8.323 Problem Set 6 Solutions

March 21, 2023

Question 1: Particle Production by an External Source, Continued (10 points)
Consider again Problem 2 of Problem Set 4. Introduce

$$Z[J] = \int D\phi e^{i\int d^4x \mathcal{L}}, \qquad Z_0 = Z[J=0] = \int D\phi e^{i\int d^4x \mathcal{L}_0}$$

for the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J(x)\phi = \mathcal{L}_0 + J(x)\phi$$

We can show that, with an appropriate $i\epsilon$ prescription,

$$\langle 0, +\infty | 0. -\infty\rangle = \frac{Z[J]}{Z_0}$$

Use this to find the probability of no particle production

$$P_0 = |\langle 0, +\infty | 0, -\infty\rangle|^2$$

by directly evaluating the path integral. You should reproduce your answer in 2(h) of Problem Set 4.
We compute:

$$\begin{aligned}
Z[J] &= \int D\phi e^{i\int d^4x(\mathcal{L}_0 + J\phi)} \\
&= \int D\phi \exp\left[-\frac{i}{2}\int đ^4p \left(\phi^\dagger(p)(p^2+m^2-i\epsilon)\phi(p) - J^\dagger(p)\phi(p) - J(p)\phi^\dagger(p)\right)\right] \\
&= Z_0 \exp\left[\frac{i}{2}\int đ^4p \frac{|J(p)|^2}{p^2+m^2-i\epsilon}\right]
\end{aligned}$$

In the last line, we complete the square and perform a linear shift of ϕ, $\phi^\dagger$. Therefore,

$$\begin{aligned}
P_0 = \left|\frac{Z[J]}{Z[0]}\right|^2 &= \exp\left[i\int đ^4p |J(p)|^2 \,\mathrm{Im}\frac{1}{p^2+m^2-i\epsilon}\right] \\
&= \exp\left[-\pi\int đ^4p |J(p)|^2 \delta(p^2+m^2)\right] \\
&= \exp\left[-\pi\int đ^4p |J(p)|^2 \frac{1}{2\omega_{\mathbf{p}}}(\delta(p^0-\omega_{\mathbf{p}}) + \delta(p^0+\omega_{\mathbf{p}}))\right] \\
&= \exp\left[-\pi\int \frac{đ^3\mathbf{p}}{2\pi}\left(|J(\omega_{\mathbf{p}},\mathbf{p})|^2 + |J(-\omega_{\mathbf{p}},\mathbf{p})|^2\right)\right] \\
&= \exp\left[-\int \frac{đ^3\mathbf{p}}{2\omega_{\mathbf{p}}}|J(p)|^2\right] = e^{-\lambda}
\end{aligned}$$

In the line 1, we use that $a - a^* = 2i\mathrm{Im}(a)$. In line 2, we use the identity

$$\lim_{\epsilon \to 0^+} \frac{1}{x - i\epsilon} = \mathrm{PV}\left(\frac{1}{x}\right) + i\pi\delta(x)$$

where PV is the Cauchy principal value. Line 3 uses the identity from Problem Set 1 that

$$\delta(p^2 + m^2) = \frac{1}{2\omega_{\mathbf{p}}}(\delta(p^0 - \omega_{\mathbf{p}}) + \delta(p^0 + \omega_{\mathbf{p}}))$$

In line 4 we perform the p^0 integral. Finally, in line 5 we use the invariance of the $\int đ^3\mathbf{p}$ under $\mathbf{p} \to -\mathbf{p}$, along with $J(-p) = J^*(p)$ to show that the second term is equal to the first.

Question 2: Connected Diagrams (30 points)

Consider the $\lambda\phi^4$ theory discussed in lecture, with interaction Hamiltonian

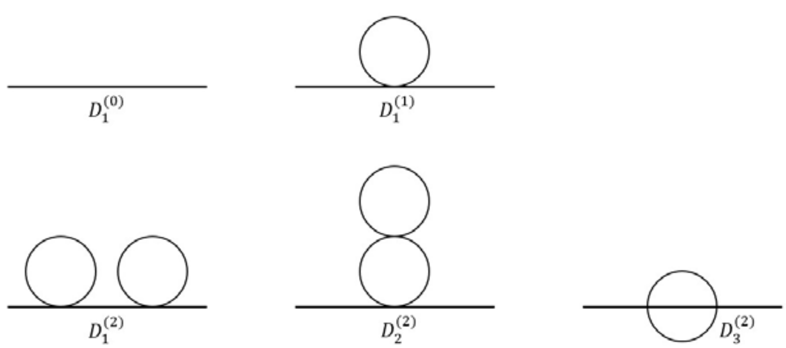
$$H_I = \frac{\lambda}{4!}\int d^3x\phi^4(x)$$

(a) List all connected diagrams for

$$\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)e^{-i\int_{-\infty}^{\infty}dtH_I}|0\rangle$$

to order $\mathcal{O}(\lambda^2)$, and give the symmetry factor for each diagram. For diagrams at orders $\mathcal{O}(\lambda^0)$ and $\mathcal{O}(\lambda^1)$, write down their expressions in both coordinate and momentum space.

Order	Diagram	S	Expressions
λ^0	$D_1^{(0)}$	1	$G_F^0(x_1-x_2)$ $(2\pi)^4\delta^{(4)}(p_1-p_2)\frac{-i}{p_1^2+m^2-i\epsilon}$
λ^1	$D_1^{(1)}$	2	$-\lambda\int d^4zG_F^0(x_1-z)G_F^0(x_2-z)G_F^0(0)$ $-i\lambda(2\pi)^4\delta^{(4)}(p_1-p_2)\frac{-i}{p_1^2+m^2-i\epsilon}\frac{-i}{p_2^2+m^2-i\epsilon}\int \not{d}^4q\frac{-i}{q^2+m^2-i\epsilon}$
λ^2	$D_1^{(2)}$	2×2	$(-i\lambda)^2\int d^4z_1d^4z_2G_F^0(x_1-z_1)G_F^0(x_2-z_2)G_F^0(z_1-z_2)G_F^0(0)^2$ $(-i\lambda)^2D_1^{(0)}(p_1,p_2)\frac{-i}{p_1^2+m^2-i\epsilon}\frac{-i}{p_2^2+m^2-i\epsilon}\left(\int \not{d}^4q\frac{-i}{q^2+m^2-i\epsilon}\right)^2$
λ^2	$D_2^{(2)}$	2×2	$(-i\lambda)^2\int d^4z_1d^4z_2G_F^0(x_1-z_1)G_F^0(x_2-z_2)G_F^0(z_1-z_2)^2G_F^0(0)$ $(-i\lambda)^2D_1^{(0)}(p_1,p_2)\frac{-i}{p_2^2+m^2-i\epsilon}\int \not{d}^4q\left(\frac{-i}{q^2+m^2-i\epsilon}\right)^2\int \not{d}^4q\frac{-i}{q^2+m^2-i\epsilon}$
λ^2	$D_3^{(2)}$	$3!$	$(-i\lambda)^2\int d^4z_1d^4z_2G_F^0(x_1-z_1)G_F^0(x_2-z_2)G_F^0(z_1-z_2)^3$ $(-i\lambda)^2D_1^{(0)}(p_1,p_2)\frac{-i}{p_2^2+m^2-i\epsilon}\int \not{d}^4q_1\not{d}^4q_2\frac{-i}{q_1^2+m^2-i\epsilon}\frac{-i}{q_2^2+m^2-i\epsilon}\frac{-i}{(q_1+q+q')^2+m^2-i\epsilon}$



The symmetry factors for the order λ^2 diagrams are obtained as follows. $D_1^{(2)}$ receives a factor of 2 from each of the propagators starting and ending on the same point (x_1 and x_2). $D_2^{(2)}$ has one factor of 2 from the propagator starting and ending on the same point, and another factor of 2 from the 2 identical propagators connecting x_1 and x_2. $D_3^{(2)}$ has 3! from the 3 identical propagators connecting x_1 and x_2.

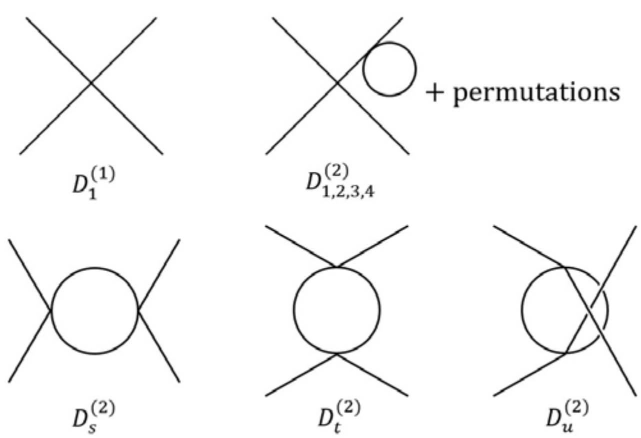
(b) List all connected diagrams of the four-point function

$$G_4(x_1, x_2, x_3, x_4) = \lambda\Omega|\mathrm{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle$$

to order $\mathcal{O}(\lambda^2)$, and choose 2 to write down their expressions in coordinate and momentum space.

Here we take all momenta to be ingoing, and use the shorthand $G_F^0(x-y) =: G^0_{x,y}$.

Order	Diagram	S	Expressions
λ^1	$D_1^{(0)}$	1	$-i\lambda \int d^4z G_F^0(x_1-z)G_F^0(x_2-z)G_F^0(x_3-z)G_F^0(x_4-z)$ $-i\lambda(2\pi)^4\delta^{(4)}(\sum_i p_i)\frac{-i}{p_1^2+m^2-i\epsilon}\frac{-i}{p_2^2+m^2-i\epsilon}\frac{-i}{p_3^2+m^2-i\epsilon}\frac{-i}{p_4^2+m^2-i\epsilon}$
λ^2	$\sum\limits_{i=1}^{4} D_i^{(2)}$	2	$(-i\lambda)^2\int d^4z_1 d^4z_2 G^0_{x_1,z_2}G^0_{z_1,z_2}G^0_{0,0}G^0_{x_2,z_1}G^0_{x_3,z_1}G^0_{x_4,z_1} + \sum\limits_{i=2}^{4}(1\leftrightarrow i)$ $-i\lambda D_1^{(1)}(p_1,p_2,p_3,p_4)\frac{-i}{p_1^2+m^2-i\epsilon}\int \text{đ}^4q\frac{-i}{q^2+m^2-i\epsilon} + \sum\limits_{i=2}^{4}(1\leftrightarrow i)$
λ^2	$D_s^{(2)}$	2	$(-i\lambda)^2\int d^4z_1 d^4z_2 G^0_{x_1,z_1}G^0_{x_2,z_1}(G^0_{z_1,z_2})^2G^0_{x_3,z_2}G^0_{x_4,z_2}$ $-i\lambda D_1^{(1)}(p_1,p_2,p_3,p_4)\int \text{đ}^4q\frac{-i}{q^2+m^2-i\epsilon}\frac{-i}{(p_1+p_2-q)^2+m^2-i\epsilon}$
λ^2	$D_t^{(2)}$	2	$(-i\lambda)^2\int d^4z_1 d^4z_2 G^0_{x_1,z_1}G^0_{x_3,z_1}(G^0_{z_1,z_2})^2G^0_{x_2,z_2}G^0_{x_4,z_2}$ $-i\lambda D_1^{(1)}(p_1,p_2,p_3,p_4)\int \text{đ}^4q\frac{-i}{q^2+m^2-i\epsilon}\frac{-i}{(p_1+p_3-q)^2+m^2-i\epsilon}$
λ^2	$D_u^{(2)}$	2	$(-i\lambda)^2\int d^4z_1 d^4z_2 G^0_{x_1,z_1}G^0_{x_4,z_1}(G^0_{z_1,z_2})^2G^0_{x_2,z_2}G^0_{x_3,z_2}$ $-i\lambda D_1^{(1)}(p_1,p_2,p_3,p_4)\int \text{đ}^4q\frac{-i}{q^2+m^2-i\epsilon}\frac{-i}{(p_1+p_4-q)^2+m^2-i\epsilon}$



For the $D_i^{(2)}$ diagrams, the symmetry factor of 2 comes from the propagator starting and ending on the same internal vertex. For the $D_{s,t,u}^{(2)}$ diagrams, it comes from the 2 identical propagators connecting x_1 and x_2.

Question 3: Vacuum Diagrams (30 points)
For a $\lambda\phi^4$ theory, consider the quantity

$$Z_0 = \langle 0|\mathrm{T}e^{-i\int_\infty^\infty dtH_I}|0\rangle$$

where the expectation is evaluated in the free theorem. We also assume the free theory vacuum $|0\rangle$ is properly normalized, i.e. $\langle 0|0\rangle = 1$.

(a) Consider

$$W_0 = \log Z_0$$

Show that W_0 can be written in a form

$$W_0 = \mathrm{cst} - i\epsilon VT$$

where cst is a constant independent of the spacetime volume, ϵ is the energy difference between the full and free theories, and VT is the total spacetime volume.

Method 1

We can write the path-integral as a ratio of matrix elements:

$$Z_0 = \langle 0|\mathrm{T}e^{-i\int dtH_I}|0\rangle = \frac{\int D\phi e^{i(S_0+S_I)}}{\int D\phi e^{iS_0}} = \frac{\langle\phi=0,\infty|\phi=0,-\infty\rangle_\Omega}{\langle\phi=0,\infty|\phi=0,-\infty\rangle_0}$$

We first compute the numerator by inserting complete sets of eigenstates of H, at very early and late times. We further take $H \to H(1-i\epsilon)$ to make the expression convergent.

$$\begin{aligned}\langle\phi=0,\infty|\phi=0,T/2\rangle_\Omega &= \lim_{T\to\infty}\sum_{n,m}\langle\phi=0,\infty|n,T/2\rangle\langle n,T/2|m,-T/2\rangle\langle m,-T/2|\phi=0,-T/2\rangle_\Omega \\ &= \lim_{T\to\infty}\sum_{n,m}\langle\phi=0,\infty|n,T/2\rangle e^{-iE_m(1-i\epsilon)T}\delta_{n,m}\langle m,-T/2|\phi=0,-T/2\rangle_\Omega\end{aligned}$$

For very large T, the dominant contribution to this sum is that with lowest E_m, i.e. vacuum E_Ω. Hence,

$$\langle\phi=0,\infty|\phi=0,T/2\rangle_\Omega = \lim_{T\to\infty}\Psi_\Omega[\phi=0]\Psi_\Omega^*[\phi=0]e^{-iE_\Omega T}$$

where $\Psi_\Omega[\phi=0]$ measures the ground state overlap of the $\phi=0$ state.

By the same procedure, we have

$$\langle\phi=0,\infty|\phi=0,T/2\rangle_0 = \lim_{T\to\infty}\Psi_0[\phi=0]\Psi_0^*[\phi=0]e^{-iE_0 T}$$

Putting both pieces together, and using that for a perturbation $E_\Omega - E_0 \ll 1$,

$$W_0 = \log Z_0 = \log\frac{|\Psi_\Omega[\phi=0]|^2 e^{-iE_\Omega T}}{|\Psi_0[\phi=0]|^2 e^{-iE_0 T}} \approx \mathrm{cst} - i(E_\Omega - E_0)T$$

This is of the desired form, where we identify $\epsilon = (E_\Omega - E_0)/V$. It is implied that $T\to\infty$.

Method 2

Alternatively, we can compute this directly by expanding in eigenstates of the full Hamiltonian:

$$\begin{aligned}Z_0 &= \langle 0|\mathrm{T}e^{-i\int dtH_I}|0\rangle = \lim_{T\to\infty}\langle 0|e^{iH_0T}e^{-iHT}|0\rangle = \lim_{T\to\infty}e^{iE_0T}\langle 0|e^{-iHT}|0\rangle \\ &= \lim_{T\to\infty}e^{iE_0T}\sum_n\langle 0|e^{-iH(1-i\epsilon)T}|n\rangle\langle n|0\rangle = e^{i(E_0-E_\Omega)T}|\langle\Omega|0\rangle|^2\end{aligned}$$

As in Method 1, we take $H \to H(1 - i\epsilon)$ to make the expression convergent. For very large T, the dominant contribution to this sum is that with lowest E_m, i.e. the vacuum E_Ω.
For a perturbation $E_\Omega - E_0 \ll 1$, so we have

$$W_0 = \log Z_0 \approx \log |\langle\Omega|0\rangle|^2 - i(E_\Omega - E_0)T$$

Again, we identify $\epsilon = (E_\Omega - E_0)/V$.

(b) The Feynman diagrams in the perturbative expansion of Z_0 have no external lines, and are often called vacuum diagrams/bubbles. We thus say that Z_0 is obtained by summing over vacuum diagrams. Show that W_0 is the sum of connected vacuum diagrams.

Let $\{V_i\}$ be the set of connected vacuum diagram contributions (including symmetry factors), and $\{\tilde{V}_I\rangle$ be the set of all vacuum diagram contributions.

Then, a general diagram $\tilde{V}_I$ consists of n_i^I V_i sub-diagrams, for each $V_i \in \{V_i\}$. In particular, we have explicitly

$$\tilde{V}_I = \frac{1}{S_I}\prod_i (V_i)^{n_i^I} = \prod_i \frac{1}{n_i^I!}(V_i)^{n_i^I}$$

Note that the expressions V_i already contain symmetry factors with associated with exchanging internal elements of subdiagrams. Therefore, the symmetry factor S_I above comes only from exchanging identical connected subdiagrams, of which there are n_i^I of type V_i. Hence $S_I = \prod_i n_i^I!$.

The full vacuum contribution comes from summing over all possible topologically distinct diagrams $\tilde{V}_I$. By the previous discussion, equivalently we may sum over all sets $\{n_i\}$:

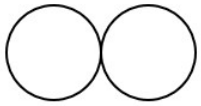
$$Z_0 = \sum_{\{n_i\}} \tilde{V}_I = \sum_{\{n_i\}} \prod_i \frac{1}{n_i!}(V_i)^{n_i} = \prod_i \sum_{n_i=0}^{\infty} \frac{1}{n_i!}(V_i)^{n_i} = \prod_i e^{V_i} = e^{\sum_i V_i}$$

Since $Z_0 = e^{W_0}$, we recover $W_0 = \sum_i V_i$ as desired, the sum of all connected vacuum diagrams.

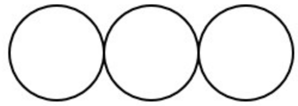
(c) Write down the expression for ϵ to order $\mathcal{O}(\lambda^2)$, in either coordinate or momentum space.

We first write down all the connected vacuum diagrams to order $\mathcal{O}(\lambda^2)$, using $\int d^4x = VT$.

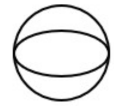
Order	Diagram	S	Expressions
λ^1	$V_1^{(1)}$	2^3	$-i\lambda VTG_F^0(0)^2$ $-i\lambda\left(\int đ^4q \frac{-i}{q^2+m^2-i\epsilon}\right)^2$
λ^2	$V_2^{(2)}$	2^4	$(-i\lambda)^2 VTG_F^0(0)^2 \int d^4z G_F^0(z)^2$ $(-i\lambda)^2\left(\int đ^4q \frac{-i}{q^2+m^2-i\epsilon}\right)^2 \int đ^4q \left(\frac{-i}{q^2+m^2-i\epsilon}\right)^2$
λ^2	$V_3^{(2)}$	$2 \times 4!$	$(-i\lambda)^2 VT \int d^4z G_F^0(z)^4$ $(-i\lambda)^2 \int đ^4q_1 đ^4q_2 đ^4q_3 \frac{-i}{q_1^2+m^2-i\epsilon}\frac{-i}{q_2^2+m^2-i\epsilon}\frac{-i}{q_3^2+m^2-i\epsilon}\frac{-i}{(q_1+q_2+q_3)^2+m^2-i\epsilon}$



$V_1^{(1)}$



$V_2^{(2)}$



$V_2^{(2)}$

The symmetry factor of $V_1^{(1)}$ has 2^2 from propagators starting and ending on the same vertex, and another factor of 2 permuting the loops. The symmetry factor of $V_2^{(2)}$ has 2^2 from propagators starting and ending on the same vertex, a factor of 2 permuting the identical vertices, and another factor of 2 due to the 2 identical propagators connecting the internal vertices. The symmetry factor of $V_2^{(2)}$ has a factor of 2 permuting the identical vertices, 4! from the 4 identical propagators connecting the internal vertices.

Hence, we have

$$\begin{aligned}
\epsilon &= \frac{i}{VT}(V_1^{(1)} + V_2^{(2)} + V_3^{(2)}) + \mathcal{O}(\lambda^3) \\
&= \frac{\lambda}{8}(G_F^0(0))^2 - \frac{i\lambda^2}{16}(G_F^0(0))^2 \int d^4x (G_F^0(x))^2 - \frac{i\lambda^2}{48} \int d^4x (G_F^0(x))^4 + \mathcal{O}(\lambda^3)
\end{aligned}$$

Question 4: General n-point Function (10 points)
Prove that in evaluating the n-point function $G_n(x_1, \ldots x_n)$, diagrams that contain factor(s) of vacuum diagrams all cancel. That is, G_n is obtained by summing over diagrams without any vacuum diagram factors. This stateement is true for any H_I, but it is enough to prove for the $\lambda\phi^4$ theory.

Method 1

We start with the expression

$$G_n(x_1, \ldots, x_n) = \frac{\langle 0|\mathrm{T}\phi(x_1)\cdots\phi(x_n)e^{-i\int dtH_I}|0\rangle}{\langle 0|\mathrm{T}e^{-i\int dtH_I}|0\rangle}$$

The numerator can be expanded as:

$$\langle 0|\mathrm{T}\phi(x_1)\cdots\phi(x_n)e^{-i\int dtH_I}|0\rangle = \sum_{m=0}^{\infty}\frac{1}{m!}\left\langle \mathrm{T}\phi(x_1)\cdots\phi(x_n)\left(i\int d^4xL_I\right)^m\right\rangle$$

Observe that bubble diagrams must come from contractions between the L_I's themselves. We define $\langle\cdots\rangle_{\text{n.b.}}$ to be the contribution to a contraction from non-bubble diagrams. We can thus write the numerator as:

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}\left\langle \mathrm{T}\phi(x_1)\cdots\phi(x_n)\left(i\int d^4xL_I\right)^{m-k}\right\rangle_{\text{n.b.}}\left\langle\left(i\int d^4xL_I\right)^k\right\rangle \\
&=\sum_{k=0}^{\infty}\sum_{m=k}^{\infty}\frac{1}{k!(m-k)!}\left\langle \mathrm{T}\phi(x_1)\cdots\phi(x_n)\left(i\int d^4xL_I\right)^{m-k}\right\rangle_{\text{n.b.}}\left\langle\left(i\int d^4xL_I\right)^k\right\rangle \\
&=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{k!m!}\left\langle \mathrm{T}\phi(x_1)\cdots\phi(x_n)\left(i\int d^4xL_I\right)^{m}\right\rangle_{\text{n.b.}}\left\langle\left(i\int d^4xL_I\right)^k\right\rangle \\
&=\sum_{k=0}^{\infty}\frac{1}{k!}\left\langle\left(i\int d^4xL_I\right)^k\right\rangle\times\sum_{m=0}^{\infty}\frac{1}{m!}\left\langle \mathrm{T}\phi(x_1)\cdots\phi(x_n)\left(i\int d^4xL_I\right)^{m}\right\rangle_{\text{n.b.}}
\end{aligned}$$

In lines 2 and 3, all we have done is changing the summation indices (discrete change of variables). This allows us to factor out the vacuum contribution, which we do in line 4. The first factor is precisely the denominator of the Gell-Mann Low formula, thus

$$G_n(x_1, \ldots x_n) = \left\langle \mathrm{T}\phi(x_1)\cdots\phi(x_n)\left(i\int d^4xL_I\right)^m\right\rangle_{\text{n.b.}}$$

That is, the n-point function is obtained by summing over diagrams without any vacuum bubbles, as desired.

Method 2

Again, we start with the Gell-Mann Low formula. The idea is to factor the numerator into a sum of diagrams with no vacuum bubbles, times a factor containing all the vacuum-bubble dependence.

Let $\{V_i\}$ represent the connected vacuum diagram contributions (including symmetry factors), as in Problem 3(b). Further, let $\{D_i\}$ represent diagrams (including symmetry factors) contributing to the numerator in the Gell-Mann Low formula, that contain no vacuum bubbles.

For any diagram D_i, the sum of all diagrams contributing to the numerator which contain D_i as a subdiagram is precisely $D_i\prod_j V^j$. This follows from the argument in 3(a), and that all symmetry factors are accounted for–there are no additional symmetry factors between the D_i and V_j's. Furthermore, every contribution to the numerator must contain some D_i as a sub-diagram, therefore the numerator is

$$\langle 0|\mathrm{T}\phi(x_1)\cdots\phi(x_n)e^{-i\int dtH_I}|0\rangle = \sum_i \prod_j e^{V_j} D_i = Z_0 \sum_i D_i$$

where in the second equality we use from Problem 3(b) that $Z_0 = \prod_i e^{V_i}$. Therefore,

$$G_n(x_1,\ldots,x_n) = \frac{Z_0 \sum_i D_i}{Z_0} = \sum_i D_i$$