[SQUEAKING]

[RUSTLING]

[CLICKING]

PROFESSOR: So at the end of last lecture, we start talking about this propagator. So in non-relativistic quantum mechanics, in the Heisenberg picture, we can introduce this position eigenvector corresponding to the eigenvalue of the position operator at time t with eigenvalue x. And then from this object, you can construct a propagator.

So this object is powerful if you know how to compute this object, because if you load the function at t prime, and then by-- yeah, by just integrate it with this object, and then you can find your function at a later time. And so this is a powerful object in non-relativistic quantum mechanics. So now, we can ask what are the analogous objects-- say whether they exist analogous object in field theory.

But in QFT, there is no natural way to define a localized state such as-- so there's no counterpart of this object. So one reason is that now there's no position operator anymore. So remember, now, x is actually a label-- no longer a operator.

It's the phi is the operator. x is just a label. So there's no-- in quantum field theory, there's no position operator. So you cannot-- and then you cannot define-- yeah, so the position operator, then you can-- no natural way to define the eigenvector associated with localized state. And there's also a fundamental barrier-- there's a fundamental contradiction with Lorentz symmetry.

So there's also a fundamental barrier-- fundamental obstruction from Lorentz symmetry. So you can show that localized-- you can show a localized state in space is not Lorentz covariant. So this does not-- so suppose-- yeah, you can do this by proof by contradiction.

Suppose they exist-- there exist states like some states-- say, x t. So by definition, a local state should satisfy in-- if you evaluate them at the same time, then that should be proportional to delta function because that's by definition what it means what we mean by a localized state-- localized at the point.

And that's the property satisfied by this. If you set t prime equal to t, and then this will be a delta function. But now, suppose you can define such a state, but then this notion cannot be covariant defined. It's because this is not a covariant object.

So the three-dimensional spatial delta function that does not transform nicely under Lorentz transformation. And so you can show that if you act-- suppose U lambda is the unitary operator generating a Lorentz transformation, which we discussed last time. If you have the momentum-- if you have the generated Lorentz transformation.

And then you can show that the U lambda acting on x and t is not the same as the lambda x and the lambda t. So lambda x and lambda T denotes a Lorentz transformation of this vector. Here, I'm using a shorthand notation. So lambda-- x means the Lorentz transformation acts on x, and here, means the Lorentz transformation acts on t.

And you actually will do this explicitly in your pset. You will check this explicitly. This does not transform. So this cannot be a covariant concept.

So that means that you can most define in some frame, but if you change your frame, and then this state is no longer localized. So this concept is not the Lorentz-- it's not compatible with Lorentz symmetry. So the closest analog you can define in field theory-- closest-- sorry, not closed-- closest analog in QFT in field theory is this object.

So let's define an object x could be defined by-- so again, whenever I write something like x, you should view this as a shorthand for the full vector including both space and time. And so this second, define an object by this-- acting-- let's just consider-- suppose we have a real scalar field theory. So for example, let's just consider real scalar field theory.

So we can define an object like this. So this is the closest we can get into a position eigenstate. And then-- yeah, its conjugate. Then it's given by phi. So phi is a hermitian, because it's real, so it's a hermitian operator.

So now, you can check this object by definition because phi is a scalar field will transforms nicely on the Lorentz transformation. And the vacuum is invariant under Lorentz transformation. And so this object actually transforms nicely.

So under Lorentz transformation, x actually it goes to lambda x. So this object actually transforms nicely. But you can also ask, what is the overlap between these two such kind of position eigenstate if we call this quoted position eigenstate?

So now let's consider the overlap between two such states. And then this will be given by-- from this definition will be given by 0 phi x and phi x prime on there. So it's given by the expectation value of x times phi x prime between the vacuum states just from the definition.

So the reason we say this is the closest analog is because even though this state is Lorentz covariant, but this state is not localized. So if you want a state to be localized, this has to be proportional to delta function when you take the them at the same time. But now you can show-- so this is the object we can compute explicitly, because we already solved the phi completely in terms of a and a dagger. You can just plug-in here to compute this object explicitly, and we will do a towards the end of today's lecture.

Here, let me just tell you the results. So you can see that this guy actually is non-zero for t equal to t prime and x not equal to x prime. So that means this is not-- this means that this cannot be a delta function. So when you evaluate them at the same time, if it's a localized state, then it should be proportional to delta function. But this is not the case.

So this object is not quite localized, but we will see actually in some sense, it's localized, and we will see in what sense later when we compute this object explicitly. So right now, let's just say the conclusion that this is not localized. So this is not-- so this means this is not a localized state.

Good. So heuristically, we can say-- yeah, so let me just make a side remark here. So if we-- so as we say, this is not a position eigenstate. But if we treat it as some kind of approximation to the position eigenstate, and then we can talk about the wave function of a particle.

So now let's suppose-- so given a single particle state, psi-- so psi would be say some kind of superposition, which we wrote last time-- so some arbitrary superposition of the single particle state integrate over the momentum. And then we can define its wave function.

So I put this function as a quote as psi x equal to just overlap between the psi and this x object. And then this is given by just from that definition-- just given by 0 phi x psi. So we can define a function-- so this is closest analog you can define a wave function for particle in the field theory is to use this object.

And now, you can check-- so it's a very simple exercise for you to check yourself. So if I just take this to be the pure momentum state-- say, for example, if I take psi to be the pure momentum state of k-- remember, this notation k is the square root 2 omega k on this state. So this is the pure momentum state.

And then if you calculate this object, and then you get-- then k x then just becomes 0 phi x k. Now, you can again-- you know explicitly how phi expressed in terms of a and a dagger, and you know how this is obtained from the vacuum by acting a dagger on it. You can just compute this object. So take a couple of minutes.

And you can also just essentially guess the answer. So this just gives you the plane wave-- just gives you an initial plane wave. So with your energy are given by omega k. So you just get the plane wave function. So in this sense, it's like a wave function. It makes sense to call it the wave function. Yes?

STUDENT: So in this state here, we call this the wave function. If we take it, and you multiply it by its own complex transpose-- can we interpret it as a probability?

PROFESSOR: Yeah, this is the comment I'm going to make now. Yeah, this is a good question. So we cannot really-- so we cannot-- so in general-- so psi x prime, in general, cannot be interpreted as giving the probability of the particle at x.

So the reason we cannot do that is because this is not-- because x is not the genuine position eigenstate. And also in quantum field theory, we don't really have this-- we don't really have this proper sense of talking about-- you cannot really-- in general, you cannot separate the single particle with multiple particle exactly, in general, in quantum field theory.

In free theory, you can because there is no interaction. But in general, you cannot. So the reason this is a side remark is that this is often used by people, and they just call it a wave function. But you should keep in mind what's the content of this object, and it's not something you can actually rigorously define both mathematically and physically as a wave function. Any questions on this?

STUDENT: So could you explain how you got e to the negative i?

PROFESSOR: Oh, you just calculate. But it's very easy to see how you get that because remember, phi x is-- phi x can be expanded in terms of a u plus a dagger u star. And the k is obtained by a dagger acting on 0. Essentially, just a and a dagger part, and give you this object, and a part proportional to u, and that's how you get it. Good?

So when we compute this object explicitly, you will get a more precise sense what I mean that this is an approximation. You will get a more precise sense. And so this-- I'll now defer that a little bit because I want to talk about something else first.

Good. Other questions? So here-- so this actually is a very important object. Forget about this definition. So this object-- yeah, let me just write this object again. So this G plus-- so this object 0 phi x and phi x prime 0-- so we motivated as this overlap between these two localized states-- these two approximate localized states.

But in quantum field theory, actually, this is a very important object, and actually, we normally assign a rotation G plus with this object. And so the application of this object, we will not talk too much about it. But in condensed matter-- for example, say in phi, describes some approximate-- some continuum description of a spin system.

And then this would be measure the correlation between the spins at different locations-- at different locations and different time. So this is what people call the correlation functions. And so in condensed matter, this actually plays a very important role, and matches the correlation between different physical objects.

And so here-- so when x-- for x and x prime, at non-equal time-- in general, phi x and phi x prime do not commute. Remember our commutation relation-- canonical commutation relation is phi-- phi x and phi x prime, they commute when they at equal time.

When they are not equal time, in general, they don't have to commute. So the ordering here is actually important. So the ordering of phi in G plus is actually important.

So if I define some-- I can consider a lot of objects with a different ordering. Say if I put the phi x prime first, I can also consider this object for the phi x prime first. So this is normally called-- now, this is a different function in general because they don't commute and we call it G minus. So this is also appropriately just x prime x.

So this is the x prime x and this is using that notation is x prime. And so in general, G plus and G minus, they are not the same. So you can also define some other functions-- say since the phi and phi x and phi x prime don't commute, then you say-- then we can consider more general orderings.

We can consider the superposition of these two. So you can consider the retarded function so-called, which is defined to be the theta t minus t prime-- the commutator of phi x and phi x prime. So this is some other object you can define, which is the difference between the two, but it only takes value when t is greater than t prime.

So when t is smaller than t prime, then it's 0. So this is another other object we often use. And so this often is denoted by theta t minus t prime, and call this object-- this commutator to be delta t, t prime.

So sometimes, people also use the following object-- the G A equal to minus t, t prime minus t. So only it's non-zero when t prime is greater than t and the delta. Oh, sorry. Here, it should be x, x prime.

And finally, you can define an object called G F which is half-- so this is called the retarded. This is called advanced, which is like half of the retarded and half advanced. So when t greater than t prime, you take the G plus. And when t is greater than-- t prime greater than t-- let me do it here-- you do G minus.

Again, this G F is a function of x and x prime. So because they don't commute, so these are various objects you can define. So one second. So at the moment, they may seem not very intuitive to you-- why should we be worried about those objects.

And later, you will see actually some of those objects play a very important role, and some of these objects will play a very important role. Yeah, for now, they're just some definitions which we will use potentially later. Yes?

STUDENT: So considering spins on a lattice, would x and x prime be the positions of the spins on the lattice? So why would it physically differ one I define which-- it seems arbitrary to switch them right?

PROFESSOR: Yeah, if you do them at equal time, then it doesn't matter. But remember, x and x prime, they don't have the same time, and if they're not at the same time-- and then actually, sometimes, the ordering matters. Yeah, whether you do this measurement first, and then you wait a while to do that measurement, or you do this measurement first and then-- yeah, then that can be different. Yeah?

STUDENT: So G minus is x' inner product with x?

PROFESSOR: Oh, yeah.

STUDENT: G plus just uh, wouldn't it just be--

PROFESSOR: Yeah, just they're switched. Yeah, remember this order is important. So here, I have x first and x prime second. So this order is different-- is important.

STUDENT: But is G minus then just the complex congugate of G plus?

PROFESSOR: Yeah, it is the complex congugate of G plus. Other questions? Yeah?

STUDENT: Yeah, so like the retarded and advanced ones aren't like Lorentz covariant because of the theta function?

PROFESSOR: Yeah, we will talk about that. Yeah, they are actually. Yes?

STUDENT: Can you go back and explain more about why the non-equal time non-equal position operators don't commute?

PROFESSOR: Yeah, it just say our canonical-- yeah, our canonical commutation relation only requires them to commute at the equal time. And when they are not equal time, and each of them will evolve. And remember, the phi itself is-- phi itself depends on time.

Yeah, so let me just-- if I just write the-- yeah, let me just write the-- so this is the k, say a k u k plus a k dagger u k dagger u k star. So this is my expression for phi. So canonical commutation relation only requires that they are equal when t equal to t prime.

But when t is not equal to t prime, I just have some general expression. I do the commutator-- does not guarantee you will get 0. Yeah, you just get-- yeah, you have different terms. They may not cancel each other.

So at t equal to 0, we guarantee that they cancel each other, but when t is not equal to 0, then not guaranteed to cancel each other. But we will figure out when they cancel, will not cancel. Yeah, we will talk about that more explicitly. Yes?

STUDENT: Is it correct that if the point are spacelike separated then we can rotate so that t -- to a frame where t --?

PROFESSOR: Yeah, we will talk about it. Yeah--

STUDENT: [INAUDIBLE]

PROFESSOR: Yeah, we will talk about that. Good? So let me just emphasize again, this is something in principle you can define. And it's phys-- some of its meaning will be clear later.

And in the next pset-- not in this pset-- in the next p set, you will see an example, which this thing actually play a very important role. So this is the analog of the retarded function in E and M? Do you still remember retarded function E and M?

And yeah, so this is quantum version of this retarded function in E and M. And you will see application of this in the pset four. And later, we will use this thing all the time. So this G F, we will use it all the time when we consider interactions, and so that's why we want to introduce it here.

And this G plus G minus are called the Wightman functions. So sometimes, called the Wightman function. The G F is called the Feynman function.

So Feynman was the first person who introduced it. And yeah, so G plus G minus was introduced by other people, but Wightman made many important use of it in the early days of quantum field theory, so it's called the Wightman function. So now, let's just study a little bit the properties of those functions.

So you can also think these are essentially the simplest observables of the system. And in particular, if you think of them as correlation functions, and then that just tells you the correlation between the phi at a different point. So this is in some sense, also the simplest observables in your theory. Good? Do you have other questions? Yes?

STUDENT: So this is the correlation for the vacuum state, right?

PROFESSOR: Yeah, this is the correlation vacuum state.

STUDENT: So you have a different state?

PROFESSOR: Yeah, if you're not in a vacuum state, of course, you get a different function.

STUDENT: So is there a way to measure for other states?

PROFESSOR: Yeah, you can compute. So in this theory, we can compute in any state. But just we often use -- we probe in the vacuum state. Other questions? Yeah, for example, in the lab, if you can prepare in some whatever state you prepare, you calculate this object in that state, but the vacuum is the simplest one. It's the simplest. Yes?

STUDENT: [INAUDIBLE] we can, we can define a useful localized state in space and [INAUDIBLE] other [? state ?] defined is also [INAUDIBLE], spacetime is useful [INAUDIBLE]?

PROFESSOR: No, it's not quite in spacetime. We will see in what sense it's approximate to the local in space. Good. Now, let's just discuss the property of those objects.

And so the first thing is that the-- remember the phi satisfy the following equation. So this is the operating equation satisfy phi. And so if you act this on the-- so suppose this is-- so here, you have two arg-- here, you have two arguments, x and x prime.

So here, when we write differential operator, we always refer to x. It's always derivative of x. So we must satisfy this equation. So if you act this on the G plus and the G minus, you immediately get 0.

So you find-- because of [INAUDIBLE] equation, then G plus/minus are just x and x prime immediately gives you 0 just because phi satisfies this equation. Oh, sorry, it should be a minus sign, I think. And similarly, with this-- similarly with delta-- so the delta is the difference between the two. It's the commutator.

So the delta is essentially the difference between G plus and G minus, so the same thing with delta. So this is-- the first property is that they satisfy the same equation as the classical equations of phi. So the G plus and delta, they satisfy. And we will talk-- and then you can now also look at the G F and the G R.

So now, if you act this object on G R, G A, or G F, the story is a little bit tricky because the partial square containing time derivatives. And when you add the time derivatives-- and time derivatives do act on those theta functions. And when you add time derivative on theta function, you will get some delta functions, et cetera. And so the story is a little bit more intricate.

But if you just carry it through-- it just carries through-- then you can show. And so this is a simple exercise you should try to show yourself-- we'll not do it here-- that you find, you act this on the G R A, and F. You get actually the-- any of them, you get the same equation. You get the right-hand side becomes-- it's non-zero-- actually gives you a delta function.

So this is a four-dimensional delta function. So in both in spatial location and in time. So heuristically, you can understand that this delta function in time comes from taking derivative on theta functions and then the derivative in the spatial direction-- oh, delta function in the spatial direction can come out in the following way.

So here, you have two-- so here, you have two time derivatives right in the partial square. So imagine you have one time derivative acting on the theta function, and then give you a delta function in time. And then if you have one time derivative acting on this phi, and then taking it into pi.

And then equal time commutator then give you a delta function in the spatial direction, and so that's how you get that. Yeah, but it's easy to check yourself. So this is the first property-- is that they satisfy those nice equations.

The second property is that phi x and phi x prime, even though in general they don't commute, but they actually commute for space-like separations. So that means that if you look at the commutator of phi x and phi x prime, this is actually equal to 0 for x minus x prime greater than square greater than 0 for space-like separations. So this can be checked by expression computation.

So we know the mode expression of phi. You can just do it commutator-- yeah, you can just check it. But you can also-- but there's also a simple argument to make it.

So you can do-- you can check by expression computation. So here, I give you alternative argument. Here, I give you an alternative derivation.

So yeah, it's very simple, but let me just divide it into three small steps. So essentially, by definition-- so let's just consider-- yeah, we consider the commutator, which here we call to be delta. So the delta x plus/minus-- oh, x and x prime, which is the commutator.

Yeah, let me define-- yeah, sorry-- yeah, so delta here is the expectation value of 0, so let me call this delta hat. So the delta hat is defined to be the commutator. So the first claim is that this is actually a C number.

So this is C number, and the reason is very simple. So just some constant-- even if it's non-zero can, at the most, be a constant. Be some-- yeah, but C number means that it's not an operator.

So the reason this is C number is because if you look at the phi, it's linear in a and a dagger. If you look at the commutator between phi and phi itself, you always just get commuted between a and a dagger. a and a dagger just give you a constant, and so this just-- you get some C number.

So in fact-- so this delta hat is actually the same as delta because the, if your C number, you take this expectation value doesn't do anything. So the second step is that now let's consider the Lorentz transformation-- the operator, which generates the Lorentz transformation, which is given by-- remember, we discussed last time. So this is the generator. So this is the unitary-- so this is operator of Lorentz transformation.

Let's look at this object. And then by definition-- so this object acting on phi x dagger will give you phi prime x. And phi prime x transforms phi under the Lorentz transformation, and this is the same as you just do an inverse Lorentz transform on the coordinates.

So this is the-- so by the way, just-- do you feel comfortable about this equation? Do you know where this equation comes from? So you're all comfortable with this? OK, good. So by definition, U acts on phi as this.

So now, let's just act both-- act U and U lambda dagger on both sides of this equation. And then we get-- so on the right-hand side-- so this is a unitary operator -- because the M is Hermitian, this is a unitary operator.

So the right-hand side, you just go back to this C number. It does not change. So the equation does not change. And the left-hand side-- so the delta is the same.

So you find that the-- yeah, so you find the delta x prime, which is equal to this C number. Then, it's the same as U Lambda phi x phi x prime and the U lambda dagger. And that gives you-- me phi lambda minus 1 x commutator with phi lambda 1-- minus 1 x prime. So that's the same as delta lambda x and the lambda x prime.

Good? So now, we can look at the last step. So you see, when we make a Lorentz transformation on delta, actually nothing changes, and the value is the same because the commutator is a C number.

And then-- now remember for any space-like separated x and x prime, we can always choose to find the some lambda-- some Lorentz transformation-- that the lambda x and the lambda x prime to be at the equal time. They have the same time. We can always make a transformation. Yeah, and let me call it something else-- t tilde.

So your original x is for-- you can always transform it to equal time. And then according to this one, then will be identically 0. And then we find delta x and x prime equal to 0, because now, this is evaluated at equal time.

So this is very intuitive. When you make a Lorentz transformation, you can-- yeah-- but this is a precise proof of the statement. Any questions on this? Yes?

STUDENT: What's the x on [INAUDIBLE]?

PROFESSOR: Sorry?

STUDENT: [INAUDIBLE]

PROFESSOR: Yeah, for x minus x prime squared greater than 0. So this is a four distance square. You're asking about this notation? Yeah, this is just-- there are four distance square-- the separation between them. Yes?

STUDENT: I have a question on the first property. So is that equation up in the corner, so that defines the-- it says that G is like the Green's function for your Klein-Gordon operator?

PROFESSOR: Yeah.

STUDENT: So why does that-- is there an intuition for why the correlator in the vacuum state gives you the Green's function? Because then you could use this to get any classical solution right?

PROFESSOR: Yeah, so the intuition is that essentially-- yeah, so for example, this is just how we-- yeah, first is that this kind of thing is simple enough. Essentially, you can just see by observation, but in a more complicated theory. And you can actually derive it-- and the retarded Green function always have this form. Yeah, so that's a thorough more elaborate derivation.

STUDENT: When you're talking about classical field theory, usually solving the Green's function is kind of hard and you have to expand-- like, here, we quantize it and then got out--

PROFESSOR: Yeah, this is different. So classically, we just define those functions using those equations. So here, we just say-- in quantum field theory, the object can be written this way.

Yeah, those objects can be written this way. And this definition just follows from the standard-- yeah, if you do quantum mechanics and you-- a retarded function can always be written like this. Yeah, this is also just in quantum mechanics, but it requires a little bit of calculation to see that. Other questions? Yes?

STUDENT: Should it be lambda minus 1?

PROFESSOR: Oh, yeah. Sorry. Yeah, it's-- lambda minus 1. Yeah, so here also should be lambda minus 1. Other questions?

OK, good. Yeah, so this also shows-- yeah, so when we talk about the canonical quantization-- so we mentioned that we need to impose this equal time commutator to be 0. That's our canonical commutation.

Yeah, we impose the canonical commutation relation at a single time. And some of you were asking that actually-- does that actually break Lorentz symmetry because we have to choose a single time? But now, you can see actually this does not break Lorentz symmetry because the only space-like separated, and the commutator is always the same.

So no matter what frame you choose and the-- in any frame, if it's equal time, and it's always space-like separated in some other frame. Good? So that means-- so immediate conclusion-- it means that for space-like separated, x and x prime G plus equal to G minus is equal to G F.

So the time ordering does not matter, so the G plus-- then you can do G minus. And then the G F equal to the sum of them. And then when-- here, it's for t greater than t prime.

Here, it's for t prime greater than t. And then this should just add equal to 1 because these two become the same. And the G R and the G A equal to delta equal to 0 for space-like surface case. So those functions are pretty simple.

So the-- yeah. So now-- so the last property-- now, you can show due to the spacetime translation symmetry and the Lorentz symmetry of the vacuum-- of the vacuum state. So you can show-- I think this will be-- this is a pretty simple argument, but I will leave it to show yourself.

You can show that any of those functions only depend on the difference between of them-- and in particular, only depend on the four distance between them. And this G here can be any of the G R, G A, G plus/minus, G F, et cetera.

So all of them have very nice properties. Even naively depend on two arguments. So x have four components. x prime have four components. So naively, this is a function of eight variables, but this tells you once you use all the symmetries, this is actually a function only a single variable. This is a very powerful statement.

So they have very-- yeah, so this answers one of the question some-- one of you asked earlier-- that the-- despite the theta functions, you can show that they still have very nice properties under Lorentz transformation. You can show they have a nice properties under Lorentz transformation. Good. Other questions? Yes?

STUDENT: [INAUDIBLE]

PROFESSOR: Yeah, so this is very easy. When you have a translation symmetry, and then the reference point you choose is not-- you can choose any point to be the reference point. And so you can just choose x, and then only the separation between x and the x prime matters, because you can just choose x prime, for example, to be at the origin-- and then only the separation.

And then the Lorentz symmetry tells you that only the distance matters. It doesn't matter the direction. Yes?

STUDENT: Does the commutator have this property too, or is it just G?

PROFESSOR: Yeah, the commutator also have this property. Yeah, the delta also have this property. Yes?

STUDENT: Is this only true when x and x' are spacelike separated?

PROFESSOR: No, this is true in general. This is always true, but it's only in the vacuum state. So the key is that you have to use that the vacuum state are invariant under those transformations.

Yeah, it's a-- once you understand how to do it, it's a very simple argument, so that's why I will leave it into one of the exercises. Other questions? OK, good.

So now, let's compute this object G plus. We have been talking about it. And now, finally, after discussing all these general properties-- so let's compute them explicitly.

So let's just first compute the G plus. Did I just erase the definition of G plus? Yeah. So this is easy to do.

So those properties you can just get by symmetries. Of course, you can also get by doing explicit calculations, but you can get them without doing any explicit calculation just based on symmetries.

So here, we do-- calculate this object explicitly, we will see-- satisfy those properties. So we just put in the expansion of phi and phi x and the phi x prime. And then-- yeah, then we have the 0, and then we have a k u k evaluated at x plus a k dagger u k star evaluated at x times a k prime u k prime evaluated at x prime plus a k prime dagger u k prime star at x prime, and then acting on 0.

So this decay with 2 omega k, and this essentially gives you phi. And the other one gives you the phi-- so this gives you the phi x, and this one gives you the phi x prime. I just calculated this. So now, this is easy to do.

This is just now become an exercise in harmonic oscillator. So only this term-- multiply this term. It's non-zero. The rest of terms are 0, because this will annihilate the vacuum, and this 2a dagger together will annihilate that vacuum.

So the only term not vanishing is this term combined with this term. So you just get-- and there's a delta function when you evaluate them between 0. And so you find-- And then you have -- t minus t prime.

So this is the object. So we can do this integral further, and then we also rewrite this in a slightly different form. So you can also write this as integrating-- so here, omega k is on shell, so I can also write it in a more general form-- in the form like this-- 2 pi theta k 0 delta k squared plus m square exponential i k dot minus x.

So in this form, the k is unconstrained, but I have a delta function to enforce it on shell. And then I put the theta k 0 to make it to be positive. And then you can see that this will lead to that.

You can see that. And similarly, you can find-- similarly, you can find the integral expressions for G R, G A, and G F. I will not give the expression here. Yeah, we will-- yeah, you can easily find them yourself.

So let me just make one remark. So you can now actually calculate this object. So now, let's calculate this object. Remember earlier, we say this object is the overlap between x and x prime.

So let's see in what sense this is like a delta function. In this sense, it's like a delta function when we say them to be equal time. So now, note-- so let's consider the equal time t equal to t prime. So for t equal to t prime-- so the integral becomes following-- G plus x and x prime just equal to-- just become this.

And this integral clearly is non-zero. So that's why we said earlier, this is not a localized state. So this is equal to this x and x prime. So this is non-zero as equal to, and so this is not a localized state.

Yeah, so this is not proportional to a delta function if the thing is here. If you don't have this factor, then it's a delta function, but this one, it's not. But you can nevertheless you can calculate this integral exactly.

And also just based on the here-- based on the spherical symmetry-- rotational symmetry, you can immediately see that only depend on the distance between them. Only-- so this G plus will only be a G plus a function of r, and r is the distance between the-- yeah, just based on symmetry, you can easily convince yourself. And this only depends on the distance between them just from the symmetry of the integral. Yes?

STUDENT: So when we added the 2 omega k [INAUDIBLE]?

PROFESSOR: Yeah, it's for the-- for example, for calculating the-- you wan a, a dagger equal to 1. Yeah, because that's the normalization for the a dagger. Yeah, that's a good question. I forgot to mention that.

Other questions? So this object at t equal to 0 only depends on the distance between them, because this is a spherical integral because this one only depends on the magnitude of k. And then this is just a standard Fourier transform.

So now, you can just evaluate this integral explicitly. I urge you to do it yourself. And then you can show this is actually proportional-- you can actually evaluate this using Bessel function.

So this gives you a modified Bessel function-- K0 times m r. So do any of you-- are any of you expert of Bessel function? So do you know the behavior of this function for when r is large.

STUDENT: Goes to 0.

PROFESSOR: Yeah, it does to 0. That's a good intuition, but it goes to 0 in what way? Actually, exponentially.

So it actually goes to exponential minus mr, so for r large.

For m r greater than 1. Now, you can see in what sense this is approximately localized state. You see this is not 0, but this is pretty small at large distance. So at distance r, it's much greater than 1 over m.

So when r is much greater than 1 of m-- so that object is very small, so this exponential is small. So this tells you that even though this is not a perfectly localized state, this is localized to the distance of 1 over m. So if these two points are separated more than 1 over m, and then the overlap becomes very small.

And so this is-- yeah, so sometimes, we call this xi. And then can in condensed matter-- language which, when you interpret this as a correlation function-- which the correlation between the x and the phi x prime-- and then this is also called the correlation length. So when the phi-- the distance beyond 1 over m-- or beyond this xi and then the phi no longer correlated.

So this a sense-- this is approximately localized object, and it's essentially the Compton wavelength of the particle. It's the Compton wavelength of the particle. And once you go outside the Compton wavelength of the particle, and then the overlap becomes very small.

So this makes perfect physical sense. It makes perfect physical sense. There was a hand?

STUDENT: Yeah, I had a question, but then I realized that my thinking was wrong.

PROFESSOR: OK, good. Yeah, so this is just [INAUDIBLE] of what we said earlier. So you can similarly get such kind of integral expression for those things. But rather than-- the coordinate space expression is actually often-- will be often-- actually lead to the expression momentum space. So it often leads there.

So often-- in the future, we'll often lead the expressions momentum space. So we can just consider the Fourier transform. So here is the convention of the Fourier transform.

So dx-- so any object or function of x-- again, this is a four vector, and the minus i k x. So I use the same notation, G, to denote its Fourier transform, and distinguish them only by k and x because it's just annoying to always put some tilde above it.

And the inverse transformation would be the G x x prime. So you see here explicitly-- yeah, there, we already see explicitly that is only the function of their separations. And then if you use the symmetry here further, then you see the distances.

So we can also just write this in terms of the-- this is i k x x minus x prime G k. So this is our convention for the Fourier transform. So from here, if you just look at this expression and compare with this expression, and then immediately we conclude for this Whiteman function G plus, k is just given by 2 pi theta k 0 delta function k squared plus m squared. So this is the expression for the G plus. Yes?

STUDENT: So this localization is [INAUDIBLE], which is greater for higher momentum, so is this a manifestation of the uncertainty principle?

PROFESSOR: Yeah, this is a-- so here, this is, in some sense, the uncertainty principle. This-- here, we just consider in the vacuum state, so there's not-- the state itself does not have any energy. And so this is just the Compton wavelength of the particle itself.

And it's just like a static Compton wavelength. Other questions? Yes?

STUDENT: If this is the vacuum state, then what does it mean to be [INAUDIBLE]?

PROFESSOR: Yeah, but still you have fluctuations. This is quantum mechanics. Yeah, for example, you can calculate an analogous object for the harmonic oscillator. So if you have a harmonic oscillator, the analogous object is this object, and you can also calculate this non-zero just because of the fluctuations.

Other questions? OK, good. So this is the expression for the G plus. And for those object, we can get a momentum space expression by going through the same procedure. Just plug them in, labor through, and then write the final answer in this four momentum integral form, and then you just read the answer.

For them, we can do the same thing. But it's actually much easier. Instead of going through that procedure to calculate those quantities, it's actually much easier to start from here, because they satisfy these equations.

So let me call this star. So to find G F k the same with G R, A k, it's actually easier to use star. Unfortunately-- so now, if you look at this equation-- if we look at this equation, then let's use Fourier transform on both sides.

We do a Fourier transform on both sides. So the left-hand side, the partial-- each derivative just gives you a k. So this is the standard rule for Fourier transform.

And so here, you just get k squared plus m squared. And let me just call it G, so denote any of them, equal to minus i. Because the right-hand side, when you do a Fourier transform, it just becomes one. It's a delta function.

So now, we can immediately write down the expression in momentum space. But now, you see we have a problem. So what problem do we have? Yes?

STUDENT: When you have k squared equals negative m squared it's like a singularity.

PROFESSOR: Good. Well, we have a lot of problems. So this is one of the problems. We have another problem. Yes?

STUDENT: Is this on shell?

PROFESSOR: Hmm?

STUDENT: [INAUDIBLE]

PROFESSOR: So k is not on shell, because when we do a Fourier transform, it's for the general k. Yeah, it's defined for the general k. Indeed, for the on shell value of k, then you will have a singular behavior. But for the-- but we still have another problem. Yes?

STUDENT: They're not supposed to be complex?

PROFESSOR: Yeah, it is complex. It's OK. Yeah, that aspect is OK. Yeah, so since-- [INAUDIBLE].

So you see here, we have defined the three different functions, but there, how many solutions we have? We only have one. We have unique solution seems like.

Then we have a problem. It seems like-- that seems to say all these three are the same. But if you calculate them going through this theta function [INAUDIBLE], they are not the same. So it turns out these two problems are related.

It turns out these two problems are related because now, if you consider the coordinate space expression, which obtained by the Fourier transform of this guy-- again, by Fourier transform of this guy-- so in the-- so if you write down explicitly 1 over k squared plus-- so 1 over k squared plus m squared, if we write it explicitly.

So this is 1 over minus omega squared plus k squared plus m squared-- write it explicitly. And then this is equal to minus omega squared plus omega k squared. So this is just the omega k squared.

So as you said, this actually has a singularity when omega equal to plus minus omega k. When you satisfy this on shell condition. But this singularity is actually along the integration contour of omega.

Because in these four integrals is given by d omega dk. So the omega-- so this from minus infinity to plus infinity. And these two singularities are the real value of omega.

So they actually-- so if you look at omega axis-- so at these two points, actually, they become singular. The integrand is singular. So the integral is actually not well-defined. So the integral is not well-defined.

So in order to define the integral-- in order to define this omega integral, then we have to do your standard trick in complex analysis. So what is that trick? Yes?

STUDENT: [INAUDIBLE]

PROFESSOR: That's right. You go around the singularity. So now, you go to the complex plane-- complex omega plane.

So now this is a real omega. This is imaginary omega. You go to the complex omega plane. Now, the integration constant is along the real axis.

And now, you have four different choices. You can either go up or going down for each one of them. And now you have four different choices. And this 3 equals 1 into three choices of them. Then there's a fourth one, which is not frequently used, so we normally don't give them a name.

And the fourth one, sometimes, we call them G F tilde. It's just-- anyway, so there are four different choices of going around the singularity, and then that gives you four possible solutions. And then that gives you the-- yeah, so now let's talk about the G R-- do them one by one.

So for G R-- so remember, G R is defined to be when theta t greater than t prime, G R should be proportional to t, to t minus t prime. So that means this should be 0 for t minus t prime smaller than 0. This should be 0 for t minus t prime smaller than 0.

So how do we achieve that by going around the poles? So remember, in this integral, it's a peice minus-- the omega dependent piece is minus i omega t minus t prime. So when t minus t prime smaller than 0.

If you want to do this integral using contour integration, then you can close-- so this is smaller than 0, and then this is positive. And then you can close the contour in the upper half plane. You can close the contour in the upper half plane. In order for this to be identically 0, you need your integration-- counter don't enclose any singularity. So that means for the retarded, you need to going around the singularity along which direction? We only have one minute left.

[LAUGHTER]

For it to be 0, when you integrate-- so you want to close in the upper half plane. You want to close it because the t minus t prime is smaller than 0, and then this is positive i, and the omega in the upper half plane. This is a decay exponential.

And so you wonder, there's no singularity inside the contour, and then you want to go around the singularity this way. Going above the singularity. When you're going above the singularity, and then this contour don't include-- and then within this contour-- and then there's no singularity.

Then, this is identically 0 from the Cauchy theorem. So it tells you that for the retarded Green function, you need the contour like that. So similarly, for the advanced-- because the advanced, we just change the direction.

So this is for the retarded. For the advanced, it's proportional to t minus t prime. Then you just go in the opposite direction. So this is for the retarded-- for the advanced. And then, finally, if we want to do it for the Feynman-- and then you choose one of them going up and one of them going down.

I always forget which one going up, which one going down. Yeah, so actually, this one you go down, and this one you go up. So this is for the Feynman for the G F. That's all for today.