8.323 Problem Set 4 Solutions

March 7, 2023

Question 1: Properties of Wightman and Feynman Functions (20 points)

(a) For a free scalar field theory, using

$$U_\Lambda a_{\mathbf{k}} U_\Lambda^\dagger = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a_{\Lambda\mathbf{k}}, \qquad U_\Lambda a_{\mathbf{k}}^\dagger U_\Lambda^\dagger = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a_{\Lambda\mathbf{k}}^\dagger$$

show that

$$U_\Lambda \phi(x) U_\Lambda^\dagger = \phi(\Lambda x)$$

We use the mode expansion for $\phi(x)$, and the transformation properties of the ladder operators:

$$\begin{aligned}
U_\Lambda \phi(x) U_\Lambda^\dagger &= U_\Lambda \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{-ik\cdot x} + a_{\mathbf{k}}^\dagger e^{ik\cdot x} \right) U_\Lambda^\dagger, \qquad k^0 = \omega_{\mathbf{k}} \\
&= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \left(\sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a_{\Lambda\mathbf{k}} e^{-ik\cdot x} + \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a_{\Lambda\mathbf{k}}^\dagger e^{ik\cdot x} \right) \\
&= \int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}} \sqrt{2\omega_{\Lambda\mathbf{k}}} \left(a_{\Lambda\mathbf{k}} e^{-ik\cdot x} + a_{\Lambda\mathbf{k}}^\dagger e^{ik\cdot x} \right) \\
&= \int \frac{đ^3\mathbf{k}'}{2\omega_{\mathbf{k}}'} \sqrt{2\omega_{\mathbf{k}}'} \left(a_{\mathbf{k}'} e^{-ik'\cdot \Lambda x} + a_{\mathbf{k}'}^\dagger e^{ik'\cdot \Lambda x} \right) = \phi(\Lambda x)
\end{aligned}$$

where in the last line we make the change of variables $\mathbf{k}' = \Lambda\mathbf{k}$ (with $k'^0 = \omega_{\mathbf{k}'}$), and use that the measure $đ^3\mathbf{k}/2\omega_{\mathbf{k}}$ is Lorentz invariant.

(b) Now in the subsequent parts, let us consider a general interacting theory, where the mode expansion for ϕ given in lecture does not apply. But as far as the system is Lorentz and translation invariant,

$$U_\Lambda \phi(x) U_\Lambda^\dagger = \phi(\Lambda x), \qquad U_y \phi(x) U_y^\dagger = \phi(x+y)$$

are valid. Use this to prove that for the Wightman function $G_+(x.x') := \langle 0|\phi(x)\phi^\dagger(x')|0\rangle$, we have

$$G_+(x,x') = G_+(x-x')$$

Note that the same is true for the advanced, retarded, and Feynman two-point functions.

Using the translation-invariance of the vacuum, we have:

$$\begin{aligned}
G_+(x,x') = \langle 0|\phi(x)\phi^\dagger(x')|0\rangle &= \langle 0|U_y\phi(x)U_y^\dagger U_y\phi^\dagger(x')U_y^\dagger|0\rangle \\
&= \langle 0|\phi(x+y)\phi^\dagger(x'+y)|0\rangle = G_+(x+y,x'+y)
\end{aligned}$$

Since y is arbitrary, we can take $y=-x'$, which gives the desired result:

$$G_+(x,x') = G_+(x-x',0) := G_+(x-x')$$

(c) Using $U_\Lambda \phi(x) U_\Lambda^\dagger = \phi(\Lambda x)$ and the result of (b), show that one can write $G_+(x,x')$ as

$$G_+(x,x') = \theta(t-t')G((x-x')^2) + \theta(t'-t)G^*((x-x')^2)$$

where $G(y)$ is some function which satisfies

$$G(y) = G^*(y), \qquad \text{for } y > 0$$

We apply the same method in (b), except using Lorentz transformations instead of translations:

$$\begin{aligned} G_+(x,x') &= \langle 0|\phi(x)\phi^\dagger(x')|0\rangle = \langle 0|U_\Lambda \phi(x) U_\Lambda^\dagger U_\Lambda \phi^\dagger(x') U_\Lambda^\dagger|0\rangle \\ &= \langle 0|\phi(\Lambda x)\phi^\dagger(\Lambda x')|0\rangle = G_+(\Lambda(x-x')) \end{aligned}$$

Therefore, we have $G_+(x-x') = G_+(\Lambda(x-x'))$, i.e. our 2-point function is Lorentz invariant. For a given vector v^μ, there are 2 independent quantities we can build which are invariant under proper orthochronous Lorentz transformations: v^2 and $\theta(v^0)$. This follows immediately from definition. Lorentz transformations are (linear) transformations preserving a norm, and orthochronous restricts to those keeping the sign of v^0 fixed. Note that preserving parity gives no additional invariant quantity, as with 3D rotations. Therefore, we have the decomposition

$$G_+(x,x') = \theta(t-t')G_1((x-x')^2) + \theta(t'-t)G_2((x-x')^2)$$

Note further that G_1 and G_2 are not independent: complex conjugating this equation and observing $G_+(x,x') = G^*(x',x)$, we have $G_1(y) = G_2^*(y)$. Hence

$$G_+(x,x') = \theta(t-t')G((x-x')^2) + \theta(t'-t)G^*((x-x')^2)$$

Finally for $y = (x-x')^2 > 0$ one has $x - x'$ spacelike. Therefore, we can use an $\mathrm{SO}^+(3,1)$ transformation to go to a frame where $t' > t$, and a frame where $t > t'$. Since our expression is Lorentz-invariant, this demands the desired result,

$$G(y) = G^*(y)$$

(d) Now take ϕ to be real, and show that for the Feynman function G_F

$$G_F(x,x') = G((x-x')^2)$$

where the function G in the above equation is the same as that in part (c).

By definition, for ϕ real we have

$$G_F(x,x') = \theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle = \theta(t-t')G_+(x,x') + \theta(t'-t)G_+(x',x)$$

Now we substitute our result from (c) for G_+.

$$\begin{aligned} G_F(x,x') = \theta(t-t')&\left(\theta(t-t')G((x-x')^2) + \theta(t'-t)G^*((x-x')^2)\right) \\ &+ \theta(t'-t)\left(\theta(t'-t)G((x-x')^2) + \theta(t-t')G^*((x-x')^2)\right) \\ &= \theta(t-t')G((x-x')^2) + \theta(t'-t)G((x-x')^2) = G((x-x')^2) \end{aligned}$$

where we use that $\theta(t-t')\theta(t'-t) = 0$, and $\theta(t-t') + \theta(t'-t) = 1$.

Question 2: Particle Production by an External Source (60 points)

Consider a free scalar field theory with external 'source' $J(x)$, whose Lagrangian density can be written

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J(x)\phi = \mathcal{L}_0 + J(x)\phi$$

where $\mathcal{L}_0$ is the Lagrangian density for a free scalar, and $J(x)$ is a fixed function. We assume it has the properties

$$J(t,\mathbf{x}) \to 0, \qquad t \to \pm\infty, \qquad |\mathbf{x}| \to \infty$$

and its Fourier transform

$$J(p) = \int d^4x e^{-ip\cdot x} J(x)$$

is analytic in the complex ω-plane.

Since $J(x)$ depends on time, the system does not have time translation symmetry. In particular, the vacuum at past infinity $|0,-\infty\rangle$ will be different from the one at future infinity $|0,+\infty\rangle$. Suppose we start with the vacuum state $|0,-\infty\rangle$ at $t=-\infty$. In the Heisenberg picture, the system remains in the same state $|0,-\infty\rangle$ at all times. At $t=+\infty$, the system is then not in the ground state (as $|0,-\infty\rangle \neq |0,+\infty\rangle$), and contains particle excitations. In other words, turning on a source $J(x)$ has produced particles. Below we will find the relation between $|0,-\infty\rangle$ and $|0,+\infty\rangle$, and calculate the probability of producing particles.

At $t=\mp\infty$, since $J=0$ we have a free theory, and ϕ can be written

$$\phi(x) \to \phi_{\text{in/out}}(x) := \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\left(a_{\text{in/out}}(\mathbf{k})e^{ik\cdot x} + a^\dagger_{\text{in/out}}(\mathbf{k})e^{-ik\cdot x}\right), \qquad t \to \mp\infty$$

where

$$[a_{\text{in/out}}(\mathbf{k}), a^\dagger_{\text{in/out}}(\mathbf{k}')] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

We then define the past/future vacuum as

$$a_{\text{in/out}}(\mathbf{k})|0,\mp\infty\rangle = 0, \qquad \langle 0,\mp\infty|0,\mp\infty\rangle = 1$$

Particles (at past/future infinity) can be defined by acting $a^\dagger_{\text{in/out}}$ on this vacuum. For example, and n-particle state at past/future infinity can be written as

$$|\mathbf{k}_1,\ldots,\mathbf{k}_n,\mp\infty\rangle = \prod_{i=1}^n \sqrt{2\omega_{\mathbf{k}_i}}\, a^\dagger_{\text{in/out}}(\mathbf{k}_i)|0,\mp\infty\rangle$$

with the normalization for a single particle state

$$\langle \mathbf{k},\mp\infty|\mathbf{k}',\mp\infty\rangle = 2\omega_\mathbf{k}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

Due to the presence of the external source $J(x)$, the past and future annihilation operators a_{in} and a_{out} are different. The field $\phi(t,\mathbf{x})$ with $-\infty < t < \infty$ interpolates between ϕ_{in} and ϕ_{out}.

(a) For general t, solve the classical equation of motion for this Lagrangian, and show that the solution can be written in a form

$$\phi(x) = \phi_0(x) + i\int d^4x' G(x-x')J(x') \tag{1}$$

where $\phi_0(x)$ is a solution of the homogeneous equation

$$(-\partial^2 + m^2)\phi_0(x) = 0$$

and $G(x - x')$ is a Green function satisfying

$$(-\partial^2 + m^2)G(x - x') = -i\delta^{(4)}(x - x')$$

We discussed various types of Greens functions. Which one should be used here?
The Euler-Lagrange equations pick up an extra term due from $J(x)\phi$ in the Lagrangian, and the equation of motion is given by

$$(\partial^2 - m^2)\phi = -J$$

We substitute the form for $\phi(x)$ given above into the left-hand side of equation of motion, and show it reproduces the source term.

$$\begin{aligned}(\partial^2 - m^2)\phi &= (\partial^2 - m^2)\phi_0(x) + (\partial^2 - m^2)i\int d^4x' G(x - x')J(x') \\ &= -i\int d^4x'(-\partial_x^2 + m^2)G(x - x')J(x') \\ &= -i\int d^4x'(-i)\delta^{(4)}(x - x')J(x') = -J(x)\end{aligned}$$

In the second line we use that ϕ_0 solves the homogeneous equation of motion. In the 3rd line we use the defining property of the Greens function as satisfying the inhomogeneous equation of motion with δ-source.

Here we should use the retarded Green's function, as causality requires that the field $\phi(t, \mathbf{x})$ at time t can only be influenced by the source at times $t' < t$.

(b) Now consider the quantum theory, and promote ϕ to a quantum operator, with (1) now an operator equation. Show that ϕ_0 in this equation should be given by ϕ_{in}.
We evaluate (1) as we take $t \to -\infty$, with $G = G_R$ the retarded Green's function. In this regime, $\theta(t - t') = 0$ for all t' on which $J(x)$ has support. Hence, $G_R(x - x') \propto \theta(t - t')$ also vanishes for these t'. We thus compute

$$\phi(x)|_{t\to-\infty} = \phi_0(x) + i\int d^4x' G(x - x')J(x') = \phi_0(x)$$

Finally, we know $\phi(x)|_{t\to-\infty} = \phi_{\text{in}}(x)$. Combining these equations we have $\phi_0(x) = \phi_{\text{in}}(x)$, as desired. Note that this equation only makes sense for $t \to -\infty$, since this is the domain of $\phi_{\text{in}}(x)$.

(c) Evaluate (1) at $t = +\infty$ to find the relation between $a_{\text{out}}(\mathbf{k})$ and $a_{\text{in}}(\mathbf{k})$.
Recall from class that the retarded Green's function is

$$G_R(x - x') = \theta(t - t')\int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}}\left(e^{ik\cdot(x-x')} - e^{-ik\cdot(x-x')}\right)$$

For $t \to +\infty$, for all points in the support of $J(t', \mathbf{x}')$ we have $\theta(t - t') = 1$. Using this and the result of (b), we compute

$$\begin{aligned}
\phi_{\text{out}}(x) = \phi(x)|_{t\to\infty}(x) &= \phi_{\text{in}}(x) + i\int d^4x' G(x-x')J(x') \\
&= \phi_{\text{in}}(x) + i\int d^4x' \int \frac{d\!\!^-{}^3\mathbf{k}}{2\omega_{\mathbf{k}}}\left(e^{ik\cdot(x-x')} - e^{-ik\cdot(x-x')}\right)J(x') \\
&= \int \frac{d\!\!^-{}^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\text{in}}(\mathbf{k})e^{ik\cdot x} + a^\dagger_{\text{in}}(\mathbf{k})e^{-ik\cdot x}\right) + i\int \frac{d\!\!^-{}^3\mathbf{k}}{2\omega_{\mathbf{k}}}\left(e^{ik\cdot x}J(k) - e^{-ik\cdot x}J(-k)\right) \\
&= \int \frac{d\!\!^-{}^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left[\left(a_{\text{in}}(\mathbf{k}) + i\frac{J(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}\right)e^{ik\cdot x} + \left(a^\dagger_{\text{in}}(\mathbf{k}) - i\frac{J^*(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}\right)e^{-ik\cdot x}\right]
\end{aligned}$$

Note that the 4-momenta are implicitly evaluated at $k = (\omega_{\mathbf{k}}, \mathbf{k})$, which is what we mean when we write $J(k) = J(\mathbf{k})$. In the last line, we also use that $J(x)$ is real for a real scalar, so its Fourier transform satisfies $J(-k) = J^*(k)$.

Finally, we can compare this to the mode expansion

$$\phi_{\text{out}}(x) := \int \frac{d\!\!^-{}^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\text{out}}(\mathbf{k})e^{ik\cdot x} + a^\dagger_{\text{out}}(\mathbf{k})e^{-ik\cdot x}\right)$$

to relate the incoming and outgoing ladder operators,

$$a_{\text{out}} = a_{\text{in}}(\mathbf{k}) + i\frac{J(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}, \qquad a^\dagger_{\text{out}} = a^\dagger_{\text{in}}(\mathbf{k}) - i\frac{J^*(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}$$

(d) Using (c), show that the expectation value λ for the total number of particles produced is given by

$$\lambda = \int \frac{d\!\!^-{}^3\mathbf{k}}{2\omega_{\mathbf{k}}}|J(k)|^2$$

The expectation value for the total number of particles produced is given by taking the expectation of the outgoing number operator with respect to our state. In the Heisenberg picture our state is constant, and is just $|0,-\infty\rangle$. Therefore,

$$\begin{aligned}
\lambda &= \langle 0,-\infty|\int d\!\!^-{}^3\mathbf{k}\, a^\dagger_{\text{out}}(\mathbf{k})a_{\text{out}}(\mathbf{k})|0,-\infty\rangle \\
&= \int d\!\!^-{}^3\mathbf{k}\langle 0,-\infty|\left(a^\dagger_{\text{in}}(\mathbf{k}) - i\frac{J^*(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}\right)\left(a_{\text{in}}(\mathbf{k}) + i\frac{J(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}\right)|0,-\infty\rangle \\
&= \int d\!\!^-{}^3\mathbf{k}\frac{J^*(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}\frac{J(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}\langle 0,-\infty|0,-\infty\rangle = \int \frac{d\!\!^-{}^3\mathbf{k}}{2\omega_{\mathbf{k}}}|J(\mathbf{k})|^2
\end{aligned}$$

(e) Show that we can write

$$a_{\text{out}}(\mathbf{k}) = S^\dagger a_{\text{in}}(\mathbf{k})S$$

with S a unitary operator given by

$$S = e^{iB}, \qquad B = \int d^4x J(x)\phi_{\text{in}}(x)$$

We compute this using Baker-Campbell-Hausdorff:

$$S^\dagger a_{\text{in}}(\mathbf{k})S = e^{-iB}a_{\text{in}}(\mathbf{k})e^{iB} = a_{\text{in}}(\mathbf{k}) - i[B, a_{\text{in}}(\mathbf{k})] - \frac{1}{2}[B,[B,a_{\text{in}}(\mathbf{k})]] + \cdots$$

This series terminates after the second term, since $[B, a_{\text{in}}(\mathbf{k})]$ is a c-number:

$$[B, a_{\text{in}}(\mathbf{k})] = \int d^4x J(x)[\phi_{\text{in}}(x), a_{\text{in}}(\mathbf{k})] = \int d^4x J(x) \int \frac{đ^3k'}{\sqrt{2\omega_{\mathbf{k}}}}[a^\dagger_{\text{in}}(\mathbf{k}'), a_{\text{in}}(\mathbf{k})]e^{-ik'\cdot x}$$
$$= -\frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \int d^4x J(x) e^{-ik\cdot x} = -\frac{1}{\sqrt{2\omega_{\mathbf{k}}}} J(\mathbf{k})$$

Therefore,

$$S^\dagger a_{\text{in}}(\mathbf{k}) S = a_{\text{in}}(\mathbf{k}) + \frac{i}{\sqrt{2\omega_{\mathbf{k}}}} J(\mathbf{k}) = a_{\text{out}}(\mathbf{k})$$

(f) Use the result in part (e) to show that

$$S|0, +\infty\rangle = |0, -\infty\rangle, \qquad S|\mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty\rangle = |\mathbf{k}_1, \ldots, \mathbf{k}_n, -\infty\rangle$$

This indicates that S is in fact the S-matrix operator of the system, i.e. for free theory states $|\alpha\rangle$, $|\beta\rangle$

$$S_{\beta\alpha} := \langle \beta, +\infty|\alpha, -\infty\rangle = \langle \beta, -\infty|S|\alpha, -\infty\rangle$$

To show that the state $S|0, +\infty\rangle$ is the incoming vacuum, we need to show that it is annihilated by all $a_{\text{in}}(\mathbf{k})$, and has the correct normalization. We compute

$$a_{\text{in}}(\mathbf{k}) S|0, +\infty\rangle = SS^\dagger a_{\text{in}}(\mathbf{k}) S|0, +\infty\rangle = S a_{\text{out}}(\mathbf{k})|0, +\infty\rangle = 0$$

Furthermore, S defined in part (e) is unitary (as both J and ϕ are real), so

$$\langle 0, +\infty|S^\dagger S|0, +\infty\rangle = \langle 0, +\infty|0, +\infty\rangle = 1$$

Therefore $S|0, +\infty\rangle$ can only differ from $|0, -\infty$ by a phase, which we can absorb into our definition of S. We have shown $S|0, +\infty\rangle = |0, -\infty\rangle$.

Now we show the desired equality for n-particle states:

$$\begin{aligned} S|\mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty\rangle &= \sqrt{2\omega_{\mathbf{k}_1}} \cdots \sqrt{2\omega_{\mathbf{k}_n}} S a^\dagger_{\text{out}}(\mathbf{k}_1) \cdots a^\dagger_{\text{out}}(\mathbf{k}_n)|0, +\infty\rangle \\ &= \sqrt{2\omega_{\mathbf{k}_1}} \cdots \sqrt{2\omega_{\mathbf{k}_n}} SS^\dagger a^\dagger_{\text{in}}(\mathbf{k}_1) S \cdots S^\dagger a^\dagger_{\text{in}}(\mathbf{k}_n) S|0, +\infty\rangle \\ &= \sqrt{2\omega_{\mathbf{k}_1}} \cdots \sqrt{2\omega_{\mathbf{k}_n}} a^\dagger_{\text{in}}(\mathbf{k}_1) \cdots a^\dagger_{\text{in}}(\mathbf{k}_n)|0, -\infty\rangle \\ &= |\mathbf{k}_1, \ldots, \mathbf{k}_n, -\infty\rangle \end{aligned}$$

In the 3rd line we use both $S|0, +\infty\rangle = |0, -\infty\rangle$ and the result in (e).

(g) In the following parts we will compute the probability of computing n particles. Before doing that, in this part we develop a technical tool to make that task easier. Show that we can write S as

$$S = e^{iB} = e^F e^G e^{-\lambda/2}$$

with λ as in (d), and

$$F = i \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} J(\mathbf{k}) a^\dagger_{\text{in}}(\mathbf{k}), \qquad G = i \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} J(-\mathbf{k}) a_{\text{in}}(\mathbf{k})$$

Again we use the Baker-Campbell-Hausdorff formula, which in one of its forms states that

$$e^F e^G = \exp\left(F + G + \frac{1}{2}[F, G] + \frac{1}{12}[F, [F, G]] - \frac{1}{12}[G, [F, G]] + \cdots\right)$$

We first compute the commutator of F and G:

$$
\begin{aligned}
[F, G] &= -\int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \frac{đ^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}} J(\mathbf{k}) J(-\mathbf{k}') [a_{\text{in}}^\dagger(\mathbf{k}), a_{\text{in}}(\mathbf{k}')] \\
&= \int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}} J(\mathbf{k}) J(-\mathbf{k}) = \int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}} |J(\mathbf{k})|^2 = \lambda
\end{aligned}
$$

In the last line we use that $J(x)$ is real, so $J(-k) = J^*(k)$. This is a c-number, so all further commutators with operators vanish. The Baker-Campbell-Hausdorff formula reduces to

$$
e^F e^G = e^{F+G+\lambda/2} \quad \Rightarrow \quad e^F e^G e^{-\lambda/2} = e^{F+G}
$$

It remains to compute the right-hand side. We have

$$
\begin{aligned}
F + G &= i \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \left(J(\mathbf{k}) a_{\text{in}}^\dagger(\mathbf{k}) + J(-\mathbf{k}) a_{\text{in}}(\mathbf{k}) \right) \\
&= i \int d^4x J(x) \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \left(e^{-ik\cdot x} a_{\text{in}}^\dagger(\mathbf{k}) + e^{ik\cdot x} a_{\text{in}}(\mathbf{k}) \right) = i \int d^4x J(x) \phi_{\text{in}}(x) = iB
\end{aligned}
$$

Putting everything together, we have the desired result,

$$
e^F e^G e^{-\lambda/2} = e^{iB} = S
$$

(h) Use the results of (f) and (g) to find the vacuum to vacuum probability,

$$
P_0 = |\langle 0, +\infty | 0, -\infty \rangle|^2
$$

P_0 is the probability of no particle production.

We compute

$$
P_0 = |\langle 0, +\infty | 0, -\infty \rangle|^2 = |\langle 0, -\infty | (S^\dagger)^\dagger | 0, -\infty \rangle|^2 = e^{-\lambda} |\langle 0, -\infty | e^F e^G | 0, -\infty \rangle|^2
$$

Note that G is a linear combination of a_{in}'s, so $e^G|0, -\infty\rangle = e^0|0, -\infty\rangle = |\infty\rangle$. Similarly, F is a linear combination of $a_{\text{in}}^\dagger$'s, so $\langle 0, -\infty|e^F = \langle 0, -\infty|$. We thus find

$$
P_0 = e^{-\lambda} |\langle 0, -\infty | 0, -\infty \rangle|^2 = e^{-\lambda}
$$

(i) Use the results of (f) and (g) to show that

$$
\langle \mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty | 0, -\infty \rangle = i^n J(\mathbf{k}_1) \cdots J(\mathbf{k}_n) e^{-\lambda/2}
$$

We compute:

$$
\begin{aligned}
\langle \mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty | 0, -\infty \rangle &= \langle \mathbf{k}_1, \ldots, \mathbf{k}_n, -\infty | S | 0, -\infty \rangle = e^{-\lambda/2} \langle \mathbf{k}_1, \ldots, \mathbf{k}_n, -\infty | e^F e^G | 0, -\infty \rangle \\
&= e^{-\lambda/2} \sqrt{2\omega_{\mathbf{k}_1}} \cdots \sqrt{2\omega_{\mathbf{k}_n}} \langle 0, -\infty | a_{\text{in}}(\mathbf{k}_1) \cdots a_{\text{in}}(\mathbf{k}_n) e^F | 0, -\infty \rangle \\
&= \frac{e^{-\lambda/2}}{n!} \sqrt{2\omega_{\mathbf{k}_1}} \cdots \sqrt{2\omega_{\mathbf{k}_n}} \langle 0, -\infty | a_{\text{in}}(\mathbf{k}_1) \cdots a_{\text{in}}(\mathbf{k}_n) F^n | 0, -\infty \rangle
\end{aligned}
$$

In line 1, we use from (f) that $S|\mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty\rangle = |\mathbf{k}_1, \ldots, \mathbf{k}_n, -\infty\rangle$, and the decomposition of S in (g). In line 2, we use from (b) that $e^G|0, -\infty\rangle = |\infty\rangle$. In line 3, we Taylor expand e^F explicitly, and keep only the term F^n, as all other terms vanish (a term vanishes unless the number of creation operators equals the number of annihilation operators).

To finish the computation, we note that

$$\begin{aligned}
a_{\text{in}}(\mathbf{k}_n)F^n &= F^n a_{\text{in}}(\mathbf{k}_n) + [a_{\text{in}}(\mathbf{k}_n), F^n] \\
&= F^n a_{\text{in}}(\mathbf{k}_n) + i^n \int \frac{đ^3\mathbf{k}_1'}{\sqrt{2\omega_{\mathbf{k}_1'}}} \cdots \frac{đ^3\mathbf{k}_n'}{\sqrt{2\omega_{\mathbf{k}_n'}}} J(\mathbf{k}_1') \cdots J(\mathbf{k}_n')[a_{\text{in}}(\mathbf{k}_n), a_{\text{in}}(\mathbf{k}_1') \cdots a_{\text{in}}(\mathbf{k}_n')] \\
&= F^n a_{\text{in}}(\mathbf{k}_n) + n\frac{iJ(\mathbf{k}_n)}{\sqrt{2\omega_{\mathbf{k}_n}}} F^{n-1}
\end{aligned}$$

acting both sides on the state $|0, -\infty\rangle$, the first term on the right vanishes. We can use this result to simplify our previous equation:

$$\begin{aligned}
\langle \mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty | 0, -\infty\rangle &= \frac{e^{-\lambda/2}}{n!}\sqrt{2\omega_{\mathbf{k}_1}} \cdots \sqrt{2\omega_{\mathbf{k}_n}} \langle 0, -\infty | a_{\text{in}}(\mathbf{k}_1) \cdots a_{\text{in}}(\mathbf{k}_n) F^n | 0, -\infty\rangle \\
&= \frac{e^{-\lambda/2}}{(n-1)!}\sqrt{2\omega_{\mathbf{k}_1}} \cdots \sqrt{2\omega_{\mathbf{k}_{n-1}}} iJ(\mathbf{k}_n) \langle 0, -\infty | a_{\text{in}}(\mathbf{k}_1) \cdots a_{\text{in}}(\mathbf{k}_{n-1}) F^{n-1} | 0, -\infty\rangle
\end{aligned}$$

By iterating this step n times in total, we obtain the desired result,

$$\begin{aligned}
\langle \mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty | 0, -\infty\rangle &= i^n J(\mathbf{k}_1) \cdots J(\mathbf{k}_n) e^{-\lambda/2} \langle 0, -\infty | 0, -\infty\rangle \\
&= i^n J(\mathbf{k}_1) \cdots J(\mathbf{k}_n) e^{-\lambda/2}
\end{aligned}$$

(j) The probability dP of finding exactly n particles, with one particle each in the ranges $d^3\mathbf{k}_i$ around $\mathbf{k}_i$, $1 \le i \le n$, can be shown to be

$$dP = |\langle \mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty | 0, -\infty\rangle|^2 \prod_{i=1}^{n} \frac{đ^3\mathbf{k}_i}{2\omega_{\mathbf{k}_i}}$$

Use this formula to show that the probability of finding exactly n particles is

$$P_n = e^{-\lambda}\frac{\lambda^n}{n!}$$

This is a Poisson distribution with average particle number λ.

We compute

$$\begin{aligned}
P_n &= \frac{1}{n!}\int \prod_{i=1}^{n} \frac{đ^3\mathbf{k}_i}{2\omega_{\mathbf{k}_i}} |\langle \mathbf{k}_1, \ldots, \mathbf{k}_n, +\infty | 0, -\infty\rangle|^2 = \frac{1}{n!}\int \prod_{i=1}^{n} \frac{đ^3\mathbf{k}_i}{2\omega_{\mathbf{k}_i}} |J(\mathbf{k}_1) \cdots J(\mathbf{k}_n)|^2 e^{-\lambda} \\
&= \frac{1}{n!} e^{-\lambda} \left(\int \frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}} |J(\mathbf{k})|^2 \right)^n = e^{-\lambda}\frac{\lambda^n}{n!}
\end{aligned}$$

Note that the integration over phase space contains a factor of $1/n!$ to avoid overcounting identical particles. In the second equality we use the result from (i).

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Spring 2023