[SQUEAKING]

[RUSTLING]

[CLICKING]

PROFESSOR: So last time, we talked about Dirac equation. So the Dirac equation has the following form: gamma mu partial mu minus m psi equal to 0.

So here, we have two spaces. So one is your standard physical spacetime, so with x mu. And then we have a lot of internal space which is labeled by the index of the psi and the gamma, which has suppressed here.

So psi should be considered as a four vector. So alpha equal to 1, 2, 3, 4, and the gamma mu should be considered as a matrix-- a 4x4 matrix. And so this space labeled by alpha-- so this is called the spinor space.

This is called spinor space. So you have to be careful that now we have two spaces intertwined together. One is your ordinary spacetime, and so this is a function of the spacetime, but it's carried on index-- yeah, carries on indices.

And so this is the matrix equation. So altogether, there are four equations here for each component of psi. And yeah, so let me pull-- let me just write this a little better. So this is x-- let me call this equation 1, which I will use later.

And this gamma mu, they are not ordinary matrices. They satisfy this condition. Gamma mu, the anticommutator with nu equal to 2 eta mu nu.

So that means different gamma matrices when they are indexed are not the same. They anticommute with each other. So if you pass them through, you get a minus sign. So they anticommute with each other because when mu not equal to nu, the right-hand side is 0.

And when you pass through each other, you get the minus. And the square of them-- so gamma 0 square, you get minus 1 because this is 0, 0. It's minus 1. And the gamma i squared, you just get one. So this is very simple. Yes?

STUDENT: When you say gamma i squared is 1, you mean like its the identity matrix?

PROFESSOR: Yeah, its identity matrix. Yeah, when I say gamma 0 square is minus 1, it's also minus times identity matrix. Yeah, exactly.

And also not all gamma mu are Hermitian So the relation is that the gamma mu dagger is equal to gamma 0 gamma mu gamma 0. So again, from here, you can-- yeah, so this is just a compact way to write its properties. When you have-- when mu is equal to 0-- and then you have gamma 0 times gamma 0 squared is minus 1. And then you get the minus gamma 0-- tells you that the gamma 0 is anti-Hermitian.

And if you take the index to be i here, and then you have index i here-- and then when 0 and i-- they are not the same, so they anticommute. You can pass through this gamma 0 through this gamma i. You get the minus sign, and then gamma 0 square gives you another minus sign, so you get 1.

So that means that the gamma i dagger equal to gamma i. So gamma i is hermitian. Yes?

STUDENT: So is there supposed to be an i in front of the derivative or do we absorb the i into the gamma?

PROFESSOR: So in my convention, there is no i. So with this-- so eta mu mu here is mostly plus metric. So some people, when they use, say, mostly negative metric, then there may be an i in there. Yeah, so it depends on your convention in defining the gamma matrices.

STUDENT: So for minus plus plus plus, no i-- OK, sorry.

PROFESSOR: Yeah. Good. Other questions? Yeah, as I mentioned last time, different conventions are annoying, so-- but you just stick one convention-- should be fine. Stick to my convention-- would be fine. Yes?

STUDENT: Sorry, how can we get the expression for gamma mu dagger?

PROFESSOR: Oh, yeah, this is just a compact way to write down that gamma 0 is anti-Hermitian and gamma i is Hermitian. Yeah, just-- this just compact. Yeah, just this is useful so that you can treat all gamma mu in the same way so that you don't have to always separate them.

Good. Other questions? So last time, we also said you have many, many choices for gamma mu, but they are all physically equivalent. And so different choices, they are convenient for different purposes. So often we just pick one of them, and depend on the problem we are-- yeah, so that comes with a little bit experience.

So for certain problems, certain choices of gamma become more useful-- become more easier to manipulate. Good. So now, let's talk about the Lorentz covariance of the Dirac equation, which we started a little bit last time.

So Lorentz covariance means that the equation looks the same in all Lorentz frames. So when you make a Lorentz transformation-- so imagine you make a-- go to another frame. You start with your frame defined by this x mu.

So now, imagine you go to another frame which you make a Lorentz transformation. So the Lorentz covariance means your equation when written in this frame in terms of x prime mu have the same form. So last time, we talked about the scalar equations at the end.

And another equation we have seen before is the Maxwell equation, so let me also very quickly mention for the Maxwell equation. So when you have a Maxwell equation, which is the story for the vector potential-- so now, in contrast-- remember for the scalar case-- so last time-- let me just write it down here.

Last time we said for the scalar case, when you go to a different frame, the phi prime should be equal to-- evaluate the new coordinate should be equal to phi x. So this is the transformation for scalar field. But now, if you want to do Maxwell equation-- so we have to think about how this a mu transform under Lorentz transformation.

And now a mu is a vector-- is a four vector in spacetime. And so that means that the a mu should transform also as the vector. So this means that when you go to a new frame, A mu transforms as lambda mu nu A nu x. So here, not only-- so when A mu evaluate at new position-- it can be written as a linear superposition of the value at your original position.

And this linear superposition by itself is a Lorentz transformation because the A mu is a vector. And so that means just under Lorentz transformation, the different components of A mu should also change under the transformation. So given this-- so I will not show it now.

Just will take a couple of minutes, but you should convince yourself-- try to check yourself that the Maxwell equation, which can be written-- so for example, as F mu-- mu nu equal to 0. This implies that the partial mu prime in the new frame, and then the f mu mu prime. f mu mu prime is obtained from this new A mu prime, and these two equations are the same-- are equivalent. Just in one frame, and then when you go to a different frame, then you get this equation.

So the equation has exactly the same form, but now it's in the new frame. Good? So now, let's go to Dirac equation. So now, go to Dirac equation. So in the Dirac equation, again, this psi-- we need to ask how the psi should transform under Lorentz transformation.

But now, this psi is a completely new object. So this is not like in the Maxwell case. We know this is a spacetime vector, so you can easily guess how you should transform. So now this psi-- this spinor space is completely new. So now, we have to figure out how psi transforms. So let's suppose under a Lorentz transformation lambda that psi transforms as follows.

Psi prime alpha x prime-- because now, again, you evaluate in the new position-- new frame. Since this carries index, in principle, this can be a superposition. You can mix them in the internal space.

In principle, we can have form like this. Say for some matrix S, which depends on lambda. So in principle, you can have a transformation like this. Say when you evaluate new position, it's given by the value at the old position, but now you can make some rotations in the internal space just as here for the Maxwell field.

So except-- and so suppose this and then the Lorentz covariance would be the statement, say, for some matrix. And then the Lorentz covariance then is a statement starting from equation 1-- you should be able to find going to the new frame-- you have an equation like this-- gamma mu partial mu prime minus m psi prime x prime equal to 0. So you should get an identical equation-- not identical equation 2. You should get an identical equation, but now everything is in terms of prime.

So I want to emphasize that gamma mu does not change despite a carry index mu. Mu gamma mu just some constant matrices-- just the-- it's just four matrices. It does not transform under any spacetime trans-- it's not dynamic variable. It does not transform on the spacetime transformation, so gamma mu does not change.

Good. So now the question of the Lorentz covariance-- where the Dirac equation is Lorentz covariance boils down whether we can find such S. Whether-- yeah, the question is, can we find, such an S? So if we can find such an S, which this is true, then we say the Dirac equation is Lorentz covariant.

And this is how the Dirac field should transform. So this psi is normally called the Dirac field, and this is how a Dirac field should transform. So psi is often called Dirac field. It's also often called spinor field.

And it's also often called Dirac spinor field. And anyway, and then that should be the way how it's transformed. Yes?

STUDENT: Yeah, I was just confused, because before, for the scalar fields, when we get transformations, we would either say-- like, transform the fields, like phi goes to phi prime or transform the coordinates, like x goes to x prime. So why did we transform both in this case?

PROFESSOR: Yeah. No, no, no, in this case-- so when you talk about transform of the scalar field itself, of course, it's just phi equal to phi prime and-- because that function changes. But here, we are asking about the form of the equation in the new frame. And so when you go to the new frame, your function form changes, but your coordinate also changes.

So that's why you need to-- when we say the covariance, you need to evaluate your new function in the new position. Yeah, that means they look the same in different frames. Yes?

STUDENT: So should I think of S as the representation of lambda in the spinor space?

PROFESSOR: Yeah, exactly. Good. That's exactly the right mathematical language to talk about this, which we'll mention later. At the moment, I didn't want to use mathematical language.

Yeah, for those people who are familiar with group theory, indeed. So this S would be the representation of the Lorentz group in the spinor space. Good.

So now, let's try to find this S. So before we do that, let's first write what is a partial prime x mu. So recall, partial prime mu, it should be partial, partial x prime mu. This transform the new coordinate. And from that transformation, you can easily figure it out-- just chain rule.

So this is the same equal to partial, partial x mu and the lambda minus 1 nu mu. So this is very easy to figure it out because they have to be-- x will multiply the x. Partial x will have to be a singlet, so this transform as a inverse matrix.

So this implies that the gamma mu partial mu prime should be equal to lambda minus 1 nu mu gamma mu partial nu. Let me just write it better. so with this preparation, then we can try to see whether we can find the S from this equation. No-- yeah, find this equation from 1.

So what we can do is let's multiply the equation 1 from the left by S. So let's imagine we multiply S-- such a matrix. Again, I always just suppress the spinor indices and just write in the matrix form.

So let's just write gamma mu. So take that equation 1, we multiply it by s. So we have two terms here. So this is a matrix in the same space as S. S act in the same space as gamma mu, so we cannot easily commute here-- normally, they don't commute.

But this-- but m is a constant, so we can pass S through here. So we can rewrite this equation as follows. We can write it as S gamma mu S minus 1 partial mu minus m S psi 0.

So for m, I just pass S through, but for this term, I just inserted S minus 1 and S-- and so yeah, it's the same. So now, we can use this equation in here, because S times psi just equal to psi prime x prime. So now, we get S gamma mu S minus 1 partial mu minus m psi prime x prime equal to 0. We just use that equation. Yes?

STUDENT: I'm a little confused how you can pass S through partial mu because given that partial mu acts on psi and it has to have a representation in this inner space [INAUDIBLE].

PROFESSOR: Yeah, so that's the key I emphasized earlier. So x mu and psi alpha, they are in different space. And so all this S and gamma, they are all constant.

They don't depend on spatial location. And so partial mu does not act on S. Yeah, for partial mu point of view, S is just a constant.

It's a spacetime constant. It's only rotate psi at a single point-- the different component of psi at the single point. Good? So now, we have this equation.

So now we can compare this object with this object. Yeah, in particular, this object is equal to that. So we conclude that 1 goes to 2 if S gamma mu S minus 1 equal to this object lambda minus 1 nu mu.

I think-- yeah, I think I'm messing up a little bit notation here. Gamma mu-- I think my index is a little bit wrong. Let me just make sure. Oh, yeah, here is a partial nu. Sorry, here is partial nu, so I need to exchange the mu and the nu index here.

So mu and the nu, then gamma. So this has to be equal to that. So this has to be equal to that, and you need to compare partial nu with partial nu-- you have to exchange and nu and the nu here anyway. So this is the equation we have to satisfy.

So we have to find a matrix S which acts on gamma mu. Gamma mu is some bunch of matrices. And gives you like this. So as if the gamma mu actually-- when you act on Sophia as if gamma mu transforms as some Lorentz transformation-- yeah, inverse Lorentz transformation.

So we need to-- now let's try to find this S. So to do this, again, we use a trick which we have been using before. So how would you approach this problem?

STUDENT: The identity.

PROFESSOR: Good. Yes, just do infinitesimal transformations. So once you learn how to do infinitesimal transformations, then you always know how to do finite ones. And infinitesimal ones, it's much, much easier to do.

So now, again, we consider Lambda mu nu close to your identity. So that means we write lambda mu nu as delta mu nu, which is the identity. Then plus omega mu nu, and take the omega to be small.

And also remember previously, we discussed that omega mu nu, when you know the index, is actually antisymmetric. And this is infinitesimal, so we take this to be infinitesimal and work everything to first order in omega. So similarly, the lambda minus mu nu just equal to delta mu nu just [INAUDIBLE] minus omega mu nu.

So yeah, just to leading order in omega expansion-- the inverse metric is just given by that. So now, we can try to-- so since the S-- so on the right-hand side, when lambda is close to the identity-- so the right-hand side is just the gamma mu. Just the identity does not do anything. Just gamma mu, and then plus something proportional to omega.

Then, that means that S must also-- when lambda is close to identity, it means S must also have the structure to be identity and proportional to something linear in omega. So the S must also have this structure. Yeah, let me just don't write the index. Just directly write as identity.

And then it should be something proportional to omega mu nu. So from convention, we write this way. So it should be linear in omega mu nu and sigma mu nu. So sigma mu nu are a bunch of matrices.

So remember, S is a 4x4 matrix, and so omega mu nu is just some number. So this will be a bunch of 4x4 matrices. So for each omega can in principle independently multiply some matrix. So this is the most general way to expand this linear order in omega. So this sigma essentially just the first derivative of x with respect to each omega mu nu.

And this i over 2 is just convention. And similarly, the inverse is just corresponding when you change the sign here to leading order in omega. Yeah, so emphasize each sigma mu nu is a matrix. It should be understood as some matrix in the spinor space.

So now, you just need to plug in this equation. Plug in this S and S minus 1 into this equation. Yeah, just expand the both sides to the omega, and you equate the coefficients. You equate the coefficients.

From that way, you determine the sigma. So let me call this equation star. Yeah, I'll just call this equation 3. So then to order omega mu nu to linear order. Then, the equation 3, if-- when you expand on both sides to omega-- to order of omega and equate both sides, then you find the following equation.

This is just a couple of lines algebra, so I urge you to do it yourself. So given by i commutator lambda rho, gamma mu equal to eta lambda mu gamma rho minus rho mu gamma lambda. So you get this equation.

So the left-hand side is very easy to understand. So essentially, you just-- whenever you have a commutator-- for people have done this Baker-Hausdorff et cetera, the first order, you always get the commutator. So that's where this commutator come from.

So the right-hand side, when you expand this, essentially, you just get omega mu nu because we have to lower the indices. So you have some eta here. You have eta here, and yeah, so that's how you get the right-hand side.

And the reason you get two terms-- because it should be antisymmetric in the-- yeah. Good. So now, it just boils down to solve this equation. So if we can find sigma-- satisfy this equation, and then we are done.

So yeah, of course, this is-- now, you do it by trial and error. And the bottom line is that there's a solution. So let me just write down the solution.

And the nice thing about other people found the solution is that you can just check it. So you can check this quantity-- this solves the equation. So just can plug this. So this is the commutator of gamma lamda and gamma rho.

You plug into that, and then you just evaluate this gamma matrices and use this kind of equation over and over, and you will find this is satisfied. Again, I will leave it as an exercise for yourself. Yes?

STUDENT: I have a question about this board. So you start off with saying that partial mu is transformed and that's how you get this lambda inverse. Then, the last thing is now actually-- this gamma's transformed and it's not partial mu transformed. I'm a bit confused how it-- it seems like you're linking a transformation in partial mu with a transformation in--

PROFESSOR: No. So here, we want to match 1 and 2. So the step is we do-- well, have each equation-- from equation 1, we reach here. From equation 2, we just plug this into there. And so each equation has done one step, and then I equate them.

STUDENT: Right, but I guess what I'm asking is equation 1 has partial mu transformed, and then that's how you get your lambda inverse. And then in equation 2, you are transforming your gamma.

PROFESSOR: No, we want to show this-- you want to match this equation with this equation. This equation is derived from 1. So we want to derive-- we want to find the 2 from 1.

This is equation 1. So I just slightly rewrite the equation 2 by inserting this transformation here, then I matched them. Other questions? OK, good.

So now, we can just-- now we can just immediately-- so given this equation, and now we can immediately write the final transformation. So with the finite transformation-- so for each lambda mu nu, each finite-- you can obtain the corresponding omega mu nu, and then this is also finite. And now, we can just obtain the S by exponentiating this.

And then the corresponding S would be S equal to exponential minus i over 2 omega mu nu sigma. And the sigma mu nu is just given by this one. And you can check yourself. This satisfies-- you can check this satisfies equation 3.

Yeah, that's finite equation. Good. Any questions on this? OK, good. So now, let me-- was there some-- so now, let's make some remarks.

So then this S, it just generates Lorentz transformation in spinor space because it only act on this alpha and beta. So sigma i j-- or sigma mu nu according to our standard terminology, this is called the generator of the transformation. So these are the generators.

So when omega mu nu-- when the mu nu equal to spatial directions, and then that's corresponding to a spacetime-- spatial rotation and the 0 i corresponding to a boost. So remember. So that means for sigma ij, which is i equal to 4, gamma i, gamma j-- this generates generators of rotations.

So remember, previously-- so if you remember how we do the Lorentz transformation, the omega ij would correspond to the rotational angle in the ij plane. So you just rotate in the ij plane by angle omega ij. So then this just corresponding now-- corresponding to the generator of the rotations in ij plane.

So you can-- because the gamma i and the gamma j are Hermitian, recall that the gamma i is Hermitian. And then the sigma ij is also Hermitian because when you take the dagger of this i gives you a minus sign, but the commutator gives you a minus sign. and so this is Hermitian.

So that means that S, which corresponding to rotation, is unitary. So this is unitary. So this is a unitary matrix. So now let's consider the sigma 0 i, which corresponding to the generator for boost.

So this will have the form gamma 0 gamma i, so this is the generators for boost. In i-th direction in the spinor space. And now, because the-- remember gamma 0, when you take the dagger, you get the minus sign. So the sigma 0 i, now if you take the dagger, you actually also get the minus sign because the-- yeah, so now it's anti-Hermitian.

So that means the boost matrices-- so this is a boost transformation-- so this is not unitary. So this is not unitary. So in general-- so normally, as we said before, normally, when you do a symmetry transformation, the transformation is a unitary transformation.

But in this case, actually-- yeah, this is a classical transformation. Here, it's actually not a unitary matrix. So this implies that S dagger, in general-- for general Lorentz transformation, say, which including both rotation boost-- say S and S dagger is not equal to 1.

S dagger-- S is not equal to 1. So this has very important consequences, for example, for writing down the action for the Dirac equation. So so far, we only wrote down the Dirac equation, but we did-- remember, previously, we normally started with the action first, and then from the action, we derive the equation of motion.

But in this case, since this spinor is a completely new concept, we started with actually the equation. But now, if we want to write down an action-- which is by definition, it should be Lorentz invariant-- then we should construct quantities which are invariant under Lorentz transformations. And so this property then becomes a key. Yes?

STUDENT: So like, we're-- so these are operators on the spinor space, and we're talking about Hermiticity and stuff. But Hermitian is with respect to an inner product, and we haven't talked--

PROFESSOR: No, here, we are not talking about quantum mechanics. Here, we're just talking about the equations-- classical equations. We are just talking about-- they're matrix-- they're just matrix. We are talking about whether they are unitary matrix or they are not unitary matrix. They are just ordinary matrices-- 4x4 matrices.

Other questions? So that means psi dagger psi-- so psi dagger is a row vector, and this is a column vector, and altogether this is a number. So this means this is not a scalar.

So this transform under Lorentz transformation as psi dagger S dagger S psi. So that means this is not-- since this is not equal to 1, then this is not scalar under the Lorentz transformation. So now, we have to search a little bit harder to find the scalar.

So in order to write the action, we need to find something which is invariant under the Lorentz transformation. So the easiest thing to think about is this quantity, because this automatically gives you a number. But this thing won't work, so we have to search it a bit harder. So to do that, let's-- we can get some hint from the following identity.

So let's look at what this S dagger really is. So let's look at the property of S dagger. So recall that the-- I think I erased it, so let me just write it again.

So gamma mu dagger is equal to gamma 0 gamma mu gamma 0. So now, let's try to find what mu nu dagger. So let's write mu nu dagger in the uniform-- so even though we wrote it separately.

So you can easily check yourself because of this property. So this is just given by minus gamma 0 sigma mu nu gamma 0. Because this is just a commutator, so you can easily work it out that you just get this.

So now we can find what S dagger. So S dagger is equal to the exponential 1/2 i omega mu nu, then sigma mu nu dagger. And then this is equal to i over 2 omega mu nu minus gamma 0 sigma mu nu gamma 0.

So now, if you remember, gamma 0 squared is equal to minus 1. So whenever you have such a situation-- and because gamma 0 squared is equal to minus 1, then you can actually take the gamma 0 outside of the exponential. So this is actually equal to minus gamma 0 exponential i over q comma 0 mu nu sigma mu nu.

So if you just think-- you do a Taylor expansion of this exponential, and then when you take the power of this, then for each term, the gamma 0 at the end will pair with a gamma 0 at the beginning of the other term, and then they give you minus 1.

And then you have only the first gamma 0 and the last gamma 0 left. And because that gives a minus 1, that changes this minus sign to a plus sign. So you just do a Taylor expansion you will find here.

And now, we find the nice relation. So we find that this is just minus gamma 0, and this is just equal to S minus 1 S gamma 0. So we find that the S dagger is actually minus gamma 0 S minus 1.

Then, this tells us from this property-- tells us that this quantity psi dagger gamma 0 psi should transform as a scalar. So now, let's take a look. So this, when you do a transformation-- so this gives you the psi dagger S dagger. Then, you have gamma 0, then you have S psi under Lorentz transformation.

And now, you plug this into-- plug in S dagger equal to this into here. You just get minus psi dagger, so this is gamma 0 S dagger S minus 1 gamma 0, then gamma 0 S psi. So this gives you minus 1-- cancels with the 1 here.

And the S one cancels with-- S minus 1 cancels with S, and then you have dagger goes here. So this actually is Lorentz invariant. So now, we find a nice Lorentz invariant quantity.

So since we use this all the time, it's convenient to introduce a new notation. So now, I introduce -- convenient to introduce recall psi bar equal to psi dagger. So here, it's all just clustered. This is dagger, and it's all just matrix manipulation.

Right now, we're considering a classical theory, and so it's convenient to introduce objects like this. And then we know that psi bar-- psi is a scalar. So that thing just becomes psi bar and psi. So it's convenient to work out the-- how psi bar transforms by itself.

So you just use this relation so you can check yourself. So as an exercise for you to check yourself under Lorentz transformation psi bar x is equal to psi bar prime x prime. So if you go to the new psi bar prime x prime is equal to psi bar x S minus 1.

So you can easily check yourself this relation. Similarly, using this transformation of gamma partial mu, you can also check that the gamma mu partial mu so prime psi prime x prime equal to S gamma mu partial mu psi.

So you can also check yourself-- check this equation. So yeah, just by using these properties of S dagger. Yeah, just-- yeah, this, you don't use the-- you only use for above the S dagger. Just use how partial mu prime transform and how psi transform, and then you can show this is true.

Good. So this-- also this gamma mu partial mu will appear a lot because this appears in the Dirac equation. So it's convenient to introduce a new notation. Say gamma mu partial mu we define to be partial slash. So essentially, anything slash corresponding to that thing contract with gamma mu.

And then this equation then can be written as the partial prime slash psi prime x prime equal to S partial slash psi x. Good? So let me just mention one more thing you can work out yourself. So you can check-- so these are all things you can check yourself once you equip those transformations.

You can also check that gamma psi bar gamma mu psi transforms as a vector. That means psi bar prime x prime gamma mu psi prime x prime is equal to lambda mu nu psi bar x gamma nu psi x.

So if you view this whole thing as a vector, and then you see the prime-- the quantity is equal to just a Lorentz transformation lambda on itself. So again, this is something just based on the transformations of S and the relations between different gamma matrices you can just show that. Good. Any questions on this?

So now, with these preparations, we can now write down the actions which gives rise to the Dirac equation. So now, we can write down the Dirac action. The action which gives rise to the Dirac equation. So I will just write down the answer. It's very intuitive.

So you have just essentially S equal to minus i d4 x psi bar gamma mu partial mu minus m psi. So this is the answer.

Yeah, so there are various things to uncompact here. So first, just based on those relations, based on this is a scalar, this is a scalar, and the transformation of this, you can immediately see that this is a Lorentz scalar because it only involves two variables-- one quantities-- psi bar, psi with this m term, and then psi bar gamma mu partial mu psi. And that we already show here-- that the transform as S.

So yeah, you can easily check just based on those equations that this is a scalar. And this is Lorentz invariant. So the second thing is that this i here is for-- to make the action is real, because you can show if you take the complex conjugate of this, you actually get the minus sign, and so you need the i to make it real. And this minus sign, which cannot be explained now, which we will talk about it later when we quantize the theory and we see actually, we need to put the minus sign here. Yeah?

STUDENT: And the definition of psi bar has psi dagger gamma 0. It feels like we're singling out the time dimension and the definition, because gamma 0-- is that true that we're singling out the time dimension?

PROFESSOR: We are not really singling out the time direction. just due to the property that gamma 0 is not a Hermitian. Yeah, just because if you look at all this complicate-- just related to that this S dagger is not-- yeah, S is not unitary. And the reason S is not unitary is because gamma 0 is not a hermitian.

Yeah, so you have to put in the gamma 0 in various places to compensate for that. Other questions. Yes?

STUDENT: So I guess the action, if you wanted a Lorentz invariant-- a Lorentz scalar, you could have constructed some dot product of that object that transforms as a vector, right? So the psi bar gamma-- so that would be allowed in principle. I guess this gives the right answer, but that could've appeared in the action.

PROFESSOR: Which-- yeah, this appeared in action. Yeah, so this is the-- so because this is a vector-- so when this contracts with partial mu, that gives you a scalar. Yeah, so that gives you a scalar.

So yeah, you can understand that this is-- in both ways, this transform-- the fact this transform vector is also related to here. I mean, just-- yeah. Other questions? So-- yes?

STUDENT: To follow up on that, if I contract that with itself, would it give like another [INAUDIBLE].

PROFESSOR: What?

STUDENT: if I contract that [INAUDIBLE] with itself.

PROFESSOR: Yeah, but then you will have four psi's. So four psi's will give you rise to an interacting theory. Yeah, that's right.

Here, I'm writing down a free theory right now. So you can also see that this equation gives rise to the Dirac equation. Just imagine psi bar is independent of psi, because this corresponding to psi dagger.

So if you do the variation of this one, you just automatically get the Dirac equation. Then you can also easily check by integration by part. And if you were psi, you get the complex conjugate version of the Dirac equation acting on psi bar.

So now let me just say a few words on why you need the i here. So you can check. So here, I'm listing some relations. Again, each of them, you need to really write down the paper yourself, stare at it, maybe do a derivation to get intuition about yourself.

Right now, I'm just writing down those relations. So you can also check yourself that when you take the psi bar psi dagger, again, just go through all these things. You find it's equal to minus psi bar psi. So this is just two lines here.

I will not write them explicitly for you. And similarly, you can check related to this-- the other term that the gamma psi bar gamma mu partial mu psi-- if you take the dagger-- so here-- so this one is slightly more complicated, but you can still-- you can just walk it through. You find it's just equal to negative of itself, then plus total derivatives.

So this one is not exactly minus sign, but you have to throw away some total derivatives. So total derivatives which give you-- when you plug into the action, it gives rise to boundary terms, but it will always assume the field vanish at infinity. So that explains why you need this i, because when you take the complex conjugate of the quantity here, you always get the minus sign.

Good. Any questions? Good. So now, it's getting a little bit awkward because our printer today broke. So I didn't bring enough of my notes, so now I have to look at my computer to remember my notes. One second.

Good. Ugh. I have to find the location. OK, good. So any questions on this? No?

OK, so now let's move to the next topic. So now, this concludes our discussion of the Dirac equation. So we have derived the Dirac equation by-- and also we have discussed how various quantities in Dirac equation should transform, and also finally the Dirac action.

So the next step-- the logical next step is to quantize to go to the quantum field theory. So right now, so far, everything is classical. So now, the next step is that we want to quantize this theory.

And when we quantize the theory, we will see remarkably fermions. So we will see fermions. We will see Pauli principle. But before-- but as we said before, that when we quantize the theory, we first need to-- the simplest way is to first find this all is classical solutions, and then the classical solutions then become solutions to the operator equations, and then we can automatically quantize.

So then before, we actually do the quantization. It's better we try to find all the classical solutions of Dirac equation. And so it's not like a Klein-Gordon equation-- we can just immediately write down the solutions. And the Dirac equation is a little bit more intricate, so we need to spend a little bit of effort to write down the-- to find all the solutions of the Dirac equations.

So that's what we will do. Unfortunately, we are not going to finish today, and then we will have a long spring break. So I hope you still remember what we talked about today when you come back.

So this is [INAUDIBLE]. So now, we have classical solutions. So by construction, the Dirac equation has solutions. We know must be proportional to e to the i k x with k squared equal to minus m squared. So this is by construction.

Because we square it, we get the Klein-Gordon equation. So this k mu would be omega k and k. But this is not enough because the Dirac equation-- because psi have four components.

So this just determines one factor of it. And we also have to determine it's-- the behavior of all its four components. And so now, that's we are going to do now.

So we will separate the solutions into two types. So one type is we call psi plus x, which corresponding to uk x, the expression ik x. And another we call the psi minus x, which corresponding to vk x. vk-- yeah, sorry-- no x-- because it's minus ik x.

Yeah, so uk vk, they just-- they're all four-component spinors which have the same thing as psi, because this is just some number. So because this is-- because k have the positive frequency, so this is some kind of-- called positive energy solution.

And this is called the negative energy solution. But it's the same thing as in the scalar case. They don't really-- there's not really-- when we quantize the theory, there's really no negative energy excitations.

It's just a name. So we call the positive energy and negative energy just a name. So the actual physical excitations always have positive energy. And this uk and vk, they are all four-component complex vectors. Yes?

STUDENT: So it seems like the psi should be labeled by k.

PROFESSOR: Yeah, psi-- so this is just the basis of solutions. Indeed, we should label them by k. Good. So this-- so essentially, we just-- in the scalar case, you just expand it in terms of plane wave, and then you just get some constant.

So here, its a little bit indicate we have a vector. Now, we need to solve these vectors. So our goal is to solve these vectors. And then you just plug them into the Dirac equation.

Just plug these two into the Dirac equation, and then you get the equations for the uk and the vk. And I will also suppress the k in the uv and v just for notational simplicity. So when you have the gamma mu-- partial mu minus m, psi equal to 0, you just plug the psi plus minus m, and then you find that the following equations for u if you get i k slash minus m u equal to 0, and the i k slash and the minus-- yeah, I think-- let me just double-check the sign.

Yeah, indeed. So you get i k slash plus m V equal to 0. So you get these two equations. So our goal is just to solve those equations. And the k slash, just as defined before, it just defines as in the partial slash k slash equal to mu gamma.

Good. So you can also work out the complex conjugate. So let me just write down the equations, because sometimes they will be used later. So the u bar i k slash minus m equal to 0, the complex conjugate. And v bar i k slash plus m.

So now, let's try to work out the uk and the vk by solving those equations. So we do this by-- you can, in principle, do it by brute force. So after all, you just-- these are just 4x4 equations-- 4x4 matrix equations. Just linear algebra.

You can-- in principle, we can solve it. But the physicists are often lazy, so we often-- still even for the problem, you can solve-- we still look for shortcuts. And so in this case, there are two possible shortcuts.

There are two possible ways we can do, and so let me describe both ways. Actually, I think we only have time to describe one way. So to find the explicit form of u and v-- so one simple thing to do is let's just consider in the simple case-- say, consider the particle is at rest. When the particle is at rest, then the omega is just equal to m, and then k equal to 0.

And then that equation just becomes i m gamma 0 minus m u equal to 0 and i m gamma 0 plus m v equal to 0. I think-- do I-- so I think the mu-- oh, yeah. Sorry, I should have a minus sign here because of the-- it's the k upper index.

So it's the k upper index equal to omega. Then k lower index equal to minus omega. And here, when we contract with gamma 0, we use the lower index, so we have a minus sign here.

So essentially, this just becomes-- so m can be canceled on both sides. So essentially, it just becomes the i gamma 0 u equal to minus u and i gamma 0 v equal to-- OK. So this just tells you essentially gamma u-- u-- it's an eigenvector of u and v. They are just essentially the eigenvectors of gamma 0.

And so let's just now-- to write them explicitly. So now, let's use the explicit representation of gamma matrices. Let me just copy this. So now, let's use this representation with the gamma 0 is equal to i0 0i minus i. And then gamma i is equal to 0i sigma i minus i sigma i0.

And then now, if you plug in this gamma 0 into those equations, and then you find the equation for u and v becomes very simple. For the u, it just becomes 0, 0, 0, 1 u equal to 0. So again, this is 2 by 2 blocks because this here is 2 by 2 blocks. And for v, just the 1, 0, 0 v equal to 0.

So that means u-- you can just take it to be the upper-- so that means the solution of this equation-- so that means that the solution of this equation for u-- so now let me write the upper index 0 means that this is for the 0 momentum. So here, we can just choose to pick psi 0. So psi is some arbitrary two vector.

For v, we can just choose the lower 0 eta. So psi eta are arbitrary two vectors-- two complex vectors-- two vector-- two component vectors-- two component complex vectors. So once you have u0 and v0, and we can choose a basis. For example, u0 1 equal to 0, 1, -- 1, 0, 0, 0, and u0 2 equal to 0, 1, 0, 0.

And similarly, for v0 1 and v0 2. So you can choose to be here 1, and-- so you can just choose as a basis-- you can just choose [INAUDIBLE]. And now, once you have-- so once you have the vector at k equal to 0, for general k, what do you do? Yes?

STUDENT: Lorentz boost.

PROFESSOR: Yes, just do a Lorentz boost. So we know the matrix S. And then you just-- for the general k, you just S u0 and vk Sv 0.

And then you can find the behavior general k. But this is-- but this is easily said than do. To do this actually is not quite easy. Even though this sounds like a great idea-- OK, let's find the 0 momentum and then let's just do a boost.

But this step is still a little bit tedious. But still it's doable and a little bit simpler than solve the original equation by brute force. But there's, again, still another simpler methods, which you can actually just guess the answer.

You don't have to do any calculations. You can just guess the answer-- guess the solution for the full equation. And I think we don't have time to talk about it today, and so we will talk about it next time.

And yeah, so hopefully, you still remember what we talked about today when we come back. And hope you have a good spring break. Yeah.