[SQUEAKING]

[RUSTLING]

[CLICKING]

HONG LIU: Yeah, so, today, we are going to start a new topic. OK? So, first, we talk about chiral fermions.

So remember, say, under Lorentz transformation, lambda, the Dirac spinor fields transform as S lambda psi x. And the x prime is a Lorentz transformation of x. So x prime is lambda act on x, OK?

And then the S is given by, say, omega mu nu, sigma, mu, nu, OK? And the sigma mu, nu is driven by the commutator of the gamma matrices. So let me just write it down.

So sigma mu, nu-- just remind you-- i divided by 4. OK.

So one natural question-- so, previously, when we derived the Dirac equation, we showed that the Dirac equation requires, actually, psi to have four components, OK? But, then, we showed that the Dirac equation is covariant if the psi transform this way.

So natural question is that whether you can actually restrict to a smaller part-- say, to a subset of psi-- whether they still have well-defined Lorentz transformation, whether we actually need to have four components to have well-defined-- four complex component to have well-defined Lorentz transformation, OK?

And the answer turns out to be, no, you actually don't need to have four complex components to have well-defined Lorentz transformation. Actually, you can reduce it, OK? And so there are two ways to reduce it, and one is called chiral fermion, and the one is called the Majorana fermion, OK? So one is called the Majorana fermion.

So we first talk about the chiral fermion and one way to do it. So, for this purpose, we will look at the specific representation of gamma matrices, OK? Consider-- so now I will use a representation which is different from what you-- so we consider the following one.

Gamma 0 equal to 0, i, i, 0. Gamma i equal to 0 minus i sigma i, i sigma i. So let's look at this choice of gamma matrices, OK? So I will call this choice to be star, OK?

And so you can also work it out. You find the sigma. So sigma 0, i-- so this is the sigma corresponding to the boost. So you find the sigma i, you can just do the commutator.

From the sigma i is equal to minus i divided by 2, have this block diagonal form. Sigma i 0, 0 sigma i-- again, the small sigma i is always the Pauli matrices. And then you can also work out the sigma ij.

Then you find the is given by minus 1/2, epsilon ijk, then sigma k, 0, 0, sigma k, OK? Oh, sorry. Here, there's a minus sign. OK, here, there's a minus sign.

OK, so you find that they have the following form. So what do you observe about this? So do you see something? Yes?

AUDIENCE: They're block diagonal.

HONG LIU: Yes, they are block diagonal. So if sigma is block diagonal, then that means this S is also block diagonal, OK? So when S is block diagonal, what does that mean? Yes?

AUDIENCE: [INAUDIBLE]

HONG LIU: Hmm?

AUDIENCE: [INAUDIBLE]

HONG LIU: Exactly. So when the S is block diagonal, that means, when I write psi-- so, psi-- have four components. That means that the upper two component and the lower two components-- they don't transform to each other, OK? They only transform within themselves. They don't transform to each other, OK?

So if I choose, that means this lambda is block diagonal. So that means, if I write psi x into two component vector, psi L and psi R, so I denote the upper two components by psi L and lower two components by psi R-- are two component complex vector.

That means, under Lorentz transformation, under S lambda, psi L and psi R do not mix. So they just transform among themselves. They just transform among themselves.

So they actually have well-defined Lorentz transformation as a smaller unit, OK? You don't need four components to be able to transform under Lorentz transformation. Actually, at least two components can already transform. OK.

So this tells you, in a sense that the Lorentz covariance only requires two component spinors. Just by Lorentz transformation itself, you don't actually need four components. OK.

So, now, I'm going to tell you, we actually knew this all along. So how did we know this all along? Yes.

AUDIENCE: Why is there no [INAUDIBLE]?

HONG LIU: Sorry. Say it again.

AUDIENCE: Didn't we have to go to four components in there because there are no representation [INAUDIBLE]?

HONG LIU: Yeah, yeah. We have to go to four components because there's no two component representation of the gamma matrices. But that doesn't say it's a Lorentz transformation. Yeah. To write down the algebra for the gamma matrices, you need four components.

But we actually knew all along that two components is enough for Lorentz transformation. How do we actually know all along?

AUDIENCE: Lorentz assumption isn't just [INAUDIBLE] spin like difference between the [INAUDIBLE]?

HONG LIU: Yeah, this is maybe more complicated. We have something much simpler. Yes.

AUDIENCE: Is it the massless particle?

HONG LIU: Yeah, exactly. So we already said before, if you have a massless case when m equal to 0, and the Dirac equation reduces to the two components, and it's enough to do two components, and the Dirac equation is covariant, OK? And so that means that, actually, you should be able to do it with two components, OK? Because the massless particle-- they should be able to transform on the Lorentz transformation. And so, yeah.

So we already saw this because the massless. So the hint from before is that the massless only require two components. Since the massless case must also be Lorentz covariant so that-- and Lorentz symmetry itself should only require two components.

So it is not a Lorentz symmetry, actually. Lorentz covariance requires Dirac theory to have four component. It is the mass, OK? So it is the mass-- mass m.

If you want to describe a massive particle, then you must have four components, OK? So it is the mass which is the key. OK, good. Any questions on this?

So, now, we have shown in this particular representation of gamma matrices, for this particular choice of gamma matrices, and then the psi transform block diagonally. But then we also said, but, now, consider a different choice of gamma matrices.

And then this property will not hold, OK? This property will not hold. Now, the question is that, does this-- even with other gamma matrices, can we actually reduce psi to some smaller components?

The answer still should still be yes because we said that all representation of gamma matrices should be equivalent. So if we can do it in this choice of gamma matrices, then we should be able to do it in any choice of gamma matrices, OK? So, now, let me tell you how to do it for the general gamma matrices.

So this property that you can reduce to two components should exist for all choice of gamma matrices. Just for other choice of gamma matrices, to separate the psi into psi L and psi R is more subtle.

You no longer-- is just simple the upper two component or lower two component. So we have to do a little bit of work, OK? Actually, we don't need to do much work if you actually find the right trick, OK?

And so the beautiful trick to do this for any choice of gamma matrices is that you can introduce the following object-- what is called gamma 5. So gamma 5 is defined to be i gamma 0, gamma 1, gamma 2, gamma 3, OK?

So you take the product of all the gamma matrices together and then with a factor of i, OK? So the i there is for the purpose that if you-- you can check yourself-- that the gamma 5 is actually Hermitian. So the i is there for this purpose, OK? You need i for this to be true.

You also can check yourself that gamma 5 squared is equal to 1, OK? So, this, you can almost easily understand because it's all gamma 0, gamma 1, gamma 3. So you multiply itself again because any same gamma matrix, they multiply either 1 or minus 1.

So you multiply them together, in the end, they can be either 1 or minus 1. Just turns out, for this choice of i, it's 1, OK? So yeah. And then you can also check the gamma 5 anticommute with any gamma matrices.

So mu here is, of course, from 0 to 3. And so this is of-- you can see immediately from here-- you can immediately from here-- because gamma matrices whose indices are not the same, they anticommute, OK? So if you try to commute this with any gamma matrices, you have three of them.

Yeah, because this runs over all gamma matrices. So the one in which-- yeah, so if you take with some gamma mu, and that particular one, which is the same as gamma mu, of course, commutes with gamma mu. But then you have three others. But three others will give you minus sign, OK?

And you can also check yourself. Gamma 5 actually have 0 trace, OK? So, this, I will leave as an exercise for yourself, what you can do is you did before with other-- yeah, in your homework-- yeah, similar to the exercise you have done in your homework.

So, now, from this properties-- now we can say the following things about the gamma 5 matrix. First, because gamma 5 squared, squared to 1. And, also, this is Hermitian. So it is Hermitian means its eigenvalue is all real, OK?

So its eigenvalues are all real, and gamma 5 squared equal to 1-- that means its eigenvalue is either plus or minus 1, OK? So have eigenvalues plus, minus 1.

And then from the property that this is traceless, they tell you the number of the eigenvalues, which is plus 1, and the number of minus 1. They should be the same. Otherwise, they won't cancel. It won't be traceless.

And so each eigenspace is two-dimensional. OK? So you have four eigenvalues. So there's 2 plus 1, 2 minus 1. It must be.

So since you have eigenvalues 2 plus 1, 2 minus 1, and then we can introduce a projector to project into the eigenspace, say, with plus 1-- with eigenvalue plus 1 or the eigenvalue minus 1, OK? So I can introduce a projector which, for historical reasons, is called PL. It's defined to be 1/2 1 plus gamma 5. And the PR is 1/2 1 minus gamma 5, OK?

So this will project into eigenspace with the eigenvalue plus 1 squared. And this will project into an eigenspace with a minus 1, OK? Because when 1 minus 1 plus-- yeah, anyway. So you can check, OK?

So you can check they are really projectors. So PL squared equal to PR squared equal to 1 and PLPR equal to 0 and PL plus PR equal to identity, OK? And then-- OK.

So, now, I introduce-- now I can project-- define the projection of psi L of the projection of PL psi-- project to the left space, OK? And psi R to be the projection to the other space. OK. I define them this way.

And then you can easily see, by definition, you can easily convince yourself gamma 5 acting on psi then just equal to psi L. And the gamma 5 psi R is equal to minus psi R, OK? So they project into the eigenvalues of plus, minus 1. Yes?

AUDIENCE: Wait, why is PL squared equal to PR squared plus 1?

HONG LIU: Oh, sorry. Sorry, sorry, sorry. No, no. This is good. This is completely wrong, OK? This is completely wrong. I was dreaming.

So PL squared equal to PL, PR squared. Sorry. Yeah, thank you. Yeah, so you can check their projectors, OK?

So, indeed, you see-- so, from here, from this definition, you can check this is true, OK? This is a one-second check. And so, indeed, they project to the eigenspace of gamma 5 or plus, minus 1. OK.

So then, by definition-- OK, so now this psi L, psi R, which is now defined for any choice of gamma matrices-- so, again, they have to two independent complex component, OK?

And so they call the chiral spinors-- sometimes also called Weyl spinors. And so this is the analog of psi L and psi R here for the general choice of gamma matrices.

So, now, we will check this actually, indeed. So now the claim is that psi L and psi R defined this way will transform under themselves under the Lorentz transformation. They will not mix with each other, OK?

So, again, psi L and psi R here each have four components, OK? They just have only two independent compact components. So they still have four complex components, OK? And there's still four component spinors because just there's only two independent ones. There's only two independent ones.

OK, so, now, it's easy to check they actually transform among themselves. So you can check the gamma 5 actually commutes with sigma mu, nu. OK.

So this is very easy to see. So, from here, gamma 5 commutes with any gamma mu or anticommute with any gamma mu.

And the sigma mu, nu is just the sum of two-- the product of two gamma mus-- have even gamma mus. So the gamma 5 will compete with them, OK? So gamma 5 will commute with them.

So if gamma 5 commutes with sigma mu, nu, then gamma 5 commutes with S lambda because S lambda just generated by sigma mu, nu. And then that means-- so we commute with S. That means, under transformation, S will not-- under transformation by S will not change the eigenvalues of gamma 5, OK?

So that means that psi L prime, S lambda, psi L, and gamma 5 acting on psi L prime is still gamma L. So it's still within the same space. And, similarly, we say-- OK, so that tells you that psi L and psi R-- they transform separately because the gamma 5 commutes with Lorentz transformation. And so each eigenspace, they transform separately from each other. OK.

Good. Any questions on this? So you can also find in the-- so in the chiral representation, you can check in this star-- so the star-- this particular choice of gamma is called the chiral representation, OK? Because in that choice of gamma-- things simplify, we just have upper two components and lower two components.

So you can check yourself, just by working it out, that the gamma 5 indeed just have block diagonal form-- 1, 0, 0 minus 1, OK? So that's why. In that case, it's very simple, OK?

But in other representation, gamma 5 can be more complicated. Good. Any questions on this? Yes, you have a question? OK. Yes?

AUDIENCE: Another way to do that, as you said, would be to try to find the unitary matrix that shows-- under which this arbitrary representation is equivalent to the chiral representation. From this argument, can we figure out what that unitary [INAUDIBLE] looks like?

HONG LIU: Yeah, yeah. Yeah, you can. Yeah, yeah. No it's not a unitary transformation, just a similar transformation, yeah. So each of them are related by a similar transformation. And, indeed, the gamma 5-- gamma 5 in the other representation are related to this one just by a similar transformation, too.

Yeah, so I will use that language when I talk about Majorana spinor. So, in this case, it's sufficiently simple. I don't need to use that language. Yeah, but you can use that language.

OK, so let's go back to this chiral representation and write the Dirac equation into this chiral representation. Yeah, one second.

So, now, if you write the Dirac equation in terms of psi L and psi R-- so, remember, the Dirac equation have the following form of Dirac Lagrangian density. OK? And then since psi is just equal to the sum of the-- so psi-- psi R, OK?

And then you can just write this in terms of psi L and psi R. Write this in terms psi L and psi R. And then you find that the cross term vanish. You can also check this explicitly in the chiral representation, but the expression I'm writing down is general, OK?

So you can write it as psi dagger partial sub 0, plus i sigma i, partial i, psi L. Yeah, sorry. Yeah. Yeah, actually, psi R dagger. Yeah, let me first write. I think I said something wrong. Yeah, OK.

OK? So, yeah, as I said, so this expression only applies to the chiral representation, OK? So in the chiral representation of gamma star-- in this space of star, and then we have two components. Then I can write this psi and the psi L and psi R into two components. And then, yeah. So this is just ordinary sigma matrices, OK?

And so this is the expression you get, OK? So what you notice-- is that for m equal to 0-- so there's no coupling between psi L and psi R, OK?

So it's the mass term which coupled them together, OK? Kinetic term-- psi L-- there's no cross term between the psi L and psi R, OK? And this behavior is actually general. You can write it in arbitrary representations.

But, of course, in arbitrary representation, I can no longer use this sigma i, OK? And so in this particular form-- even though this feature is general, but this particular form of the kinetic term only applies for the chiral representation. OK.

So for m equal to 0, you don't have coupling between the psi L and psi R. And then you only have to say, oh, diagonal term and psi R term. And then that gives you something else, OK?

So, again, this is reduced to our previous statement that if you have a massless case, you can describe using a two-component spinor, OK? So here, indeed. But, here, there's also something extra. So what do you see-- something extra here? Yeah. Did somebody raise your-- yes.

AUDIENCE: [INAUDIBLE]

HONG LIU: Yeah.

AUDIENCE: Why are we now taking the psi equals psi L plus psi R instead of before we had it as psi L and Psi R

HONG LIU: Right, right. Yeah, yeah. Sorry. A good question. Yeah, this expression is wrong. OK. Somehow, I was doing a-- I was trying to-- yeah. I remembered I wrote this in the general basis, but then I realized I only wrote it in that basis.

Yeah, in the general basis, I would have psi equal to psi L plus psi R. Yeah. But then I realized, I only write the kinetic term in this specific basis. Good. Yeah. Yeah, so that expression does not apply for the chiral basis but apply for the general.

So here, actually, something profound happens because, when m equal to 0, when you don't have coupling between psi L and psi R, you actually get the extra symmetry, OK? So in the Dirac Lagrangian, as we discussed earlier, so we have a U1 symmetry. Psi goes to exponential i alpha psi, OK?

So this Dirac Lagrangian is invariant under that because the psi is complex. But, now, psi L and psi R-- they are separate. So, now, I can actually transform psi L and psi R separately, OK?

So under this transformation, psi L and psi R transform the same, OK? But, now, m equal to 0-- I can have psi L goes to exponential i alpha L, psi L. Psi R goes to exponential i alpha R, psi R because they only couple to themselves, OK?

And, now, I have this symmetry, OK? And so, now-- so, here, you have U1, and now you have U1 times U1 called U1 times L and U1R. So these are called chiral symmetries because they transform the left and the right separately. OK. Yes?

AUDIENCE: I'm a little bit confused why you can't write psi as the sum of the two projections?

HONG LIU: Sorry?

AUDIENCE: Why you can't write psi as the sum of the two projections?

HONG LIU: No. No, I can write it-- no, I can write it that way. Just, now, I'm using the two-component form. When I write two-component form, then I write psi that way, it doesn't make sense because psi L is the upper two component, and psi R is the lower two component.

Yeah. Yeah, yeah. I'm using the same notation for this spaces. And so in these spaces, psi L and psi R, they only have two components. But in the general case, there have four components, OK?

So in the general case, I can write psi equal to psi L and psi R. But in these spaces, I cannot. Yeah, using this notation, I cannot.

So, now, you have a new symmetry equal to-- which you can transform them separately, OK? And the symmetries are one of the most important aspect of physics, and they have very important implications, et cetera. And the chiral symmetry actually has also very important effect in particle physics-- for example, the pions.

The pions has to do with-- I will not go into detail. The pions-- they essentially come from the chiral symmetries. Without the chiral symmetries, there's no pion. There's no pions, OK?

And actually understanding how the pions come from the chiral symmetries, et cetera-- there was a Nobel Prize to Nambu a number of years ago, which our colleague Goldstone also made a very important contribution to that. And, also, nature-- also interesting about this chiral symmetry is that they are there.

Say, if you have a classical massless-- say, if you have a massless Lagrangian, then you can have the symmetry in the classical level, in the Lagrangian level. But once you quantize this theory, you find the symmetry goes away. It becomes anomalous. It could become anomalous.

Symmetry is only present in the classical level but not in the quantum level. And, again, that plays a very important role in particle physics, actually. OK. Yeah, the bottom line is that the chiral symmetry is very important in many aspects of physics. It's also important in many condensed matter systems, like liquid helium, et cetera.

So you can also write this symmetry for general gamma. So for general gamma, you can have your previous symmetry. So, now, I'm using the four-component notation, which psi L and psi R transform the same way.

And then, now, you have a new symmetry. Gamma 5-- OK. And now you can put the gamma 5 in the exponents. Good. Any questions on this? Yes.

AUDIENCE: [INAUDIBLE]

HONG LIU: Yeah, yeah. Alpha tilde is just some other constant--

AUDIENCE: Oh, alpha tilde.

HONG LIU: If tilde is just some other constant, then you multiply gamma 5. So the way to understand that these two are related-- so think about the transformation here. So, here, we can rewrite a little bit differently.

We can consider rewrite this alpha L and alpha R in terms of the following. Let's consider the two transformation-- one transformation, psi L, and psi R transform the same. And the other transformation is that they transform oppositely. They transform in the opposite phase, OK?

So I write the alpha-- and this writing is like that, OK? So, in this way, psi L and psi R transform the same, but-- psi L and psi R, they transform opposite because they have opposite eigenvalue on the gamma 5. And so this is equivalent to that. Good. Any questions on this? Yes.

AUDIENCE: Why is it called gamma 5 rather than, say, gamma 4?

HONG LIU: I think, again, it's a historical reason. So people often like to go to Euclidean space. So when you go to Euclidean space, you continue gamma 0 to gamma 4.

Yeah, just-- yeah, to gamma 4. And then you reserve gamma 4 for that. Then the gamma 5 is the next one you take. Other questions? Yes.

AUDIENCE: Is there a physical reason why the massless case is special? Is it because it becomes scale-free, or something like that?

HONG LIU: Yeah, yeah. So massless case-- it's always special. And you will see, in physics, actually, the massless case actually gives you very much richer structure, normally, than the massive particle. Mathematically, it's because the massless case-- in the massless case, the representation of the Lorentz group is very different from the massive case.

For example, if you have a vector field, say, for Maxwell field, for the photon, massless photon have two polarizations. But if you have a massive vector-- say, for the photon is massive-- then we'll have three polarizations. And so the massless case and the massive case are very, very different.

The fermion case is the same. So if you have a massive fermion, you have four complex components. But if you have a massless case, then you have two complex components.

AUDIENCE: How about very small mass? Doesn't that cause a problem if you go from x [INAUDIBLE]?

HONG LIU: Yeah, yeah. That's a very interesting question. And there are a lot of subtleties associated with that kind of questions, indeed. Yes?

AUDIENCE: Is there any sense in which we can treat psi L and psi R as different dynamical fields, like a complex scalar field the case phi and phi star?

HONG LIU: Oh, psi L and psi R certainly are independent-- you can certainly treat them as independent dynamical, yeah. Other questions? OK.

So let's conclude our discussion of the chiral spinors. And now we can talk about the Majorana spinors. OK, so the so the Dirac spinor, which we have talked about so far, is four components. So you have four times two real components, OK?

And then the chiral spinor we talked about-- essentially, you have two complex components. It's, say, 2 times 2 real components, OK? 2 times 2 real components.

So the next one I'm going to talk about is the Majorana, in which case, I would argue, we have 4 times 1 real component. So it has four independent real components, OK? Yes?

AUDIENCE: Sorry going back real quick. So I'm assuming R and L are for right-handed and left-handed in this [INAUDIBLE].

HONG LIU: Yeah, yeah, yeah, yeah.

AUDIENCE: So can you ever have a spinor where-- what does it mean to have one of [INAUDIBLE] non-zero? So both of them are non-zero, I guess. What is that-- because, in my mind, it is either right-handed or right handed. What is the mixture of--

HONG LIU: No, no, no, no. Physically, electron contains both left and right. Yeah, so for massive particle, because they always couple together-- so the left hand, so psi L turning into psi R, psi R turning to psi L all the time. And for massive particle, you cannot really separate them.

But, now, for massless particle, then psi L and psi R-- yeah, this is a very good question-- psi L and psi R are preserved. So massless particle is either psi L or psi R. And then if you look at their so-called helicity, it's either left-handed or right-handed. So that's where the name psi L and psi R come from.

But, for the massive particle, you cannot make this separation. But for the massless case, then it's preserved. And then it's generally left-handed or right-handed.

OK, good. And, now, let's talk about the last case-- this case, which is-- you have four real components, OK? So what do you do? Again, we follow the similar strategy to see whether it's possible to have four real components.

Again, we first-- the idea is that you first try to find the special representation of gamma matrices so that a spinor that can be real, OK? And then you try to generalize to any representation of gamma matrices. OK.

So, now, if you look at the Dirac equation-- now, look at the Dirac equation.

So if I want to find the real spinor, then I ask myself whether real spinor is compatible with this equation, OK? So if the real scalar is not compatible with this equation-- because I take the complex conjugate, I then take a gamma mu star.

If psi is real, then psi also have to satisfy this equation, OK? But, in general, gamma mu is complex, as we wrote before-- yeah, I just erased it. For example, in these spaces, it's complex.

With complex, then these two equations are not compatible, and then psi cannot be real, OK? Simple as that. But there's a way out. The way out is that, if they exist, the representation of gamma mu, the question is whether there exists a representation of gamma mu so that gamma mu is real.

So if this is real, and then these two equations become the same, and then for psi being real is compatible with the Dirac equation, OK? It's compatible with the Dirac equation.

So, now, let's-- so then this becomes a question of trial and error, OK? So you try to find the representation of gamma matrices so that it's real OK. And then it turns out, you can find it, and here is the answer.

So let me just write down the answer. I don't know how he originally found it-- Majorana originally found it, but here is the answer.

So this is four gamma matrices. And you can see there, each of them is real because sigma 2 is pure imaginary, and sigma 1, sigma 3 are real. And so this is purely real. And you can check. This satisfied the algebra of the government matrices.

They satisfy the algebra of gamma matrices. They anticommute with each other, and each of them square-- so each of them squaring 1. So these three square them into 1. This square equal minus 1.

So, now, this is compatible with Dirac equation, but this is actually not enough. We also have to be sure this is compatible with the Lorentz transformation, OK? So, now, let's check whether this is compatible with Lorentz transformation.

So, now, we have this sigma mu-- gamma mu. And, remember, the sigma mu, nu-- I just erased it. So now this is pure imaginary because if gamma mu and gamma nu are real, their commutator is also real. And then sigma mu, nu will be pure imaginary.

And then that means S lambda-- so this is now purely real. Now, this means this is real. And, now, we are done, OK?

That means that if we take the psi is real, then after Lorentz transformation, this remains to be real, OK? And so that means that it's compatible with Lorentz transformation. So if we were not compatible with Lorentz transmission, then we were finished, OK? So this shows that this remains real.

So such a spinor is called a Majorana spinor, OK? So it has four real components. Yeah, we wrote there-- it has four real components.

So you can quantize it, which, I think I will give it as an exercise for you to do, OK? You can quantize it. And then, in this case, then the fermions are its own antiparticle, rather than for the Dirac spinor. You have particle and antiparticle, so this is the analog of the real scalar in the spinor case.

So this was discovered by Majorana in 1937, OK? And he was very young. At the time, he was 31. And, yeah, a brilliant physicist, complete genius. And then, in 1938-- so he lived in Sicily, OK? So his hometown was in Sicily.

So he boarded a ship from Naples to Sicily. And then he just disappeared on the ship, never seen again. At the age of 32, he just disappeared.

Yeah, it's a quite-- yeah, extremely brilliant physicist. Yeah. And there are all kinds of stories about his disappearance-- that he may be killed by Mafia or maybe suicide, et cetera. But just nobody knows. Yes?

AUDIENCE: You said that all the choices of gamma matrices were equivalent.

HONG LIU: Yeah.

AUDIENCE: This doesn't really feel equivalent. It is still equivalent to the other representations?

HONG LIU: Yeah, yeah. We will talk about that. So now we have chosen a very specific representation for gamma matrices, which psi can be chosen to be real, OK? But how about for the general representation? So now we talk about the general.

Majorana spinor-- yeah. Yeah, also, let me just make a remark. Majorana spinor, of course, also plays a very important role in modern-day physics. Say, for example, people suspect a neutrino could be a Majorana spinner, OK?

So to check whether a neutrino is a Majorana spinor-- yeah, it's a forefront experimental program-- has been pursued by many years. And, also, in condensed matter, in quantum information, and Majorana spinor play a very important role.

And so, in condensed matter, you only have electron. So electron and the Majorana spinor is, like, half electron, OK? Because the electron have eight components, right? Remember. And Majorana only have four components.

So Majorana is, like, half electron. So precisely because it's heuristically half electron, it has very stable topological properties, which a single electron does not have. And whether you can engineer in your condensed matter systems, Majorana spinor then became a Holy Grail. Because if you can do it, and then you can do lots of-- yeah, you can achieve more stable quantum computation, et cetera.

Yeah. During the last number of years, there have been various experimental reports. People say they have engineered Majorana spinor in the lab, which I think has never been fully confirmed, I think. None of them has been fully confirmed.

Anyway, so, yeah, so, now, let's talk about Majorana spinor in general basis, for general gamma mu. So the idea would be similar to the case of the chiral spinor. For the chiral spinor-- in the chiral basis, it's very simple, just upper and lower components.

So, for the chiral spinor, you have to introduce some other structure to isolate psi L and psi R. So you have to-- now, you have this nontrivial condition, OK? And the chiral fermions come from this nontrivial condition.

So, now, the key is that, how do you find the analogous condition to the psi to be real in the general basis, OK? Because when gamma mu is generally complex, clearly, you cannot set the psi equal to psi star, OK? That does not make sense. You have to find another equivalent equation to do, essentially, the same thing, OK? So that's the basic idea.

OK, so for this purpose, we want to-- now, we use any gamma matrices that are equivalent to each other up to a similar transformation, OK? So let's denote-- this basis by gamma m, OK? And then we have gamma m, which is this called Majorana basis.

And then any choice of gamma mu then related to gamma m by a similar transformation-- that means there exists some matrix C that the C can take any gamma mu into gamma mu m, OK? So there must exist C, and this equation is satisfied. Good.

So, now, given this C, then we can easily write down the condition for the general basis because under such a change of basis, the spinor in the Majorana basis, which is real, is related to the spinor in the gamma mu basis by this transformation C, OK?

So the C relates to the gamma matrices. But the C, of course, also relates to the spinor. It's just a change of basis, OK? And so psi M will be related to psi by C.

And now, since the psi M is equal to psi M star, so that means that C star psi star should be equal to C psi, OK? So that means the psi star should be equal to B psi with B equal to C minus 1 star C, OK? So that should be the condition which you impose in the general basis.

OK, so that should be the condition you should impose in the general basis. So you have to introduce this C. So if you find that the transformation between the general gamma mu and the gamma mu m, and then you can use that to find the B. And once you find the B, and then you can find the-- you can impose the-- yeah, so this is callec a Majorana condition in the general basis.

So let's understand a little bit. So, actually, we can understand the B more directly, OK? So, here, we expanded from C. But we actually can find the B more directly.

We can just take the complex conjugate of this equation because gamma mu m is real. So we can also take the complex conjugate of this equation. So that means that, since gamma mu m equal to gamma mu m star, but if we take complex conjugate of that equation, means that the C star gamma mu, C minus 1 star is equal to C gamma mu, C minus 1.

And now, again, you just-- sorry. This would be star, OK? So, now, if we put all the C to this side, and then we find gamma mu star is just equal to B gamma mu, B minus 1, OK? So you just put this to this side. Then you just-- so this becomes B, and then this becomes B minus 1, OK?

So, now, we see that B-- this actually makes sense. This is actually the matrix we take gamma mu to gamma mu star, OK? So B is the matrix to take star, OK? Good. Any questions on this? Yes?

AUDIENCE: [INAUDIBLE] this is [INAUDIBLE]. What group to they belong to?

HONG LIU: Well, they're just general, nonsingular matrices. Yeah. Yeah, just 4-by-4 nonsingular matrices. They often can be chosen to be unitary. But, in principle, you don't have to choose them to be unitary.

OK, so, now, let's double check. So let's call this equation star star. So let's check that star star-- so we show that, here, in this representation, this is compatible with Lorentz transformation, OK? So we still need to check star, star is compatible with Lorentz transformation.

So what do we mean by this is compatible with Lorentz transformation? We mean that, if we take a psi which satisfies this condition-- take a psi which satisfies star star, and then we make a Lorentz transformation psi prime equal to S lambda psi, and then psi prime should also satisfy that condition-- star star, OK?

So that means this is compatible with Lorentz transformation. It means that the psi prime star should be equal to the same as B psi prime, OK? So it means that psi prime star should satisfy that equation.

So, now, let's check that. Now, let's check this. So from here-- so before checking that, do you have any questions on this? OK.

Good. So, first, from this equation-- let's call this star, star, star. From this star cube equation, so we can find the B when you act on sigma mu B minus 1. So that gives you minus sigma mu, nu, star, OK?

So this is obvious because sigma mu, nu has i there. So the minus sign comes from the i. And, otherwise, the B takes each gamma matrices there into star, OK?

And then that means that the S star, lambda, which is given by exponential 1/2 omega mu, nu, sigma star-- sigma mu, nu star, now this is equal to-- yeah, you can just plug this in.

It just becomes exponential minus i omega mu, nu, B sigma B minus 1, OK? So let me just-- yeah-- B sigma mu, nu, B minus 1, OK? So I just inserted the sigma mu, nu, star. It's equal to minus that here, OK?

So, now, you can see that this B minus 1 is in the exponential. You can immediately take it down. So that is just equal to B S lambda B minus 1, OK?

So because when you expand this in power series, B and B minus 1 always cancel, except the first one and the last one. So we have used this trick many times. Yeah, and then so we get a very nice relation that, under Lorentz transformation-- so the Lorentz transformation matrix on the complex conjugation, again, is generated by this B, related by this B matrix.

And, now, it's just immediate, OK? And now just immediate, so when you have the psi prime equal to that, let's just take the star of this equation, OK? So psi prime star just equal to S star, lambda psi star, OK?

So that is equal to B S lambda B minus 1 and B S-- B psi, OK? So this is equal to B S lambda psi, OK? So this is equal to B psi prime-- precisely what we were trying to show.

Good. Any questions on this? OK, so, now, let me give you an explicit example of this matrix B in the-- so for this Majorana representation, the B is just equal to identity, OK? So B is just equal to identity in this representation.

And, now, let's try to give you an example of the B in the other representation. So suppose, in the chiral representation, which I wrote down before-- so, yeah, I should not have erased it. Yeah, anyway.

So in the chiral representation I wrote down before, so if you stare at that expression, you find that gamma 0, gamma 1, and gamma 2-- or gamma 0, gamma 1, and gamma 3 are imaginary, pure imaginary. And the gamma 2 is real. Gamma 2 is real, OK?

So this pure imaginary means, when you take the star of them, you get the minus sign. So this one, you take the star of them, you just get back to itself. So, now, if we look at this equation, so if this is pure imaginary, you get the star. You get the minus itself.

And then, essentially, you get the minus self means the B actually anticommutes with gamma mu, OK? Because you can just bring B minus 1 to this side, so it just becomes gamma mu star B equal to B gamma mu. So if this is minus gamma mu, that means B should anticommute with gamma mu, OK?

But if gamma is real, that means B will commute with mu, OK? If mu is real, then that means B should commute with mu. So, now, in this chiral basis, these three are pure imaginary.

That means B needs to anticommute with them. But this is real. It means B needs to commute with this guy. Then what is B?

AUDIENCE: Gamma 2.

HONG LIU: Hmm?

AUDIENCE: Gamma 2.

HONG LIU: Exactly. So B, in this case, can only be gamma 2, OK? And then we can work out, what is the Majorana condition in this basis-- so, essentially, this condition.

So that means that the psi star should be equal to gamma 2 psi, OK? So that's the Majorana condition here. So, now, remember, in the chiral basis, we can write psi in terms of the psi L and psi R. So, essentially, we have this condition.

So I erased my gamma 2. So I saw the gamma 2-- let me write it here explicitly. It's minus i gamma 2, 0 minus i sigma 2 and sigma 2, here, OK? So, now, if you look at this condition, this means that no longer psi L and psi R are no longer independent of each other.

So psi L star should be equal to minus i sigma 2 psi R or, equivalently, psi R equal to i sigma 2 psi L-- or L star, OK? So, in this case, the psi, then, have the following form-- psi L i sigma 2 psi L. So sigma R just can be expressed in terms of sigma L.

So this is the Majorana spinor in the chiral basis, OK? You see, there are only four independent, real components because each psi L is two complex components, OK? Yes?

AUDIENCE: So why do you [INAUDIBLE] that psi and psi star are not independent [INAUDIBLE]?

HONG LIU: Hmm?

AUDIENCE: Why are they--

HONG LIU: No, no, no. No, here, we are imposing this condition, right? We are imposing this condition. Yeah. Yeah, this is Majorana condition we want to impose in this basis.

AUDIENCE: And this is now independent of massless or massive particles?

HONG LIU: Yeah, yeah, yeah. Yeah, this is-- yeah. Good? So this concludes our discussion of the Majorana spinor. Do you have any questions on this? Yes?

AUDIENCE: So is the orthogonal component of psi-- the Majorana fermion [INAUDIBLE]?

HONG LIU: Sorry?

AUDIENCE: The orthogonal component of this Majorana species-- like, possible-- in the chiral one, like, psi L, and then you [INAUDIBLE] and then psi R. [INAUDIBLE] two components of psi.

HONG LIU: Yeah.

AUDIENCE: So we are-- we have one component of psi, and there should be another one, right? So is that one?

HONG LIU: Sorry. I don't quite understand your question. Say it again-- what?

AUDIENCE: OK, [INAUDIBLE].

HONG LIU: OK, OK. Other questions? Yes?

AUDIENCE: So, I guess, related to an earlier question, how do we consider handedness here for a massless particle that's also the Majorana, which psi L and psi R are [INAUDIBLE]?

HONG LIU: Right. So for the-- yeah, if we-- for massless particle, you can just direct it because then you don't have to think about the psi R, and this, you just have the same number degrees of freedom as a massless particle. Other questions? Good. OK.

OK, good. So let's now go to the next topic. We only have a few minutes, so we can only just make some general comments. So so far, mostly, we have been talking about continuous symmetries. But there are also discrete symmetries, OK?

So, by definition, discrete symmetries are symmetries, which don't have continuous parameters, OK? So continuous symmetries are symmetries, which-- yeah, which the transformation dependence on continuous parameter. Discrete symmetries, you just don't, OK? You don't have continuous parameter.

So simple example-- say, let's imagine we have-- so this real scalar theory-- we can see that before. So this theory has a discrete symmetry because this is invariant under phi. It goes to minus phi, OK?

So because you see all the terms are even, so it's invariant under phi go to minus phi. And this transformation is no continuous parameter, OK? So this is a discrete symmetry.

And so this is-- if you do it twice, you go back to itself. So this is often called the Z2 symmetry. OK, so this is called the Z2 symmetry.

And there are also spacetime discrete symmetries, OK? So this is an internal discrete symmetry-- have nothing to do with spacetime, OK? There are also spacetime discrete symmetries.

So spacetime discrete symmetries including, say, if we consider Minkowski spacetime-- so you can have t goes to minus t. So you can have so-called time reversal, which corresponding to your t, x goes to minus t, x, OK? You just transform the time.

You can also have the so-called parity. You take t. Then you revert all the spatial direction, OK?

So comment that you can ask why we actually reverse all three directions. How about if I just reverse one direction or reverse two directions, OK? That seems also to be a discrete symmetry. And indeed.

So if you just change the directions, say, in the x direction, that's also a discrete symmetry. And if you only change the direction in both x and the y direction, that's also a discrete symmetry.

But if you change-- if you do the reflection in two directions, that's equivalent to a 90-degree rotation-- a 180-degree rotation in that plane, OK? And so it's part of the continuous symmetries, so it's not independent discrete symmetry.

And now, when you change all three directions compared to change one direction, you differ only by changing two directions. So that means changing all three directions and changing one direction-- they differ by 180-degree rotation, OK? So that means that when you change-- this is the only independent discrete symmetry from the spatial reflection point of view. OK.

So for a complex scalar field-- so if you consider complex scalar field-- if you can see the complex scalar field, then this is no longer a discrete symmetry because, remember, we can rotate phi by a phase. When you rotate the phi by a phase, if you take that phase to be pi-- say, exponential i pi-- and then you take to be minus phi.

And so, in that case, this is part of the continuous symmetry, so it's no longer independent discrete symmetry. But, here, there's, nevertheless, another discrete symmetry. Can you see what is the other independent discrete symmetry here? Yes? Good.

You can take phi to phi star, OK? You can exchange phi to phi star, OK? It's a complex conjugation. And this is often called charge conjugation.

OK. This is often called charge conjugation because, remember, heuristically, we can think of phi as create-- yeah, it's just one of them create the particle, and the other create the antiparticle. And they have opposite charge, OK? So it's called a charge conjugation.

So, this, will give a symbol called T-- script T. And, this, we give a symbol called P-- script P. And, this, we give a symbol called script C. So, altogether, they are called CPT symmetry, OK? Yeah. Yeah, let's stop here.