[SQUEAKING][RUSTLING][CLICKING]

HONG LIU: OK. Let us start. So last time, we started talking about interacting theories. And here is the simplest interacting theory with the Lagrangian density of the form.

OK. So this is just the simplest one. And we will use this as our example to illustrate how you treat interacting theories, OK-- how you treat the particles with interactions. Once we understand how to do one theory, essentially you will know how to do all of them.

And we also said that the-- a key quantity when talking about interactions-- a key observable-- is called S-matrix. So S-matrix is defined as the following. So S beta alpha beta-- beta alpha equal to beta U plus infinity minus infinity alpha.

So you start with some alpha initial state, which are widely separated, which you can treat as a collections of free particles. And then say as T goes to minus infinity, they are very widely separated. But then as time evolves, they will come together, the scatter. Then they will separate into far future. And so that's our final state beta.

And so S alpha beta is the transition amplitude between the two. OK? So this is the written in terms of the Schrodinger picture. You can also written in terms of the Heisenberg picture, which is normally written in this form. OK.

So-- and we often-- so it's often convenient to write this S alpha beta separated into identity part delta beta alpha and then plus i T beta alpha. So this will capture the effect of the interactions. We'll capture the effect of the interactions.

And so-- and since the-- such as-- yeah, we always consider theories which are both spatial and the time translation invariant. So you have energy momentum conservation. And so i T alpha-- i T beta alpha will always contain a piece proportional-- we also put i here-- proportional to the moment-- the momentum conservation.

OK. So p alpha-- and the p alpha is the total initial momentum. So here is full momentum, OK? So I suppress the indices. And the p beta is the total final momentum.

So this part is purely kinematic. So the M alpha beta is the physical part. OK. So this is normally called the scattering amplitude. OK. So our goal is to be able to calculate the scattering amplitude. OK. So this is something-- some version of the scattering amplitude that can be measured in the experiments. So theoretically, we want to be able to calculate this quantity. So they'll be able to compare with the experiments.

So we also mentioned-- I will not repeat here-- we also mentioned that this S alpha beta, S beta alpha also have various properties, which you should go back to your notes of last class to review them.

So one of the key points is the LSZ theorem, which said, which I just mentioned, that M alpha beta can be obtained from a correlation function, like this. Say, for example, for that theory, OK? So the n would correspond to the total number of particles of the initial plus final states. So suppose you have 2 to 2 scattering, and then n would be equal to 4. So if you start from 2 particles, you scattering to 3 particles, then n will be 5. OK, just the total number, yeah.

So later, we will discuss, in more detail, how you obtain M alpha beta from here. But here, at the moment, it's enough for us to know that the scattering amplitude can be extracted from this quantity. Oh, sorry, I should-- So here, it should be time ordered.

And so let me just explain the notation here. So the omega is the vacuum of your interacting theory. And then the T is time ordering. It means that, whoever with the larger time, it will appear earlier. Whoever with a larger time will appear earlier. And so any questions on this?

So our goal would be to tell you how to calculate this quantity. So once you know how to calculate this quantity, then we will be able to calculate the scattering amplitude. And then you will be able to compare with experiments. And good.

So in principle, we already know how to calculate this quantity from what you learned in 8.06. So for people who have taken 8.06, here, you learned. Yeah, so we mentioned last time, this theory cannot be solved, exactly. And the only way we can treat this theory is to take lambda small and then treat this as a perturbation, treat this as a small perturbation, and then expand all your physical quantities in power series of lambda.

So this procedure is called perturbation theory. And in your quantum mechanics class, you should have already learned how to do perturbation theory. For example, in 8.06, that's a big part of the 8.06 class. Anyway, so just with 8.06, in principle, you already know how to do this. Let me just outline the procedure-- a computation.

So that is a correlation function. So it's called a time ordered correlation function. So what I will call this is a naive perturbation theory. So this is the perturbation theory you would do after you learned 8.06, OK, after you learned 8.06, and then you will be able to do this perturbation theory. But we will do something more clever, OK? We will do something more clever. But let me just remind you, equipped with your previous quantum mechanical knowledge, how would you treat this problem?

So now let's consider, again, that theory as example. So the equation of motion gives you partial square, minus m square phi equal to, say, lambda divided by 6 phi cubed. And so we can solve this equation even though we cannot solve this equation exactly. We can solve it for perturbatively in lambda.

So the way we do it is the following. We write phi equal to phi 0, and the lambda phi 1 plus lambda squared phi 2, et cetera. So lambda is a small number. You imagine you expand phi in power series of lambda. And then you just plug in this power series into this equation, and then you equate two side with the same power of lambda. With the same power of lambda?

So the 0th order, you just get our previous equation, phi 0 equal to 0. So phi 0 is just like our previous free field equation. And then at 1st order, then you see phi 1, and now you have 1 over 6 phi 0 to the cube, and then et cetera.

So this equation, we know how to solve. And once you know how to solve this equation, you find a solution for phi 0, then you plug it in this equation. And now this just becomes a linear equation. Now you can solve phi 1. And similarly, you can solve phi 2, et cetera. So order by order, you can solve all of them. And all of them become linear equations.

So phi 1 -- all higher orders can be solved in terms of phi 0. So once you find the phi 0, and then phi 0 can be solved in terms of phi 0.

So classically, this gives you a solution. And as we said, quantum mechanically, we just treat this, treat all of this as operator equations. And then, essentially, you can just solve the phi, perturbatively, as the operator equation-- as a quantum operator. And then you solve this quantum operator equation, perturbatively.

So similarly, we can solve this omega using perturbation theory. So first, we expand again. We expand this omega in power. So this is the vacuum of the interacting theory. And we expand it into a perturbative series. So the 0th order, you just get the free theory vacuum. And then you will have corrections from the lambda, here, due to you have interactions and because your Hamiltonian change. Because of the lambda, your Hamiltonian change. And so your ground state will also change. And so, now, you can use your 8.06 to work out those higher order corrections.

Yeah, so now when you worked out the phi, when you worked out the omega, now, you can work out this object. So let me just call this G n. So the G n then can be reduced to a calculation free theory. And you need perturbation theory, because everything can be expressed in terms of phi 0. And here, again, you expand in terms of phi 0. And the corrections can be obtained by, say, phi 0 acting on a vacuum state, et cetera. So in the end, if you do this procedure, then you should be able to calculate this G n. Conceptually, it's not very difficult to calculate.

But this procedure is actually very complicated. Yeah, I don't urge you to try it yourself. But I think you should try just to get the sense that this procedure is actually not very easy to do. So that's why, in quantum field theory, we actually need to develop a more sophisticated technique to treat this.

Because that's what people did in the early days. In the early days, before systematic perturbation technique for quantum field theory was developed, that's how people did it in the past. And that was not easy. But people did it. So if you want to calculate, the 1st order, it's actually doable. But the higher orders become increasingly complicated. And quickly becomes very complicated when you go to higher orders.

So eventually, people developed something more clever. So eventually, people developed something more clever. So there are two approaches. There are two more sophisticated approaches, which are, in the end, equivalent. OK, in the end, they're equivalent, but their derivation is different.

So the first is you use so-called interaction picture. You use so-called interaction picture. So what do you do is the following. So you separate your Hamiltonian, total Hamiltonian, in terms of a free theory Hamiltonian and the interacting part. So the free theory Hamiltonian just comes from, say, if I write this as L0 minus that plus L I-- so this term is L I, so this part is L0. And so similarly, you can write your Hamiltonian-- also separate into the free theory Hamiltonian and the interacting part.

And then the interacting picture-- it introduces the state into interacting picture, and you introduce the operator in the interacting picture. So the state, in the interacting picture, are related by the standard state in the Schrodinger picture. So let me just put S here. So this denotes the states in the Schrodinger picture, through this factor of H0-- basically, i H0 t.

And the operators in the interacting picture are related to the operator in the Schrodinger picture, also, through this factor of the exponential of H0. Already in your quantum mechanics, you may have discussed the interaction picture. Have you seen the interaction picture before? Yeah? Yeah.

So remember this evolves with the full Hamiltonian. In the Schrodinger picture, this evolves with the full Hamiltonian. And so, here, the interaction picture related this by evolution operator-- conjugate of the evolution operator for the free Hamiltonian. So this Psi I is still evolved in a complicated way. So in the interacting picture, the states still evolves in a complicated way.

But the operator evolve very simply, because, in the Schrodinger picture, the operator don't evolve. And here, the evolution just controlled by the free Hamiltonian. So in the interacting picture, the states still evolve, complicatedly, but the operator evolves in a simple way.

Anyway, with some manipulation, by introducing this interacting picture, you can design a more clever way to do this, to do the perturbation theory. And so I will not go through it, here. You should read the book, Peskin and Schroeder, section 4.2. And there they give a detailed discussion how you do perturbation theory using this interacting picture.

So a second approach is to use the path integral. And just use path integral. So path integrals, in my own opinion, have lots of advantage over this interaction picture. So when you do interaction picture, you get the feeling, somehow, you're doing some very clever tricks. And you don't know why you want to do those tricks. And you get some nice formulas. But somehow, you feel you're doing a little bit-- yeah, you're doing some kind of magic.

But the path integral treatment is automatic in the sense it's straightforward. And you don't have to think. And it just follows from the path integral. And it's much easier to generalize. And it's physically intuitive. And so that's why we will describe this approach. Because it's more physically intuitive. Sometimes, when you do too much fancy mathematics, you feel you obscure the physical picture. And yeah, this interaction picture gives you a little bit of that kind of feeling. But it's important for you to know it. We don't have time to cover both. And so we choose just to do this in class.

Yes, you have a question?

AUDIENCE: In the interaction picture, why do both state and the operator evolve according to H sub 0? Shouldn't the state evolve according to the interaction Hamiltonian?

HONG LIU: Yeah. Yeah, just by design.

AUDIENCE: Oh, OK.

HONG LIU: I just defined them this way.

AUDIENCE: Oh, OK.

HONG LIU: I just defined them this way. Turns out the quantity defined this way is actually convenient for doing perturbation theory. So that's why I say there's a little bit of magic here. And you have to introduce this kind of unintuitive quantities, which turns out actually to be quite simple when you do perturbation theory. Yeah. OK, good. Any questions on this?

Now, let's just go plug into the path integral. And now let's just plug into path integral.

So path integral is an alternative way to formulate quantum mechanics. It's equivalent, say, to the Schrodinger equation or Heisenberg equation. It's equivalent to your standard way of thinking about quantum mechanics. So path integral will give you a better conceptual way for non-relativistic quantum mechanics.

Path integral, most of the time, they give you a better conceptual way to think about quantum mechanics. But, in terms of calculations, it often does not offer any advantage. So that's why the textbooks, for undergraduate quantum mechanics, they don't just do path integral. Because to do the calculation, it's still much easier to solve Schrodinger equation. But conceptually, it gives you a better picture. And for quantum field theory, it actually become much more useful.

Is there any questions on this? OK, good. OK. OK, so first, we are going just to review the story for the non-relativistic quantum mechanics. And once we understand this and the immediate generalization-- the field theory is immediate. Generalization to field theory is immediate. So we will just use the example of just one particle, just a particle in one dimension, just with the most familiar one. So use this as an example. Once we understand this example, then you essentially understand everything. Good.

So now the key point about the path integral-- previously, in quantum mechanics, you start with the Schrodinger equation. But the path integral starts with a different object. So the object which the path integral is most convenient to treat is the propagator we discussed before.

So this is the object which we defined, in the Heisenberg picture, to be this object. So this is the transition amplitude. If you have a particle-- if you have an eigenstate, with x prime and t prime and go into a position eigenstate at the location, x, at time t. So we mentioned before, if you load this object, essentially it's equivalent. You have solved the Schrodinger equation. Because from the wave equation at t prime, then by integrated with this-- convoluted with this object, then you can get the wave function at time t.

And so this can also be written in terms of Schrodinger picture as this. Just start with the position eigenstate at x prime, then you evolve for t minus t prime, and then you ask the overlap with the x. Any questions on this? So let me call this object k. This is k x t, x prime, t prime, OK?

If you know the wave function, so the wave function at t x can be written as a wave function at the t x prime, k t x, t prime, x prime. Yeah, I think I changed the order. So just to be consistent with the order I wrote there. t prime, psi, t prime, x prime. So if you know the wave function at t prime, and then, if I integrate this, you know the wave function. So knowing this object is equivalent to solving the system.

So now we will describe a way to compute this object. We will describe a way to compute this object. So do you have any questions on this? Good. OK, so we will do this by using a trick.

So imagine this is your time axis. So we start with t prime. And then we end at t, So this is the time axis. End at t. So what we will do is we divide it into segments. Divide this interval by infinite number of segments, and each segment is infinitesimal, of course. So we label this point by 0, 1, 2, etcetera. So t0 would be just be t prime. And the last point will be n, and the tn would be t.

So we divided this by n intervals. So the width of the interval is delta t equal to t minus t prime divided by n. So here is delta t. And if we take n go to infinity, and then this integral goes to 0. So the reason for considering this? Just now, we can rewrite the expression, minus i H t prime in terms of many i H delta t, and then n times. Essentially, I have n times.

So now what I'm going to do? Then we have this x, say, exponential minus i H delta t. So now let's write this n times, separately-- x prime. So now let's insert. So the mathematical tool that we are going to use? Insert, between each such factor, a complete set of states, a complete set of position eigenstates. So we insert at t i. Yeah because here, it's t0, then going to here will be t1, et cetera. So at t i, you insert a complete set of eigenstate. You just insert one position eigenstate.

So now this object, k, then can be written as then d x 1, d x n minus 1. So at each here, you-- OK? And then you have x exponential minus i H delta t x n minus 1. And then all the way, here, into x1 exponential minus i H delta t, x prime.

So the purpose for doing so is that now we can calculate each of such things, explicitly. And now we will see. So now let's consider just one of such factor. So factor, here, have the form x i plus 1 exponential minus i h delta t x i. So any factor, here, have the form for some value of i. So i start from 0 all the way to n minus 1.

So now, let's compute this quantity. And now this quantity is actually computable, even though this object is very hard to compute, even though this object is very hard to compute. This object is very hard to compute. Because you have the potential there, et cetera. In general, we don't know how to compute this exponential. But now the trick is that, once you separate them into infinitesimal steps, then we can actually calculate the infinitesimal step. And once you can calculate infinitesimal step, you just add them together, then become the whole thing. So that's the basic idea.

So to do this contain two elements. So the first element is to recognize-- now, remember, h, here, is the operator. So let me just to emphasize here, now, let me put a hat, here, just to emphasize here is the operator. And so if you write this explicitly, this is minus i delta t. And then you have p squared divided by 2m. Let me just also put the-- yeah, to emphasize the operators, and the V hat.

So now, because the delta t is small, it goes to 0, then to leading order in delta t, so this is the first element. The first element is a point. Then to leading order in delta t, we can actually just separate the exponential. You can say this is approximately equal to exponential i delta t, p hat squared divided by 2m, and the exponential i delta t, V hat x. And the only corrections are order delta t squared.

So this, you can easily see from the Campbell House-- Campbell or Baker? Yeah, anyway BCH formula. So from the BCH formula-- let me write it here.

Exponential A plus B is equal to exponential A exponential B. And then exponential minus 1/2 A commutator B, then plus with more commutators. And now each A and B are proportional to delta t. So here is proportional to delta t, here proportional to delta t. And here, it involves at least the product of A and B. So here, it's all higher order in delta t squared.

So this is one of the key reason we want to do this. We want to separate into individual steps. Because we can factorize them. Because, in general, we cannot factorize this guy. This guy is very complicated, because p and x don't commute. In general, we cannot factorize them. It's very complicated. For a finite t, here, we cannot factorize them. But for delta t, we can factorize them. We get this structure. So once you have this structure, and then this thing becomes easy to compute.

The second thing you can show.

So the second thing you can show? Now, you can show that x i plus 1-- so now you plug in this factorized form into this expression. And now you can compute this. So you have minus i delta t, p hat squared divided by 2m, minus i delta t, V x hat and x i. So those are just eigenvalues. So there's no hat. But here, there's a hat. They are operators.

So here, you can now compute this object. So this becomes an elementary exercise in quantum mechanics. I will not. There's like two line calculation here. I will leave it to your homework, so you can calculate this object. Let me just write down the result.

So this becomes the following object-- m divided by 2pi i delta t exponential-- so you can evaluate this thing, explicitly-- 1/2 i m, x i plus 1 minus x i delta t squared delta t, minus i delta t V x. So you just get this. So this is the answer.

So this is the second element. Now this is readily computable using the standard technique. So I will remember to put it in your homework.

So now we have to use a little bit of imagination. So now we have to imagine-- use a little bit of imagination. I should have drawn this line longer or bigger. Let me just draw this a little bit bigger. Sorry, I drew this line too short. So let me just draw this longer.

So here is t prime. Here is t. Here, you have different locations. So I also draw the delta t to be big. So at each t i, you have x i. We insert an x i, OK? So this is t i. We insert an x i. So at t1, we insert x1, et cetera. So now you imagine, for each choice, for each value of x, in the end, we need to integrate over all possible x.

So for each particular value of x1, x2, x3, x n, we can view it as a function of t evaluated at that point. Now, let's consider, you have a function, x t, so that when you evaluate it at t i, the value is x i plus x i. So if you think this way, now this thing in the exponential will become a very familiar object.

So what is this object? Yes?

A Lagrangian?

Exactly, just become the Lagrangian. Because this is just like the time derivative, so this is just like x dot square. And this is just like V times delta t. So this is equal to-- so go from here to here. Now, this is just equal to-- with the same prefactor, just become exponential i delta t times your Lagrangian density. Yeah, just a Lagrangian, here. Just the Lagrangian x, x dot evaluated at t i.

So you have the Lagrangian. So remember, the Lagrangian is just 1/2 m x dot squared minus V x. So if you take all this delta t factor and this i factor, and then this just becomes 1/2 m x dot square. And this just becomes V all evaluated at x i. And x i is the value at t i. So all evaluated at t i.

So now you recognize, you just get your Lagrangian there. So now, we can write down the full t, full K. So now the full K become limit n go to infinity, because we take delta t go to 0. So we need to take the n go to infinity. So m 2 pi i delta t to the power m divided by 2. So each one gives you square root. And then you have d x1, d xn minus 1.

But now the integrand becomes i delta t. Now you just sum over i from 0 to n minus 1, L, your Lagrangian, x, x prime evaluated at t i. Because now, the product in the exponential just become the sum.

So now you recognize this essentially is the discrete version of the integral of L with dt. It's just over time. So now we can write. So the exponential, we can essentially-- so let me keep this integral here. So the exponential part, we can write it as exponential times i from t prime to t, say dt double prime, L, x, x dot. So this just becomes that. The integrand just becomes like that.

But we still have this whole bunch of complicated stuff. Take n go to infinity-- this thing. So normally, when you have something complicated, again, there's an unfailing trick in physics to do it. What is that trick? Do you know?

Give it a name?

Exactly, we just rename it into something simple. So we just call the whole thing D X(t). This is just supposed to represent this limit. And so this is the condition that x t prime equal to x prime, and x t equal to t-- or x t equal to x.

And the meaning of this D X(t), if you think about it, as I mentioned, each particular choice of x1 and to x n minus 1, you can think of some function of x evaluated at t i. So now, you integrate over all possible such values. You integrate over all possible such functions. So this just corresponding to integrate all possible functions x t satisfying-- so mathematically, this means satisfying that the x t prime equal to x prime, and x t equal to x. So because we always fix the two end points. You only integrate in the middle. You only integrate the middle.

So now this has a very simple physical interpretation. So now this has a very simple physical interpretation. So integrate over all possible functions x, between t and t prime, with the end point fixed. Physically, this just means, suppose-- so this is the t. This is the x-axis. So suppose that the t prime is here. So here is the, say, x prime at t prime. And so this is the x t. So this is the x prime, t prime. And this point is x and t.

So you integrate all possible functions between them just corresponding to-- you integrate over all paths between them. You integrate over all possible paths between them. So this is the same as integrate all paths between them. And the thing is weighted. And each path is weighted with the integrand. It's weighted by exponential i S. And the S is just your action. So this is, again, from t prime to t. Yeah, let me just write down. By this factor, OK? So the so the integration, with the Lagrangian, is just your action.

And then you find that, now, you can write this propagate this transition amplitude in terms of the expression corresponding to integrate all possible paths, sum over all possible paths, and with the weight by the action, exponential action. So this started as a technical trick, because computing this thing is complicated. And then we can make mathematical progress by dividing them into small intervals.

But once you do that-- but now we obtain a different way. But now, we actually obtain something conceptually new. You just say that quantum mechanics. So classically, remember, there is a fixed path from x t prime to t, just follow the equation of motion. But now we say quantum mechanically, we just sum over all possible paths. So that's the quantum uncertainty compared with the classical mechanics. So starting with that mathematical trick, but now we actually obtain something conceptually brand new. Yeah, because quantum mechanically, the difference from classical mechanics, just now we just sum over all possible paths, which is physically very intuitive. Because quantum mechanically, you can just go anywhere-- OK, this uncertainty principle.

Good. Any questions on this? Yes?

Are there constraints on what path is really possible? Does it have to be differentiable?

Yeah, this is a good question. So to make sense, rigorously, this quantity is actually-- yeah, to make rigorous, this quantity is not easy to do. So we will not go into there. That needs to go into lots of mathematics. So normally, we don't need to worry about the precise mathematics, how to define that path. Mostly, we think this from the conceptual picture. And also, when we do technical manipulations, as we will see, often, you don't need to know the details, how you define this measure, precisely. Yeah. Yes?

Sir, a question. There's no coming back in time. I mean, that's right?

In principle. Yeah, there's no coming back in time, just from here to there.

Yeah.

There's no coming back in time. But the path can wind around as you want. Yeah, but time, you cannot come back. Other questions? OK, good.

So there's also another form of this path integral. So I will not do it here. It's a slight variance of this. Let me just write it down. Again, I can put it in your homework. So there's also a Hamiltonian form, which is much less used. But sometimes this is also useful. It's a Hamiltonian form. So you can also write the k, same object as the following, as x. Again, the same boundary condition, x t prime equal to x prime and x t. So you integrate over all possible paths, as here. But you also integrate over all possible momentum, within this time range. And then the integrand becomes the Hamiltonian.

You write it in terms of the Hamiltonian. So t prime, t dt double prime, then p x dot minus H. So H is your Hamiltonian. And the p, now, is just some function, some arbitrary function. And so for any choice of x and t p, you can evaluate this quantity. And then you sum all of them together.

And so this alternative form, which can be easily-- again, a couple of lines, which you can show to be equivalent to this form. To show this is related to the second step, here, the intermediate step of doing this one will lead to this one. Good. Any questions?

So now let me just talk about the example. So how you can use this method to calculate, say, some systems, simple systems? So as I said, this method is not very efficient for non relativistic quantum mechanics-- and so even of a different conceptual picture. And you can see from those examples.

So now let's consider a simple case. So we just consider a free particle. So there's no V. So just consider a free particle. So here, S just 1/2 m x dot square. Let me just use q, since my notes use q. It doesn't matter. So let's try to compute, in this theory, the amplitude. Let's call Z t. Say at time 0, you are at q equal to 0. And then at time t, again, you add q equal to 0. So you just come back to the same point. Come back to the same point.

So corresponding to this situation, if you call this direction to be t and this direction to be q, so, at some initial point, you are here. And your final point, at some t, the value of t, again, you go back to the same point in q. And then we just sum over all paths here-- but free particle.

So in order to compute this guy, now, we have to work with this. Because this is only a formal notation. Remember, this is a formal notation for this object. OK, if you really want to compute this--

So this is called the path integral. So this is called the path integral. So if you want to really calculate this path integral, explicitly, then you have to go back to this form. You have to go back to this form. So the z t is equal to-- in this case, limit n go to infinity. Yeah, let me just not copy. Yeah, anyway, you can write down. You just copy this down. Just change this. Just to save time, change this x i to q. And now the integrand, you only have 1/2 m q squared. There's no return. You can just plug that in, into here.

So now, in principle, you can calculate that integral. And I will leave it to your exercise to do in your homework. And you find that it will recover the standard answer. So at the end of the day, when you do this integral, this mini-integral, then you find that this gives you-- actually, I didn't write down the final answer. Let me see.

So you find the final answer is a very simple one, just 2 pi i t square root. So you can do this, an infinite number of integrals, and then you get that answer. Of course, you can calculate that thing, easily, by solving the Schrodinger equation using your standard method. But this is a good exercise to do.

So this is one method to calculate it. But this method is not very efficient. Because each time, if we have to go through this limit, and it will be very annoying. So now there's a more abstract method. Now, there's a more abstract method but much less mathematically rigorous. You can make it mathematically rigorous, but I'm not going to make it mathematically rigorous.

But when you get used to it, it's much easier to manipulate. So now I'm going to talk about a second method to calculate this. So this is the first method. You just do honest integral.

So the second method? Let me call this one for the first method. And the second method--

So in the second method, let's first look at this. Oh, yeah, sorry, this should be integrated over dt. So this is a m and q's creates the Lagrangian. Here, we're talking about the action. So here we look at the action which you appear in the path integral in that formal form then you have s is equal to 1/2 m, then you have dt from t equal to 0 to capital T. From t equal to 0 to capital T. And then we have 1/2 m q dot square.

So we have that. So now I'm going to slightly rewrite this expression. So I'm going to write as following, m, from 0 to T, dt-- I will do integration by parts. Oh, sorry. So I'm already out. Then I have q minus delta t squared, q. So integration by parts, so the derivative will only act on one of the q. So I get a minus sign. And the boundary term is all 0, because the q, at the initial and the final value, are 0. Yeah, so you can check yourself.

So I'm going to write this expression a little bit further. I'm going to introduce two t, t and t prime. I write it as q. Yeah, let me, now, just write it as 1/2. Let me just write it as 1/2 q t. And I write here minus m delta t minus t prime, partial t prime squared, q t prime.

So these two are the same. So I introduce additional t prime, but, also, I introduce a delta function. So when you evaluate that delta function with t prime integral, you just get back to the expression above. So now I'm going to pull this into the k. Yeah, actually, sorry, I want to put this minus sign out, just for convenience. So let me just call this in k.

So what I get is the minus. Yeah, it doesn't matter, actually. So sorry. Let me put it here. Yeah, it doesn't matter. So here, we can just write 1/2 dt dt prime, q t, k, t, t prime, q t prime.

[AUDIENCE] Question.

Yeah?

[AUDIENCE] What's the integral of dt prime?

Yeah. The integral of dt prime, you can just evaluate using this delta function and then this go away. And then this here, this all become t, then you just reduce to that.

[AUDIENCE] So is it like an all time integral?

Yeah. Yeah. No, no, no. It's the same from 0 to T. Yeah. Yeah. The same range.

OK.

So now I just call this into the k. So now, I think we have done this before. And now I want you to view this t and q as an index. Imagine this is just a continuous index. And then q is just like a vector-- q t prime is like a vector. q t is like a vector. And this is the same vector. And now this is just a matrix between the two vector.

So now, we can view q t as a vector, with t as the index. And then the k is the matrix-- it's a matrix in the space of q t.

So when I write this form, then this integral, dq t, so then we have this Z T equal to dq t. So for simplicity, let me just suppress the initial and the final just for simplicity. And then you just have i, then 1/2 i dt dt prime. Also let me just suppress the-- then you have q dot k dot q. Let me just use this simplified notation. This is the same as that guy. Yes?

[AUDIENCE] Is this like ad hoc that you left the first q as a function of t and not made it a function of t prime instead?

It doesn't matter. You can do that it way, too. Yeah, it won't change.

So now you compare this form. So now, this look like a Gaussian integral. Now this look like a Gaussian integral. So now, let's recall your Gaussian integral, which you'll be familiar with. When you have the x exponential minus a divided by 2x square, then that gives you 2 pi divided by a. So this is a one-dimensional integral. But you can also have an n-dimensional integral. So here, the index is a sum. So this is equal to 2 pi to the power n divided by 2, square root delta A, of this matrix A.

So now this is just a generalization of this integral, except the index becomes continuous. And here, the k have two indices. It's just like m, n. So here is like q m and q n. And this is just like m n. So this is just like an ordinary-- this is just like a Gaussian integral but in the space of infinite dimensional vectors. Yeah, of course, not just infinite. It's uncountable infinite dimension, because it's a continuous index.

So now with this understanding, with this realization, we can just directly write down the answer. We say then the Z T must be just given by some constant divided by delta k, determinant k. Yeah, you just generalize this formula. You generalize this formula.

So just now, k is the determinant in the more complicated space, in the space of functions. So the only unfortunate thing is that both C and the delta k actually are divergent. So C is our constant. And the delta K is the determinant of K in the space of functions. So this isn't like the K is the matrix. It's a matrix defined in the space of q t. q t is the space of functions.

So this content can be divergent. But this doesn't matter, as we will later see, such a constant don't matter very much for your physics. And yeah, I warned you before, in quantum field theory, we will see divergences everywhere. So the key is to recognize which divergences are important and which divergence are not important. So unimportant divergence, you just forget them. Yeah, so we will later see that such a C will always cancel when you can see the physical quantities. And this will never matter.

But still, we need to talk about how to think about the determinant. What do we mean by determinant here? So we do it in the same way as you would do in the matrix space. So if you have a finite dimensional matrix, there's a way to calculate the determinant, yeah, the standard way, which you learned. Say a1, a1 1, a2 2, et cetera, there's a complicated formula, OK. But that way does not generalize to such a k, with continuous indices.

So we must find a way to generalize the determinant, here, in a way, which can apply to such a k, which is defined in the space of functions. So there's a lot of way we can calculate the determinant, a way without using the standard formula.

So you do this as follows. Remember, the determinant of A is also the product of all the eigenvalues of A So you can just find all the eigenvalues of A. You take the product of them. And then that is the determinant of A. So this way to define determinant actually generalize. It doesn't matter how many eigenvalues you have.

So now we can just find all the eigenvalues of k. Just find all the eigenvalues of k. And then we just take the product of them. That gives us the determinant. So this is the way which we can generalize.

So how do we define the determinant eigenvalue? So when we define the eigenvalue of A m n, we find the eigenvector X will satisfy this behavior. So the n is summed, and then give you a this, right? So now the analog of the m n, here, is t and t prime. So the analog here is from 0 to T, t prime, k, t, t prime, some function, f n. So n, here, label different eigenvalues. So let me call it i. i t prime, you go to lambda i f t.

So this would be the eigenvalue equation. Because now, the n index, here, is just replaced by t prime. The sum over n just replaced by integral. And then the rest is the same. So you just find all the eigenvalues of k the f t. And of course, it should satisfy the condition, here, the f 0 equal to f i-- yeah, f i T should satisfy the boundary condition. This is equal to 0. Because that's the integral. That's the condition of a path integral. The path integral, it's from 0 to 0.

And you just solve this eigenvalue problem. This is a well-defined mathematical problem. Because k is just a differential operator. k is just some differential operator. And now, this becomes a well-defined mathematical problem. We can find these eigenvalues. And then we just take the product. So this way, we generalize that formula.

So this is the exercise for you to do yourself. OK, k is just a quadratic. Differential is just delta t square. It's very easy. OK, it's very easy. So you find f i t. Yeah, actually, I think i is also not a very fortunate notation. Because it can be easily-- yeah, let me call it j, OK, so pi j t. So you can just solve this equation, easily, yourself. So let me just write down the answer.

So the eigenvalue of pi j t divided by capital T. And with eigenvalues lambda j equal to m j square, pi squared, divided by t square. And the j equal to 1, 2, et cetera. So even though the k-- both index of k are continuous, but the eigenvalues are discrete, OK, a discrete infinite. The eigenvalue are discrete infinite, just labeled by integer j. You can easily convince yourself. Yeah, just solve this equation. It's easy to solve.

So now we can find the determinant of k. Now we find the determinant of k. So this is given by product of n equal to 1i-- j equal to 1, to infinity, m j squared, pi squared divided by t square.

So you say, what is this beast? So we have all these infinite things. We have these things. We have all these constants multiplied infinite times. And then we have this j square. And so now I claim-- again, I will not have time to do it here. I will claim this guy, once you do proper things, and then that gives you that answer. When you do that C over delta t, that can get you back to this answer. Get you back to that answer. So you will work through that in your homework problem.

So this way, even though it looks hairy, but once you get used to it's much easier than doing those integrals and take the limit. It's much easier than doing this-- take the limit. And also, we will later see, most of the time, you actually don't need to calculate the determinant, explicitly. Later, we will see, most of the time, you don't need to calculate determinant, explicitly. You don't need to use those things. And just the path integral framework somehow provide a platform for you to do a lot of things. You don't actually need to evaluate it. It's actually provides a platform. That's the most important thing about the path integral. It's often not about how you evaluate it.

Good. So before we conclude, let me just make some further remark. Because it's better if we just make a couple of remarks here so that we don't forget next time. yeah give me like two minutes to make some remark.

So remember remarks I already said. We said that this is a new formulation of quantum mechanics, gives you a new conceptual way to think about quantum mechanics. I'll give you a new conceptual way to think about quantum mechanics. And you can show that these two formulations are equivalent. So this is equivalent. You can show that this is equivalent to the Schrodinger approach, to the Schrodinger formulation, by showing that the K, you calculate it this way. Using path integral, you find this K. You can show that the K actually satisfies the Schrodinger equation.

So this will guarantee-- I already erased-- that guarantee your wave function, when you come over to this K, it will also satisfy the Schrodinger equation. So you can show that this actually satisfies the Schrodinger equation. So you can show that the K actually satisfies the Schrodinger equation. Again, I will leave it to your homework.

Good. And the second point is just to highlight this contrast between the classical mechanics, which have a fixed path determined by equation of motion, and the quantum, you just sum over all possible paths weighted by your action. You just sum. Just a contrast, OK, just contrast. Yeah. Yeah, let's stop here.

So the next time, then we are ready to go to field theory. So with this, if you are familiar with this, if you are familiar with this, get used to everything here, then going to field theory is just immediate. You're just changing the notation. Because, as we said, the field theory just goes on into quantum mechanics with infinite number of degrees of freedom. Once you understand 1 degree of freedom, then you can just generalize to infinite number of degrees of freedom just by changing notations. So this aspect will be the same. So we can immediately write down the path integral for quantum field theory next time. And then we can talk about the interactions.