Quantum Field Theory I (8.323) Spring 2023
Assignment 11

Apr. 25, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 9.4

Notes:

Problem Set 11

1. **Quantization of Maxwell in the Lorentz gauge: null states and residual gauge transformations (50 points)**

   In the Lorentz gauge we consider the action

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\xi}{2}(\partial_\mu A^\mu)^2 \tag{1}$$

where ξ is an arbitrary real parameter (and different ξ's give equivalent theories). It is convenient to take $\xi = 1$, in which case

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu \ . \tag{2}$$

The complete set of solutions to operator equations following from (2)

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\sum_{\alpha=0}^{3}\left[\mathcal{E}_\mu^{(\alpha)} a_{\vec{k}}^{(\alpha)} e^{ik\cdot x} + \mathcal{E}_\mu^{(\alpha)*}(a_{\vec{k}}^{(\alpha)})^\dagger e^{-ik\cdot x}\right] \tag{3}$$

where $\omega_{\vec{k}} = |\vec{k}|$ and $k^\mu = (|\vec{k}|, \vec{k})$. $\mathcal{E}_\mu^{(\alpha)}$ are polarization vectors defined by

$$\mathcal{E}_\mu^{(0)} = (1, \vec{0}), \quad \mathcal{E}_\mu^{(3)} = \left(0, \frac{\vec{k}}{|\vec{k}|}\right), \quad \mathcal{E}_\mu^{(1,2)} = (0, \vec{\epsilon}_{1,2}), \quad \vec{\epsilon}_{1,2}\cdot\vec{k} = 0 \tag{4}$$

where $\vec{\epsilon}_{1,2}$ are orthogonal unit-norm spatial vectors.

With canonical commutation relations

$$[A_\mu(t,\vec{x}), \pi^\nu(t'\vec{x}')] = i\delta_\mu^\nu \delta^{(3)}(\vec{x}-\vec{x}'), \quad \pi^\mu = \partial_0 A^\mu \tag{5}$$

we find that

$$[a_{\vec{k}}^{(\alpha)}, (a_{\vec{k}'}^{(\beta)})^\dagger] = \eta^{\alpha\beta}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}') \tag{6}$$

with all other commutators vanishing. We define the vacuum as

$$a_{\vec{k}}^{(\alpha)}|0\rangle = 0, \quad \forall \alpha, \vec{k} \tag{7}$$

and the "big" Hilbert space is defined as

$$\mathcal{H}_{\text{big}} = \{|\psi\rangle \text{ obtained by acting } (a_{\vec{k}}^{(\alpha)})^\dagger \text{ on } |0\rangle\} \ . \tag{8}$$

As discussed in lecture, $\mathcal{H}_{\text{big}}$ contains states with negative norms, and thus unphysical states. Indeed $\mathcal{H}_{\text{big}}$ follows from (2), which by itself is not the Maxwell theory.

To obtain the Maxwell theory we still need to impose $\partial_\mu A^\mu = 0$ and get rid of the residual gauge freedom. In this problem we will guide you to do this. We will see that by imposing $\partial_\mu A^\mu = 0$ we get rid of the negative-norm unphysical states in $\mathcal{H}_{\text{big}}$. But this is not enough. We will observe that some states possess zero norm, which can be attributed to the presence of residual gauge freedom. By eliminating these null states, we obtain the physical Hilbert space, which contains only two transverse massless degrees of freedom rather than four.

The enforcement of $\partial_\mu A^\mu = 0$ at the quantum level is subtle. Remember that in the Coulomb gauge, classically we apply the gauge condition $\nabla \cdot \vec{A} = 0$ as part of the equations of motion, which becomes an operator equation at the quantum level. In the Lorentz gauge, classically we only need to impose "boundary conditions" to ensure that the equation $\partial^2(\partial_\mu A^\mu) = 0$ has the trivial solution $\partial_\mu A^\mu = 0$. This implies that, at the quantum level, we cannot enforce $\partial_\mu A^\mu = 0$ as an operator equation (as you will check yourself below). Instead, we will have to do something weaker, imposing a variant of it as a condition on the physical states.

(a) Calculate

$$[A_0(t,\vec{x}), \partial_\mu A^\mu(t.\vec{x}')] \tag{9}$$

and from your result explain why we cannot impose $\partial_\mu A^\mu = 0$ as an operator equation.

(b) We also cannot impose that "physical states" satisfy

$$\partial_\mu A^\mu |\psi\rangle = 0 \tag{10}$$

as $\partial_\mu A^\mu|0\rangle \neq 0$. We do want to keep $|0\rangle$ to be a physical state. So to define physical states we need a weaker condition. It turns out the condition that eliminates all negative norm states while keeping the vacuum $|0\rangle$ is

$$\partial^\mu A_\mu^{(-)}|\psi\rangle = 0 \tag{11}$$

where $A_\mu^{(-)}$ denotes the annihilation part of A_μ. We now impose that physical states should satisfy (11) and denote the set of physical states in $\mathcal{H}_{\text{big}}$ to be $\mathcal{H}_{\text{small}}$.

Show that there is no negative-norm state in $\mathcal{H}_{\text{small}}$.

(c) Show that

$$\langle\psi'|\partial_\mu A^\mu|\psi\rangle = 0, \quad \forall|\psi\rangle, |\psi'\rangle \in \mathcal{H}_{\text{small}}, \tag{12}$$

that is, $\partial_\mu A^\mu$ has zero matrix element among states in $\mathcal{H}_{\text{small}}$.

(d) Introduce

$$b_{\vec{k}}^{(\pm)} = \frac{1}{\sqrt{2}}\left(a_{\vec{k}}^3 \pm a_{\vec{k}}^0\right), \qquad (b_{\vec{k}}^{(\pm)})^\dagger = \frac{1}{\sqrt{2}}\left(a_{\vec{k}}^{3\dagger} \pm a_{\vec{k}}^{0\dagger}\right) \ . \tag{13}$$

Show that the physical state condition (11) can be written as

$$b_{\vec{k}}^{(+)}|\psi\rangle = 0 \ . \tag{14}$$

(e) We are not done yet as $\mathcal{H}_{\text{small}}$ still contain zero-norm states. To see this, first work out the commutation relations

$$[b_{\vec{k}}^+, (b_{\vec{k}'}^+)^\dagger], \qquad [b_{\vec{k}}^-, (b_{\vec{k}'}^-)^\dagger], \qquad [b_{\vec{k}}^+, (b_{\vec{k}'}^-)^\dagger], \qquad [b_{\vec{k}}^-, (b_{\vec{k}'}^+)^\dagger] \ . \tag{15}$$

and show that $\mathcal{H}_{\text{small}}$ can also be described as

$$\mathcal{H}_{\text{small}} = \{\text{all states obtained by acting } (a_{\vec{k}}^1)^\dagger, (a_{\vec{k}}^2)^\dagger, (b_{\vec{k}}^+)^\dagger \text{ on } |0\rangle\} \ . \tag{16}$$

In other words, a physical state can have no $(b_{\vec{k}}^-)^\dagger$ excitations.

(f) Show that any state $|\psi\rangle$ in $\mathcal{H}_{\text{small}}$ with nonzero $(b_{\vec{k}}^+)^\dagger$ excitations has zero norm and its overlap with any state in $\mathcal{H}_{\text{small}}$ is zero. Such states (which are called null states) clearly cannot have any physical significance.

(g) Show that any state with nonzero norm then must have the form

$$|\psi\rangle = |\psi_T\rangle + |\chi\rangle \tag{17}$$

where $|\psi_T\rangle$ contains only excitations of $(a_{\vec{k}}^1)^\dagger, (a_{\vec{k}}^2)^\dagger$ (i.e. transverse components) and $|\chi\rangle$ is a null state.

$|\psi\rangle$ should be physically equivalent to $|\psi_T\rangle$ as they differ only by a null state.

We can then forget about the null states and define

$$\mathcal{H}_{\text{phys}} = \{|\psi_T\rangle\} \ . \tag{18}$$

$\mathcal{H}_{\text{phys}}$ contains only positive-norm states and is identical to that obtained in the Coulomb gauge.

This concludes the discussion of quantization in the Lorentz gauge. For the rest the problem we explore a bit further the nature of null states and their physical interpretation.

(h) Let us call excitations of $(b^+_{\vec{k}})^\dagger$ null photons. To understand the physical interpretation of a null photon, let us consider the "wave function" $\chi_\mu(x)$ of the single null photon state

$$\chi_\mu(x) = \Big\langle 0 \big| A_\mu(x) \big| \vec{k}, + \Big\rangle, \qquad |\vec{k}, +\rangle = \sqrt{2\omega_{\vec{k}}}(b^+_{\vec{k}})^\dagger |0\rangle \ . \tag{19}$$

Note that the above definition of wave function $\chi_\mu(x)$ is the straightforward generalization to vector field of our previous discussion for a scalar field. Show that $\chi_\mu(x)$ can be written as

$$\chi_\mu(x) = \partial_\mu \lambda(x) \tag{20}$$

where $\lambda(x)$ is some function which satisfies the equation for a massless scalar

$$\partial_\mu \partial^\mu \lambda = 0 \ . \tag{21}$$

This shows that a null photon can be interpreted as a gauge transformation from the vacuum.

[Recall that the Lorentz gauge $\partial_\mu A_\mu = 0$ leaves residual gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \lambda, \qquad \partial_\mu \partial^\mu \lambda = 0 \ . \tag{22}$$

Thus $\chi_\mu(x)$ can be considered as a residual gauge transformation from $A_\mu = 0$.]

(i) Show that the conclusion of part (h) holds for any wave packet of a null photon

$$|f\rangle = \int d^3\vec{k}\, f(\vec{k}) |\vec{k}, +\rangle, \tag{23}$$

i.e. the wave function for $|f\rangle$ again has the form of a residual gauge transformation from $A_\mu = 0$.

2. **Casimir effect in one dimension (30 points)**

Until now, we have disregarded the vacuum's zero-point energy as an unobservable (infinite) shift in the energy scale. However, as Casimir demonstrated in

1948, *differences* in vacuum zero-point energies are, however, observable. This phenomenon is known as the Casimir effect. A simplest example of the Casimir effect is a small attractive force between two close parallel conducting plates, due to quantum vacuum fluctuations of the electromagnetic field. The force is caused by change in vacuum energy of the electromagnetic field that results from the boundary conditions imposed by the plates.

In this problem we will explore the Casimir effect. For technical simplicity, we will consider a toy version of the effect, which captures all the essential physics. As we have learned, after quantization the electromagnetic field has the same number of degrees of freedom as two massless scalar fields. So we will consider a massless scalar field instead of the Maxwell theory. Also instead of three spatial dimensions we consider one spatial dimension.

Consider a free, *massless*, real scalar field ϕ in one spatial dimension, i.e.

$$S = -\frac{1}{2}\int dxdt\, \partial_\mu\phi\partial^\mu\phi \ . \tag{24}$$

The vacuum of the system has an infinite zero-point energy. Let us denote it by E_0. Now imagine we put two "plates" at $x = 0$ and $x = a$ such that ϕ is required to vanish at the location of the plates,[1] i.e.

$$\phi(x = 0, t) = \phi(x = a, t) = 0 \ . \tag{25}$$

Adding plates which imposes additional boundary conditions on ϕ disturbs the vacuum, and results in a different zero-point energy $E(a)$.

Even though both E_0 and $E(a)$ are infinite, their difference turns out to be finite and physically meaningful. In fact the difference

$$U(a) = E(a) - E_0 \tag{26}$$

can be considered as the potential energy between the two plates. Changing a modifies the potential energy and results in a force between the plates which can be measured experimentally!

In this problem I will guide you to calculate $U(a)$. As you learned in calculus, taking the differences between infinities is a highly dangerous thing to do. One can in principle get any answer. So we will need to be very careful.

Both E_0 and E_a have two sources of infinities, one from the infinite volume, the other from there are infinite number of *local* degrees of freedom. It is convenient to separate these two infinities by putting the system in a box with finite size $L \gg a$. We will take L to infinity at the end of the calculation. More explicitly, we require ϕ to satisfy a periodic boundary condition

$$\phi(x, t) = \phi(x + L, t) \tag{27}$$

[1]In real-life Casimir effects, the plates are simply conducting plates.

i.e. putting the system on a circle of size L.

(a) In the vacuum (i.e. before putting the plates), write down the mode expansion for ϕ and calculate its zero-point energy E_0. Your answer should have the form

$$E_0 = \frac{1}{2}\sum_n \omega_n \tag{28}$$

where ω_n is the energy for each mode. You should specify both ω_n and the range of summation.

(b) Adding the plates separates the system into two segments, one has size a, the other has size $L - a$. In both segments one has Dirichlet boundary conditions (25) at two ends. (Remember the system is on a circle.) So it is enough to work out the story for one of them. Find the mode expansion for ϕ in the region $[0, a]$ and zero-point energy $\varepsilon(a)$. Again your answer should have the form

$$\varepsilon(a) = \frac{1}{2}\sum_n \tilde{\omega}_n \tag{29}$$

and you should specify both $\tilde{\omega}_n$ and the range of summation. The total zero-point energy of the system in the presence of the plates is thus

$$E(a) = \varepsilon(a) + \varepsilon(L - a)\ . \tag{30}$$

(c) Both sums (28) and (30) are divergent. There is not much sense in taking the difference between them. To take the difference we will first make them finite. We will do this by introducing a "UV cutoff" Λ, i.e. change the sums to[2]

$$E_0 = \frac{1}{2}\sum_n \omega_n e^{-\frac{\omega_n}{\Lambda}} \tag{31}$$

and

$$\varepsilon(a) = \frac{1}{2}\sum_n \tilde{\omega}_n e^{-\frac{\tilde{\omega}_n}{\Lambda}}\ . \tag{32}$$

Clearly taking the limit $\Lambda \to \infty$ *inside* the sum we recover (28) and (29). Now evaluate (31) and (32) with a finite Λ.

(d) Expand the answers you get in part (c) in the limit $\Lambda \to \infty$. You will find they become divergent. Keep terms which are divergent and finite, but throw away all terms which go to zero in the limit (i.e. throw away all terms with a negative power of Λ).

[2]Note that by introducing Λ, we suppress the contributions from "high" energy modes with $\omega_n, \tilde{\omega}_n \gg \Lambda$.

(e) From your answers in part (d), find

$$U(a) = E(a) - E_0 \ . \tag{33}$$

You should find $U(a)$ is finite in the limit $\Lambda \to \infty$. Now take the limit $L \to \infty$ in $U(a)$, and find the force between the plates in the $L \to \infty$ limit.

8.323 Relativistic Quantum Field Theory I
Spring 2023