8.323 Problem Set 12 Solutions

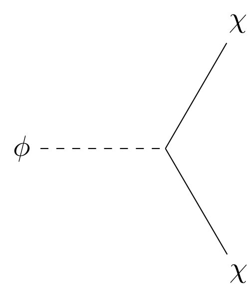
May 14, 2023

Question 1: Decay of a Scalar Particle (20 points)
Consider a theory with 2 scalar fields ϕ and χ,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}M^2\phi^2 - \frac{1}{2}(\partial_\mu \chi)^2 - \frac{1}{2}m^2\chi^2 + \frac{1}{2}g\phi\chi^2$$

Assume $M > 2m$ and the coupling g is smnall. Calculate the decay rate Γ of ϕ-particles to the lowest order in g.
We can read off the Feynman rules from the Lagrangian. They are $\frac{-i}{p^2+M^2-i\epsilon}$ for a ϕ propagator, $\frac{-i}{p^2+m^2-i\epsilon}$ for a χ propagator, and ig for the 3-point vertex.

At leading order, the decay $\phi \to \chi\chi$ is given by a single diagram. Using the Feynman rules, the amplitude is simply $\mathcal{M} = ig$.



Now we substitute this into the formula for differential decay rate in the rest frame of ϕ:

$$\begin{aligned} d\Gamma &= \frac{1}{2M}|\mathcal{M}|^2(2\pi)^4\delta^4(k_1+k_2-k_\phi)\frac{đ^3\mathbf{k}_1}{2\omega_{\mathbf{k}_1}}\frac{đ^3\mathbf{k}_2}{2\omega_{\mathbf{k}_2}} \\ &= \frac{g^2}{2M}\frac{1}{4\omega_{\mathbf{k}}^2}\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^2}\delta(2\omega_{\mathbf{k}}-k_\phi^0) = \frac{g^2}{2M}\frac{\sqrt{E^2-m^2}}{32\pi^2 E}\mathrm{d}E\mathrm{d}\Omega_2\delta(E-M/2) \end{aligned}$$

Performing the integrals, and multiplying by 1/2 for identical final state particles, we have

$$\Gamma = \frac{g^2}{32\pi M}\sqrt{1-\frac{4m^2}{M^2}}$$

Question 2: Cross-Sections (20 points)
In this problem we consider a toy model of the process in which an electron-positron collision produces a quark-antiquark final state through an intermediate photon: $e^+e^- \to \gamma \to q\bar{q}$. The role of the electron and positron is played by a massless scalar field ψ. The photon is represented by a massless scalar field ϕ. Finally, quarks are represented by a massive scalar field χ. The relevant Lagrangian density is

$$\mathcal{L}_L = -\frac{1}{2}(\partial_\mu \psi)^2 - \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}(\partial_\mu \chi)^2 - \frac{1}{2}m^2\chi^2 + \frac{1}{2}g'\phi\psi^2 + \frac{1}{2}g\phi\chi^2$$

Assume that couplings g, g' are small, and of comparable magnitude.
(a) Consider the process

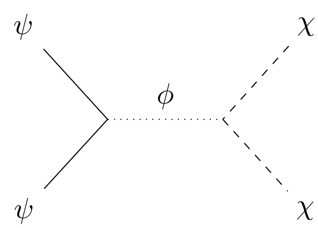
$$\psi\psi \to \chi\chi$$

Compute the total cross-section to lowest order in couplings. Express your answer in terms of s, t, u.
The Mandelstam variables for $2 \to 2$ scattering are given by

$$s = -(p_1 + p_2)^2, \qquad t = -(p_1 - p_3)^2, \qquad u = -(p_1 - p_4)^2, \qquad s + t + u = \sum_{i=1}^{4} m_i^2$$

The Feynman rules are $\frac{-i}{p^2 - i\epsilon}$ for a ϕ propagator, $\frac{-i}{p^2 - i\epsilon}$ for a ψ propagator, $\frac{-i}{p^2 + m^2 - i\epsilon}$ for a χ propagator, ig for the $\phi\chi^2$ interaction, and ig' for the $\phi\psi^2$ interaction.

At leading order, the process $\psi\psi \to \chi\chi$ is given by a single s-channel diagram.



Using the Feynman rules, the amplitude is

$$\mathcal{M} = (ig')\frac{-i}{(p_1 + p_2)^2 - i\epsilon}(ig) = igg'\frac{1}{s}$$

To compute the cross-section, we first work out kinematics. We can take the momentum to be

$$\begin{aligned} &k_1 = (E, 0, 0, E), && k_2 = (E, 0, 0, -E) \\ &k_3 = (E, 0, k'\sin\theta, k'\cos\theta), && k_3 = (E, 0, -k'\sin\theta, -k'\cos\theta), \qquad k'^2 := E^2 - m^2 \end{aligned}$$

In terms of the Mandelstam variables, this gives

$$s = 4E^2, \qquad t = -E^2 - k'^2 + 2Ek'\cos\theta, \qquad u = -E^2 - k'^2 - 2Ek'\cos\theta$$

The differential scattering amplitude is

$$d\sigma = \frac{1}{64\pi^2 s}\frac{k'}{E}|\mathcal{M}|^2 \mathrm{d}\Omega_2 = \frac{g^2 g'^2}{64\pi^2 s}\frac{k'}{E}\frac{1}{s^2}\mathrm{d}\Omega_2$$

Multiplying by 1/2 for identical final state particles and performing the angular integration, the total scattering amplitude is

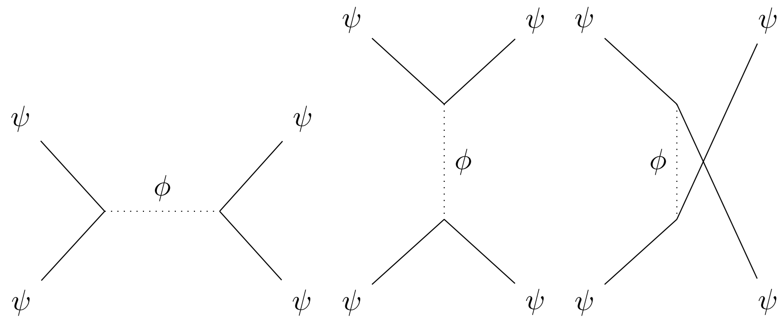
$$\sigma = \frac{1}{2}\int \mathrm{d}\sigma = \frac{(gg')^2}{32\pi s^3}\sqrt{1 - \frac{4m^2}{s}}$$

(b) Consider the process

$$\psi\psi \to \psi\psi$$

Compute the differential cross-section to lowest order. Express your answer in terms of s, t, u.

At leading order, the process $\psi\psi \to \psi\psi$ is given by s,t,u-channel diagrams.



Using the Feynman rules, the amplitude is

$$\mathcal{M} = (ig)\left[\frac{-i}{(p_1+p_2)^2 - i\epsilon} + \frac{-i}{(p_1-p_3)^2 - i\epsilon} + \frac{-i}{(p_1-p_4)^2 - i\epsilon}\right](ig) = -ig^2\left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right)$$

Proceeding in the same way as (a) with the same kinematics, the differential cross section is

$$\mathrm{d}\sigma = \frac{g^4}{64\pi^2 s}\left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right)^2 \mathrm{d}\Omega_2 = \frac{g^4}{64\pi^2 s}\left(1 - \frac{4}{\sin^2\theta}\right)^2 \mathrm{d}\Omega_2$$

Note: in principle, we would multiply by 1/2 for identical particles and perform the integral for the total scattering cross section, but the θ-integral is divergent at $\theta = 0$ and $\theta = \pi$. These are known as collinear singularities, associated with the final state particles moving along the same axis. It is related to subtleties in defining asymptotic states for massless particles, such as how to distinguish between multiple collinear massless particles with a single massless particle of the same total energy.

Question 3: Rutherford Scattering (30+10 points) The cross-section for scattering of an electron by the Coulombfield of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. Instead, treat the field as a given, classical potential $A_\mu(x)$. The interaction Hamiltonian is

$$\mathcal{H}_I = \int \mathrm{d}^3 x e \bar{\psi} \gamma^\mu \psi A_\mu$$

where $\psi(x)$ is the usual quantized Dirac field.

(a) Show the T-matrix element for electron scattering off a localized classical potential, is to lowest order

$$\langle p' | iT | p \rangle = -ie\bar{u}(p')\gamma^\mu u(p) \tilde{A}_\mu(p'-p)$$

where $\tilde{A}_\mu(q)$ is the usual 4D Fourier transform of $A_\mu(x)$.

We compute S-matrix elements:

$$\begin{aligned} {}_{\text{out}}\langle p'|p\rangle_{\text{in}} &= \langle p'|e^{-i\int \mathrm{d}^4 x H_I}|p\rangle = \langle p'|p\rangle - ie\int \mathrm{d}^4 x A_\mu(x) \langle p'|\bar{\psi}\gamma^\mu\psi|p\rangle + \mathcal{O}(e^2) \\ &= \langle p'|p\rangle - ie\int \mathrm{d}^4 x A_\mu(x) \bar{u}(p')\gamma^\mu u(p) e^{i(p-p')\cdot x} + \mathcal{O}(e^2) \\ &= \langle p'|p\rangle - ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(p-p') + \mathcal{O}(e^2) \end{aligned}$$

where in the second line we used the mode expansion for $\psi(x)$ and $\bar{\psi}(x)$.

Comparing this to the definition of the transfer matrix

$${}_{\text{out}}\langle p'|p\rangle_{\text{in}} = \langle p'|p\rangle + \langle p'|iT|p\rangle$$

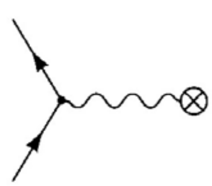
we find that to lowest order in e,

$$\langle p' | iT | p \rangle = -ie\bar{u}(p')\gamma^\mu u(p) \tilde{A}_\mu(p'-p)$$

(b) If $A_\mu(x)$ is time independent, its Fourier transform contains a delta function of energy. It is then natural to define

$$\langle p'|iT|p\rangle := i\mathcal{M}(2\pi)\delta(E_f - E_i)$$

where E_i and E_f are the initial and final energies of the particle. We adopt a new Feynman rule for computing $\mathcal{M}$:



$$= -ie\gamma^\mu \tilde{A}_\mu(\mathbf{q}),$$

where $\tilde{A}_\mu(\mathbf{q})$ is the 3D Fourier transform of $A_\mu(x)$. Given this definition of $\mathcal{M}$, show that the cross-section for scattering off a time-independent, localized potential is

$$d\sigma = \frac{1}{|\mathbf{v}_i|}\frac{1}{2E_i}\frac{d^3\mathbf{p}_f}{2E_f}|\mathcal{M}(\mathbf{p}_i \to \mathbf{p}_f)|^2 (2\pi)\delta(E_f - E_i)$$

where $\mathbf{v}_i$ is the particle's initial velocity. Integrate over $|\mathbf{p}_f|$ to find a simple expression for $d\sigma/d\Omega$.

We compute the cross section by putting our system in a large box of dimensions $T \times V$. Spatial momenta are discretized to multiples of $2\pi/L$, therefore the density of states is $V/(2\pi)^3$. There is 1 initial state excitation and 1 final state excitation, so we compute:

$$\begin{aligned}
d\sigma &= \frac{\text{scattering probability } dP}{\text{incident flux}} = \frac{1}{T|\mathbf{v}_i|/V} \frac{|\langle p_f|S|p_i\rangle|^2}{\langle p_f|p_f\rangle\langle p_i|p_i\rangle} \prod_{\text{final states}} \mathrm{d}^3\mathbf{p}_f \frac{V}{(2\pi)^3} \\
&= \frac{V}{T|\mathbf{v}_i|} \frac{(|\mathcal{M}|2\pi\delta(E_f - E_i))^2}{2E_i V} \frac{\mathrm{d}^3\mathbf{p}_f}{2E_f V} \frac{V}{(2\pi)^3} = \frac{1}{|\mathbf{v}_i|} \frac{d^3\mathbf{p}_f}{2E_i 2E_f} |\mathcal{M}|^2 2\pi\delta(E_f - E_i)
\end{aligned}$$

where in the 3rd line we used that $(2\pi\delta(E_f - E_i))^2 = 2\pi\delta(E_f - E_i)2\pi\delta(0) = T2\pi\delta(E_f - E_i)$. All factors of T and V have cancelled out, as expected.

Finally, we integrate over $|\mathbf{p}_f|$ to obtain the differential cross section:

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \int \frac{p^2 \mathrm{d}p}{(2\pi)^3} \frac{1}{|\mathbf{v}_i|} \frac{1}{4E_i E_f} |\mathcal{M}|^2 2\pi\delta(E_f - E_i) = \int \frac{p^2 \mathrm{d}p}{(2\pi)^3} \frac{1}{|\mathbf{v}_i|} \frac{1}{4E_i^2} |\mathcal{M}|^2 \frac{E_i}{|\mathbf{p}_i|} 2\pi\delta(p - |\mathbf{p}_i|) \\
&= \frac{1}{4(2\pi)^2} \frac{|\mathbf{p}_i|}{E_i |\mathbf{v}_i|} |\mathcal{M}|^2 = \frac{|\mathcal{M}|^2}{16\pi^2}
\end{aligned}$$

(c) Specialize to the case of electron scattering from a Coulomb potential, $A^0 = Ze/4\pi r$. Working in the non-relativistic limit, derive the Rutherford formula,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^4(\theta/2)}$$

Refer to (d).

(d) **Bonus.** Repeat the calculation of (c) in the relativistic regime. After averaging over the spin of the initial state and summing over the spin of the final state, show that the cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4|\mathbf{p}|^2 \beta^2 \sin^2(\theta/2)} \left(1 - \beta^2 \sin^4 \frac{\theta}{2}\right)$$

where $\alpha = e^2/4\pi$ the fine structure constant, $\mathbf{p}$ the electron momentum, and β its velocity.

Using the Feynman rules, the amplitude is

$$\mathcal{M} = e\bar{u}(p')\gamma^\mu \tilde{A}_\mu(\mathbf{p}' - \mathbf{p})u(p)$$

Therefore, the spin-summed/averaged amplitude squared is

$$\begin{aligned}
\frac{1}{2}\sum_{\text{spin}} |\mathcal{M}|^2 &= \frac{e^2}{2} \tilde{A}_\mu(\mathbf{p}' - \mathbf{p})\tilde{A}_\nu(\mathbf{p}' - \mathbf{p}) \sum_{s,s'} \bar{u}_s(p')\gamma^\mu u_{s'}(p)\bar{u}_{s'}(p)\gamma^\nu u_s(p') \\
&= \frac{e^2}{2} \tilde{A}_\mu(\mathbf{p}' - \mathbf{p})\tilde{A}_\nu(\mathbf{p}' - \mathbf{p}) \mathrm{Tr}[\gamma^\mu(\not{p} + m)\gamma^\nu(\not{p}' + m)] \\
&= 2e^2[2(p \cdot \tilde{A}(\mathbf{p}' - \mathbf{p}))(p' \cdot \tilde{A}(\mathbf{p}' - \mathbf{p})) + (m^2 + p \cdot p')\tilde{A}^2(\mathbf{p}' - \mathbf{p})]
\end{aligned}$$

Now we specialize to a Coulomb potential, $A^0 = \frac{Ze}{4\pi r}$, $A^i = 0$. The Fourier transform of $A^0(x)$ is divergent owing to the $1/r$ behavior, but we can compute it by adding a 'photon mass regulator' e^{-mr}, and take $m \to 0$ in the end.

$$\begin{aligned}
A^0(\mathbf{k}) &= \lim_{m\to 0} \int \mathrm{d}^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-mr} \frac{Ze}{4\pi r} = \frac{Ze}{2} \lim_{m\to 0} \int \mathrm{d}r\mathrm{d}\theta \, r \sin\theta e^{-mr} e^{-i|\mathbf{k}|r\cos\theta} \\
&= \lim_{m\to 0} \frac{Ze}{|\mathbf{k}|^2 + m^2} = \frac{Ze}{|\mathbf{k}|^2}
\end{aligned}$$

Therefore, using that $E_f = E_i$ for a time-independent potential, we have

$$\frac{1}{2}\sum_{\text{spin}} |\mathcal{M}|^2 = 2e^2(m^2 + EE' + \mathbf{p}\cdot\mathbf{p}')\frac{Z^2e^2}{|\mathbf{p}' - \mathbf{p}|^4} = \frac{Z^2e^4(1-\beta^2\sin^2(\theta/2))}{4|\mathbf{p}|^2\beta^2\sin^4(\theta/2)}$$

Using the result from (b), the differential cross section is thus

$$\frac{d\sigma}{d\Omega} = \frac{1}{(4\pi)^2}\frac{1}{2}\sum_{\text{spin}}|\mathcal{M}|^2 = \frac{Z^2\alpha^2}{4|\mathbf{p}|^2\beta^2\sin^4(\theta/2)}\left(1-\beta^2\sin^2\frac{\theta}{2}\right)$$

In the non-relativistic limit only the first term contributes, which is the result desired in (c).

Question 4: Electron-Muon Scattering (20+20 points)

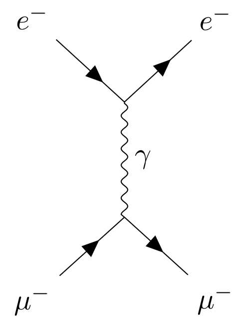
Consider the scattering process

$$e^- + \mu^- \to e^- + \mu^-$$

The electron and muon have masses m_e and m_μ, respectively.

(a) Calculate the scattering amplitude $\mathcal{M}$.

At leading order, the process $\psi\psi \to \chi\chi$ is given by a single t-channel diagram.



Using the Feynman rules, the amplitude is

$$\mathcal{M} = -ie^2 \frac{1}{t + i\epsilon}[\bar{u}_{s_1'}(p_1')\gamma^\mu u_{s_1}(p_1)][\bar{u}_{s_2'}(p_2')\gamma_\mu u_{s_2}(p_2)]$$

(b) Calculate $\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2$, averaging over the spins of the initial state, and summing over the spins of the final state. Express your answer in terms of s,t,u.

Squaring the amplitude, summing over final spins, and averaging over initial spins, we have

$$\begin{aligned}
\frac{1}{4}\sum_{s_1,s_2,s_1',s_2'} |\mathcal{M}|^2 &= \frac{e^4}{4t^2}\sum_{s_1,s_2,s_1',s_2'} [\bar{u}_{s_1'}(p_1')\gamma^\mu u_{s_1}(p_1)\bar{u}_{s_1}(p_1)\gamma^\mu u_{s_1'}(p_1')][\bar{u}_{s_2'}(p_2')\gamma_\mu u_{s_2}(p_2)\bar{u}_{s_2}(p_2)\gamma_\mu u_{s_2'}(p_2')] \\
&= \frac{e^4}{t^2}\mathrm{Tr}[\gamma^\mu \Lambda_+(p_1)\gamma^\nu \Lambda_+(p_1')]\mathrm{Tr}[\gamma_\mu \Lambda_+(p_2)\gamma_\nu \Lambda_+(p_2')] \\
&= \frac{4e^4}{t^2}[(m_e^2 + p_1 \cdot p_1')\eta^{\mu\nu} - p_1^\mu p_1'^\nu - p_1^\nu p_1'^\mu][(m_\mu^2 + p_2 \cdot p_2')\eta_{\mu\nu} - p_{2\mu}p_{2\nu}' - p_{2\nu}p_{2\mu}'] \\
&= \frac{8e^4}{t^2}(p_1 \cdot p_2 p_1' \cdot p_2' + p_1 \cdot p_2' p_1' \cdot p_2 + m_e^2 p_2 \cdot p_2' + m_\mu^2 p_1 \cdot p_1' + 2m_e^2 m_\mu^2) \\
&= \frac{8e^4}{t^2}\left(\frac{s^2}{4} + \frac{u^2}{4} - (m_e^2 + m_\mu^2)(s+u) + \frac{3}{2}(m_e^2 + m_\mu^2)^2\right)
\end{aligned}$$

In the second line we use the identity $\sum_s u_s(p)\bar{u}_s(p) = \Lambda_+(p) = i(i\not{p} + m)$. In the third line, we use the gamma trace identities

$$\mathrm{Tr}(\gamma^\mu) = \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = 0, \qquad \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}, \qquad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$$

in order to simplify

$$\mathrm{Tr}[\gamma^\mu(i\not{p} + m)\gamma^\nu(i\not{q} + m)] = \eta^{\mu\nu}(4m^2 + p \cdot q) - 4p^\mu q^\nu - 4p^\nu q^\mu$$

In the last line, we use the Mandelstam identities

$$s = -(p_1 + p_2)^2 = m_e^2 + m_\mu^2 - 2p_1 \cdot p_2, \qquad u = -(p_1 - p_2')^2 = m_e^2 + m_\mu^2 + 2p_1 \cdot p_2'$$

(c) **Bonus.** Calculate the differential cross-section in the center of mass frame.
We first work out kinematics. In the center of mass frame, we can take the momenta to be

$$
\begin{aligned}
p_1 &= (E_1, 0, 0, p), & p_2 &= (E_2, 0, 0, -p) \\
p_1' &= (E_1, 0, p\sin\theta, p\cos\theta), & p_2' &= (E_2, 0, -p\sin\theta, -p\cos\theta), \qquad & p'^2 := E^2 - m^2
\end{aligned}
$$

In terms of Mandelstam variables,

$$
\begin{aligned}
s &= -(p_1 + p_2)^2 = E_{\mathrm{cm}}^2 = (E_1 + E_2)^2 \\
t &= -(p_1 - p_1')^2 = 2p^2(\cos\theta - 1) \\
u &= -(p_1 - p_2')^2 = (E_2 - E_1)^2 - 2p^2(\cos\theta + 1)
\end{aligned}
$$

In the center of mass frame, the spin-summed/averaged differential cross-section is

$$
\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{cm}} &= \frac{1}{64\pi^2 s}\frac{|\mathbf{p}_1'|}{|\mathbf{p}_1|}\frac{1}{4}\sum|\mathcal{M}|^2 \\
&= \frac{e^4}{8\pi^2 st^2}\left(\frac{s^2}{4} + \frac{u^2}{4} - (m_e^2 + m_\mu^2)(s+u) + \frac{3}{2}(m_e^2 + m_\mu^2)^2\right)^2
\end{aligned}
$$

(d) **Bonus.** Obtain the differential cross-section in the muon rest frame. Show that in the limit $m_\mu \to \infty$, one recovers the cross section of Rutherford scattering.
In the muon rest frame, the kinematics becomes

$$
\begin{aligned}
p_1 &= (E_1, 0, 0, p), & p_2 &= (m_\mu, 0, 0, 0) \\
p_1' &= (E_1', 0, p'\sin\theta, p'\cos\theta), & p_2' &= (E_1 - E_1' + m_\mu, 0, -p'\sin\theta, p - p'\cos\theta)
\end{aligned}
$$

The Lorentz-invariant differential cross section is given by

$$
d\sigma = \frac{1}{4E_A E_B|v_A - v_B|}\frac{1}{4}\sum_{\mathrm{spin}}|\mathcal{M}|^2 d\Pi_{\mathrm{LIPS}}
$$

The spin-summed/averaged amplitude is given in (b). In the muon rest frame,

$$
E_1E_2|v_1 - v_2| = |E_1 p_2^z - E_2 p_1^z| = m_\mu p
$$

The final state phase space factor is

$$
d\Pi_{\mathrm{LIPS}} = (2\pi)^4\delta^4(p_1 + p_2 - p_1' - p_2')\frac{d^3\mathbf{p}_1'}{2E_1'}\frac{d^3\mathbf{p}_2'}{2E_2'} = \frac{1}{(2\pi)^2}\frac{1}{4E_1'E_2'}\delta(E_1 + m_\mu - E_1' - E_2')p'^2\mathrm{d}p'\mathrm{d}\Omega_2
$$

for $E_1'^2 = m_e^2 + p'^2$, and $E_2'^2 = m_\mu^2 + |\mathbf{p}_1' - \mathbf{p}_1|^2 = m_\mu^2 + p'^2 + p^2 - 2pp'\cos\theta$.

Now we take the limit as $m_\mu^2 \to \infty$. We have $E_2' \approx m_\mu$,therefore the delta function imposes $E_1 \approx E_1'$, thus $p \approx p'$. The final state phase factor simplifies to

$$
d\Pi_{\mathrm{LIPS}} \approx \frac{1}{16\pi^2}\frac{p}{m_\mu}\mathrm{d}\Omega_2
$$

The leading m_μ dependence from the amplitude is

$$
\begin{aligned}
\frac{1}{4}\sum_{\mathrm{spin}}|\mathcal{M}|^2 &= \frac{8e^4}{t^2}(p_1\cdot p_2 p_1'\cdot p_2' + p_1\cdot p_2' p_1'\cdot p_2 + m_e^2 p_2\cdot p_2' + m_\mu^2 p_1\cdot p_1' + 2m_e^2 m_\mu^2) \\
&\approx \frac{e^4 m_\mu^2}{2p^4\sin^4(\theta/2)}(2E_1^2 - m_e^2 - E_1^2 + p^2\cos\theta + 2m_e^2) \\
&= \frac{e^4 m_\mu^2}{2p^4\sin^4(\theta/2)}(2E_1^2 - 2p^2\sin^2(\theta/2)) = \frac{e^4 m_\mu^2}{\beta p^2\sin^4(\theta/2)}(1 - \beta^2\sin^2(\theta/2)), \qquad \beta = p/E_1
\end{aligned}
$$

Putting everything together, the differential cross section in the muon rest frame for $m_\mu^2 \to \infty$ is

$$\frac{d\sigma}{d\Omega} = \frac{1}{4m_\mu p}\frac{1}{16\pi^2}\frac{p}{m_\mu}\frac{1}{4}\sum_{\text{spin}}|\mathcal{M}|^2 \approx \frac{\alpha^2}{4p^2\beta^2\sin^4(\theta/2)}(1-\beta^2\sinh^2(\theta/2))$$

This matches the result from Problem 3.

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