens. So, yeah, so if we set this constant, yeah, they call this-- let me just call this Q0. And the Q0 is infinite, just as you have a constant energy for the-- just have E0, which is infinite, and here is also infinite.

But we will define the quantum version. But we can just define the Q just to include this part. And then, by definition-- so if I define the Q just by that part, the Q acting on the vacuum just give you 0 because I have c and a on the other side. So it means that the vacuum have 0 charge.

Good. But now you see something interesting happens. So do you observe something interesting here? Yes.

AUDIENCE: It should be [? posted ?] then and shown like this. But here, if we define [INAUDIBLE].

HONG LIU: That's right. Good. So naively, this quantity is infinitely-- naively, this quantity is positive definite. But if we define it in the way-- but this quantity by itself is divergent. When we say something is positive definite, if it's divergent, then it does not mean very much.

So now, when we want to make it finite, the way we make it finite is we define this, so that when you act on the vacuum, so that it's 0, so that the vacuum has 0 charge. So when we throw away this infinite constant, and then, actually, this can be either positive or negative. This can be either positive or negative.

And, yeah, so this is the magic by throwing away some constants, throwing away infinite constants. And you can make a positive quantity into an arbitrary sign.

But this is a good thing. So now, if you look at-- when you act on the a and c, because of this sign difference, when you find that when you act on ks, then you get eigenvalue 1. When you act on ks bar, you get eigenvalue minus 1.

So this has charge 1. So this has charge minus 1. So you see that this particle and antiparticle, everything else is the same. They have the same mass. They have the same spin, spin half, but they only differ charge by opposite charge. So that's why we call one of them is particle and one of them antiparticle. Yes.

AUDIENCE: Can we interpret this just like the probability charge?

HONG LIU: No, no, no. This is not probability, this is just charge. There's no probability interpretation anymore. Probability is only when you treat it as a wave function-- a wave equation.

But we don't treat it as a wave equation. We treat it as a field theory. So in the field theory, and this is just some charge particles can carry. Yeah, you have nothing to do with probability. Yes.

AUDIENCE: Is the fact that Q not is positive break any symmetry between particle and antiparticle?

HONG LIU: What do you mean by--

AUDIENCE: So just right there, Q not goes to just a positive infinity.

HONG LIU: Yeah.

AUDIENCE: Does it break-- I don't know-- particles and antiparticles are not the same?

HONG LIU: Yeah, you always have to-- because you have to choose a reference. Because the particle and antiparticle, they create something out of the vacuum. Still, even if you keep this Q0, so that means that your vacuum has a charge, which is Q0.

And, again, then the particle will increase the charge by 1 and the antiparticle will decrease the charge by 1. So this aspect is the same. Yeah.

So when you act this on the vacuum, you just increase charge. Yeah, just here, for convenience, we choose the vacuum to have 0 charge. Good. Any other questions? Yes.

AUDIENCE: Does charge have a physical interpretation, or how do I interpret the charge in this instance?

HONG LIU: Yeah, so 1 is the electron, 1 is the positron. Yeah, so these are the electric charge we observe. Yeah. So this is the electron-- so when you apply this theory to the electron, then this is just the charge of the electron.

So, yeah, so this actually my next remark. So applied to electron. So Q can be interpreted up to a sign as the charge-- as the electric charge. Yeah, up to assign and a unit.

So this is essentially the electric charge. So we will see later, this is the charge, which couples to the electromagnetism. Yes.

AUDIENCE: So I'm a little bit confused how you can define the notion of charge without even defining the notion of force or force carrier, anything like that.

HONG LIU: Yeah, the concept of charge is independent of the force. Yeah. Even though sometimes the force, you're used to this concept with a charge. But charge by itself is an independent concept.

Yeah, when we talk about the Maxwell theory and then couple this thing to the Maxwell theory and then will become clearer. Yeah, but the bottom line is that the charge by-- yeah, you can define independent of the force. It's just some quantum number. Yes.

AUDIENCE: Yeah, so how do you get out a value for what that term is? Right now, it's just one or minus one.

HONG LIU: Right. Yeah, so indeed, so here, there's a unit. Because you can multiply that j by arbitrary constant. And then, that defines your unit for your charge. And so, in principle, yeah, so that unit have to be determined by experiments.

Good? So the Dirac equation-- so it's very important, is that Dirac theory predict if you say, this is the theory of electron, so predict that electron has an antiparticle. We call it e-plus. So the e-plus have the same mass spin but just opposite charge from the electron.

And, of course, when Dirac wrote down this theory, there was no e-plus yet. People only know electron. Even though Dirac had all the wrong motivation to write down this theory, try to treat it as a wave equation et cetera, but he correctly predicted somehow electron has-- this theory predict another positive charged particle.

And so, that was in 1930-- that was in 19-- I think-- when he did that there was a-- first wrote this down. Maybe it was 1929 or 1930. So, at that time, you predict a new particle, that was considered to be crazy. And 20 years later, people try to predict one particle a day.

[LAUGHTER]

It become a fashion. But in 1929, 1930, to predict existence of new particles, people just absolutely crazy. So, yeah, I will not explain how he predicts this particle. Anyway, he tried to understand this wave equations and then predict there must be some antiparticle.

But at the beginning, he was a little bit afraid. He was worried people will just call him crazy. So he said, this particle, maybe it's a proton.

[LAUGHTER]

Say, oh, because people knew proton at the time. He said, maybe this particle is proton. But, of course, he should know immediately himself that the proton does not have the same mass as the electron.

[LAUGHTER]

So this cannot be proton. So and then he quickly gave up that idea. And he said, oh, maybe this is a new particle. Maybe this is a new particle. And so, yeah. But, luckily, so in 1931 he changed his mind. He said, this is a new particle.

So luckily, just in 1932, and Anderson discovered it in cosmic string-- oh, no, no, in cosmic rays. And then they found this new particle which have exactly the same mass as the electron but opposite charge in cosmic ray.

So that became very happy story. It became very happy story. So any questions on this? Yes.

AUDIENCE: So wouldn't any [INAUDIBLE]?

HONG LIU: Yeah, so indeed, when he first wrote it down, he treated as a wave equation. So wave equation-- wave function is complex. So he likely just treat it as a complex. He didn't even try to make it to be real. Yeah, it is very natural for it to be complex. Yeah, and then, yeah.

It turns out, if you want to write the real equation, it requires a little bit more effort, which we will describe, I think, maybe next lecture. Yeah, and you can get to the real version of this. Other questions? But that takes a little bit more effort. Yeah. And that will wait until Majorana, who discovered it. Yes.

AUDIENCE: So is this whole framework specific to charging spin one half particle? Because it's saying it has to be charged and it has to be spin 1/2 right?

HONG LIU: Yeah, exactly. Yeah.

AUDIENCE: Is there a version that could accommodate a chargeless than 1/2, or is that just a different formalism?

HONG LIU: Yeah. Chargeless spin 1/2, that is the Majorana fermion, which we will talk about maybe next lecture. Yeah.

AUDIENCE: But I guess--

[INTERPOSING VOICES]

--build out of this framework, or is it--

HONG LIU: Yeah, it's built from this breakthrough, but with a little bit more elaboration. Other questions? Good?

So now lets-- so after talk about this quantization, now we can talk about correlation functions of these Dirac fields.

So let's first look at the, say, the Wightman function, which you just don't do any time ordering. So let's look at this object. I call d plus alpha beta, which is defined, again, due to the time translation symmetry, we only depend on the-- OK. Let's look at this object.

Yeah, you can also exchange them. That's called d minus, so it doesn't matter. And so, yeah, so let's look at this object. So this is a 2-point correlation function between these two fermionic fields.

And then, you can just plug in the expansion for each of them. Then you just work it out as we did before for the boson. And so, let me just outline one step-- one intermediate step. So when you plug them in, then you find something like this.

And then you find this sum over s equal to 1, 2, and then you find u s alpha, and the u s beta bar, and exponential i k x minus y. So now, do you recognize this object? Anybody recognize it?

So this is the object we discussed last lecture. But even though, yeah, but-- so this is the projector to the positive-- to the space of the positive energy solutions. So this is the projector to the space of the U solutions. So it appears here.

So and I mentioned that this actually-- you can work it out. So this is given by this form. And then, yeah, so try to check in your notes of last week-- on Monday.

Yeah, so now, if you plug this in, and then you get d3x, d3k, i, ik slash plus m. So this is a matrix, and you take alpha beta component. And then exponential ik x minus y.

So now, so remember, in Fourier transform, any factor k, in the integrand, you can take it out in terms of a differential operator. So we can actually rewrite this as i, take this i out, and replace this by partial slash, but with derivative on x plus m alpha beta d3k 2 pi.

So you can just take this outside of this Fourier transform and then replace k slash by partial derivative on x. When you take the derivative on x, you bring down a factor of k. Yeah, bring down a factor of ik. So ik can be replaced by partial x.

But now, if you recognize this guy-- so do you recognize this guy? Yes? What is this?

AUDIENCE: [INAUDIBLE] do a function?

HONG LIU: Yeah, exactly. This is the Wightman function for a scalar field. So now we can just write it as i partial x slash plus m alpha beta and D plus x minus y. And so, this is the-- so D plus x minus y, say, is the 0 by x by dagger y of a scalar field, say, of a complex scalar.

So we see that, actually, there's a very nice relation between the complex scalar Wightman function and the fermionic ones. There's a very nice relation between them. So they just differ by this factor.

So, similarly, you can work out other kind of correlation function. So let me just write down the result for the others. So you can also define D minus alpha beta x minus y to be-- so you just exchange the order between them.

So, in general, they don't commute. So beta y psi alpha x 0. And then, you find-- in this case, you find that given by minus i partial x slash plus m alpha beta. And the D minus-- x minus y. The D minus, again, is defined as for the scalar field, you put-- you exchange these two.

And for the retarded, you can define the Dr alpha plus beta to be theta. So x minus 0 minus y 0. And now you define retarded using the anticommutator rather than the commutator we used before.

So for when we define a retarded for scalar, we use the commutator. But now you use anticommutator. And then, now you find that this is given by, again, i partial x plus m, the corresponding retarded scalar 2-point function.

So, finally, we can also define the time-ordered. So, finally, we can also define the time ordered Feynman function. So define to be DF alpha beta. So this is defined to be time-ordered, psi alpha x, psi beta bar y 0.

So now, this time-ordering is defined as following. So, again, everything for fermionic fields you replace commutator by anticommutator. You replace commutation by anticommute.

So remember, previously, when you change the order, so if the x0 is greater than y0, and then you just maintain this order. So you just have 0 psi alpha x psi bar beta y0.

But now when y0 become greater than x0, you exchange the order between them. And now you add the extra minus sign for fermion. So for boson, you would just have this. But now with fermion, you add an additional minus sign.

Then you can show that this is the same as, again, just plus m alpha beta, the scalar Gf.

So now we can go to-- so if you want to do Feynman diagrams, exactly as we discussed before, we often need the momentum space expression. So if we go to momentum space, essentially just do a Fourier transform of x minus y.

So this is a function of x minus y. Do a Fourier transform, x minus y. And then you will get dfk. So now I will suppress this alpha beta indices. And now you treat this as a matrix-- treat this as a matrix in the spinor space.

And so, you can just Fourier transform this guy. So this is easy. We know how to do this Fourier transform. And this just gives us some factor of k.

So essentially, you just get i, ik slash plus m. So i partial x slash just become ik slash. And then, we just plug in the expression for scalar propagator, just given by this.

So this is a c number. This is a matrix. This is a matrix. And now, if you remember, again, this formula, that i k that we discussed last time, ik slash m, ik slash minus m, equal to minus k squared, slash minus m squared equal to, say, minus k squared minus m square.

And so, essentially, you see that the k squared plus m square is essentially the product of these two. So this is two matrix product. And the right-hand side is the identity matrix. It is this number times the identity matrix. And so, this becomes-- essentially, you should view this as times identity matrix.

And you see that, essentially, these two up to a constant, these two are inverse matrix of each other. So we can actually rewrite this as just inverse of this matrix minus ik slash plus m minus epsilon.

So this ik slash minus m is the inverse matrix of this. And so, yeah, use this equation, you find them. Good?

So now we conclude with the momentum space, a key expression. Is that, this one. So let me just write this in very prominent position.

So this will be used over and over. So this is a matrix. And given by that. Any questions? Yes.

AUDIENCE: Is it still true that the primary function is the Green's function of some wave operator for the Dirac equation?

HONG LIU: Oh, you mean when the Dirac equation acting on it you get some delta function?

AUDIENCE: Yeah.

HONG LIU: Yeah.

AUDIENCE: All right.

HONG LIU: Other questions?

AUDIENCE: That's not the same epsilon, right?

HONG LIU: Hmm?

AUDIENCE: The epsilon is really different physically?

HONG LIU: No, the epsilon is the same. It's similar. It's just it's the m minus i epsilon. Yeah. Yeah, it is always, whether it's m squared minus epsilon or m minus epsilon, it's the same. Yeah. Just become-- yeah. Here, you only have one factor of m.

Good? So let's conclude our discussion over the quantization of the Dirac theory. And now we go to the next topic, which you have already asked several times. But we only have a couple of minutes to motivate it and to discuss it.

So the goal is chiral fermions. And [INAUDIBLE].

So, yes, we will not not have time to really start discussing. And let me just say a few remarks. So we derived before that the Dirac spinor-- so, so far, what we discussed normally is called Dirac spinor. So they transform under Lorentz transformation in the way so that the Dirac equation is covariant.

So it's a natural question-- and the Dirac spinor have eight components. So I have eight real component. You have four complex component. So altogether, there are eight variables.

So it's natural to ask whether you can reduce the Dirac spinor into a smaller set. So you have-- instead you have eight independent variables into a smaller set. That still can give rise to a Dirac equation, which transform covariantly under Lorentz transformation.

And so, the answer is, yes. So there are two ways of doing it. And one is called the chiral fermions. And the other is called the Majorana fermions. And the chiral fermions is to have two component complex vector.

So altogether you have four real components. Or you have so-called Majorana fermions, which you have four real components. So we will discuss that next time. good, that's all for today.