Quantum Field Theory I (8.323) Spring 2023
Assignment 7

Mar. 21, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 3

Notes:

- Conventions of γ matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{1}$$

- The Dirac equation has the form

$$(\gamma^\mu \partial_\mu - m)\psi = 0 \ . \tag{2}$$

- A possible choice for gamma matrices is

$$\gamma^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix} \ . \tag{3}$$

Problem Set 7

1. **Momentum conservation (10 points)**

   Consider an interacting field theory of real scalar ϕ. Assume that the theory is translation invariant. Introduce the Fourier transform of

$$G_n(x_1, \cdots x_n) = \langle \Omega | T\phi(x_1) \cdots \phi(x_n) | \Omega \rangle \tag{4}$$

   as

$$\tilde{G}_n(p_1, p_2, \cdots, p_n) = \int d^4x_1 \cdots d^4x_n \, e^{-i\sum_{i=1}^n p_i \cdot x_i} G_n(x_1, \cdots x_n) \ . \tag{5}$$

   Show that

$$\tilde{G}_n(p_1, p_2, \cdots, p_n) \propto (2\pi)^4 \delta^{(4)}(p_1 + p_2 + \cdots p_n) \ . \tag{6}$$

   Note: Recall $G_n(p_1, p_2, \cdots, p_n)$ discussed in lecture is defined as

$$\tilde{G}_n(p_1, p_2, \cdots, p_n) \equiv (2\pi)^4 \delta^{(4)}(p_1 + p_2 + \cdots p_n) G_n(p_1, p_2, \cdots, p_n) \ . \tag{7}$$

2. **Feynman rules for a complex scalar field (20 points)**

For a complex scalar field, particle and antiparticle are distinct, which we can think of as positively and negatively charged. The Feynman rules must distinguish them. We can make the distinction by including an arrow for each propagator indicating the flow of charge. Note that this is different from the arrows for momentum flows which we sometimes draw. The directions of "momentum" arrows are completely arbitrary, while those of the "charge" arrows in a diagram are not and should reflect charge conservation. *For a particle it is customary to point the arrow away from the external point in the initial state, and toward the external point in the final state.* For an antiparticle, the direction is reversed. The charge arrows for propagators of the rest of a diagram then follow from charge conservation.

In order to avoid two kinds of arrows, we can simply align momentum arrows with the charge ones.

(a) Consider a complex scalar field theory

$$\mathcal{L} = -\partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi - \frac{\lambda}{4}(\phi^* \phi)^2 \ . \tag{8}$$

Write down the momentum space Feynman rules for this theory.

(b) Draw the *connected* diagrams for the *scattering amplitude* of the process

$$\phi + \bar{\phi} \to \phi + \bar{\phi} \tag{9}$$

to order $O(\lambda^2)$, where ϕ and $\bar{\phi}$ denote particle and antiparticle respectively. You only need to draw the diagrams.

(c) Now suppose ϕ interacts with a real field χ via

$$\mathcal{L}_I = \lambda' \chi \partial^\mu \phi^* \partial_\mu \phi \ . \tag{10}$$

i.e. the full Lagrangian is the sum of (8) and (10) plus a free theory part for χ. Suppose χ has mass M. Use a solid line for the ϕ propagator and a dashed line for the χ propagator. Write down the momentum space Feynman rules for the full theory.

(d) Suppose both $\lambda, \lambda' \sim O(\epsilon)$ with ϵ a small parameter. Draw all the *connected* diagrams for the amplitude of the decay process

$$\chi \to \phi + \bar{\phi} \tag{11}$$

to order $O(\epsilon^2)$.

3. **Higgs production at LHC (10 points + 10 bonus points)**

   At LHC people collides a proton with a proton at very high energies. Each proton contains a number of quarks and gluons. So collisions of protons can also be considered as collisions of gluons. Here we consider a baby version of the Standard Model which contains three types of scalar fields:

   - real gluon field g. Use a wavy line to denote its propagator.
   - *complex* quark field q (quark and antiquark are different). Use a solid line to denote its propagator. Remember the arrow!
   - real Higgs field H. Use a dashed line to denote its propagator.

   Suppose the interaction part of the theory is given by

   $$\mathcal{L}_I = \lambda_1 g q^\dagger q + \lambda_2 H q^\dagger q \ . \tag{12}$$

   Note that there is no direct coupling between gluon g and Higgs H. You can assume that couplings λ_1, λ_2 are small and of comparable strength.

   (a) The dominant channel for Higgs production at LHC is the so-called gluon fusion process, which can be schematically written as

   $$g + g \to H \tag{13}$$

   Draw the leading Feynman diagram(s) for (13).

   (b) **Bonus problem (10 pts):** Another channel for Higgs production is

   $$g + g \to H + q + \bar{q} \tag{14}$$

   where $\bar{q}$ denotes antiquark. Draw the leading Feynman diagram(s) for such a process.

4. **Properties of gamma matrices (25 points)**

   Without resorting to a particular representation, prove the following identities

   (a) $\mathrm{Tr}\, \gamma^\mu = 0$.

   (b) $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$.

   (c) $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\lambda) = 0$.

   (d) $\not{p}\not{q} = 2p \cdot q - \not{q}\not{p} = p \cdot q - 2i\Sigma^{\mu\nu} p_\mu q_\nu$.

   (e) $\gamma^\mu \not{p} \gamma_\mu = -2\not{p}$

   where we have defined

   $$\not{p} \equiv p_\mu \gamma^\mu, \quad \Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu] \ . \tag{15}$$

5. **Conserved "probability" current of the Dirac equation (15 points)**
Starting from the Dirac equation (2),

(a) show that one can construct a current j^μ which is conserved

$$\partial_\mu j^\mu = 0 \ . \tag{16}$$

(b) Show that j^μ you constructed is real.

(c) Show that by choosing the overall sign of j^μ, the zeroth component of j^μ

$$\rho \equiv j^0 \tag{17}$$

can be made to be positive definite.