IAIN STEWART: So last time, we were talking about the chiral Lagrangian as another example of an effective field theory to illustrate some of the tools of effective field theory that we didn't meet when integrating our heavy particles. It was an example of a bottom-up effective field theory. So we were constructing it from the bottom up for the most part. That's how one should think about it.

There was several things in our program to understand from chiral perturbation theory. And we kind of got halfway through. So we want to understand nonlinear symmetry representations. And we went through how you could think about linear representations and making changes of variable and putting the Lagrangian in a form where you could see where the nonlinear realization was coming from very explicitly, as well as another way where we constructed it from the bottom up thinking about parameterizing the co-set.

And then we started talking about loops in chiral perturbation theory. And I gave you this example of this loop here. So we constructed the chiral Lagrangian. That's what I've written here.

In terms of the sigma field, this is the nonlinear version, sigma field, exponential of M field. M field has the pi n fields in it. I rewrote it in a little bit of a different notation, which will be useful today, where I introduced this thing called chi. But f squared over 8 chi is just Mq.

So it's just a rescaling. And I just wanted to have the same kind of pre-factor here and here. That's why I did that.

The power counting is in p squared derivatives or M pi squared, which is the same as Mq. And the reason that M pi squared and Mq count the same is there's this relation between pi squared and the quark masses. And then we started talking about loops.

And we said that, if we looked at this loop, the result for this loop diagram would be a factor of p squared or M pi squared times a factor of p squared or M pi squared divided by 4 pi f. This is the same size as the lowest order, so the suppression factor is this.

So we can say that loops are suppressed by p squared over some scale lambda chi squared. So this is scale of our expansion. And we can associate that scale to 4 pi times f.

OK, so we're going to continue with that story unless there's any questions from last time. OK, so we want to use dimensional regularization. And if we use dimensional regularization, we can think about having an MS bar scheme.

So how do we do that? It's a little different than gauge theory, but the idea is exactly the same. So we look at the dimensions of objects. This capital M-- supposed to be a capital M-- is like a scalar field, so it has dimension 1 minus epsilon.

The decay constant then, if you look at it, which is the coupling, has to have dimension 1 minus epsilon 2. And you can see that because this exponential better be the exponential of something dimensionless so that, whatever the dimension of M, should cancel the dimension of f.

And then that works out with the look with the measure as well, which has a ddx. So therefore, just like you did for a gauge coupling, you can say f bare in some mu to the minus epsilon times f.

And if you want to do the same thing for the second term, though I've put the V0 inside the-- sorry, there's something missing in my equation. There was a V0 here. So before last time, we've written this equation as V0 out front and then Mq inside. And I just rescaled it to this chi.

OK, so you can think either in terms of chi. Chi is kind of absorbing the parameter along with the Mq, but they're equivalent things. So you can think of also doing a rescaling that gets the dimension of this term right. And the thing that you want is this.

The P0 would then be mu to the minus 2 epsilon to compensate for the fact that the first term had an f squared and the second term just had a V0. But that's just to dimension counting.

There's also no mus in physical quantities like the pi n mass, which is an observable, which has just a definite value. And there's also, in chiral perturbation theory, no mus in the quark mass. So when you look at V0 over f squared, if you look at it bare, that's the same as renormalized.

The other thing that's different about this than in a gauge theory is, in the gauge theory, you'd have a z. You'd have g bare is equal to e to the epsilon times g, but then you'd have a zg. But in chiral perturbation theory, the way that the loops work, the loops aren't renormalizing the leading order Lagrangian. The loops are renormalizing something else because they ended up being suppressed.

So you don't need counter-terms here for the leading order Lagrangian. And that's why I didn't put a zf or a zV0 here. OK, but when you do this loop, you do get ultraviolet divergences.

So you get 1 over epsilon divergence if you're using dimensional regularization. And it comes along with the log of mu squared. And then it'll be divided by some scale that you have in your loop, like an external momentum p squared, or it could be the pi n mass. Both of those will show up generically.

And in the way we're doing counting, the way the loops are showing up, if I can bind together the factors that I told you about here, it's either p to the fourth, 4 powers of p, or 2 of p and 2 of M pi or 4 powers of M pi. So to cancel that 1 over epsilon, we need a counter-term. But it's not a counter-term in that Lagrangian.

It's a counter term in the higher order Lagrangian. So we're not done by just calculating the loop. We have to actually include some higher dimension operators that are the same order in our power counting.

So we'll talk about SU(3) a little later. I'll start out by talking about SU(2). In SU(2), the form of that Lagrangian with higher order terms is just taking what we had before and just going to a higher dimension.

So we could have two more derivatives. One way of doing that is taking what we had at lowest order and squaring it. Another way of doing it would be to take and contract the indices a little bit differently.

So I could do it like this. The trace is cyclic, so I can move things around from back to front. But I could have mu nu in one trace and mu nu in another trace rather than just contracting within the trace. That's another possible term.

And there's a bunch more terms which I decided to wait and enumerate when we do SU(3) to give you a full enumeration. These additional terms you can build from one Mq, which means one of these chi fields or chi objects and two partials. That's one possibility or two Mq's because each Mq counts like two derivatives.

So there's some other terms that we could build that are analogs of this term. You could square that term, for example. That would be a possible higher order term.

And so these coefficients of this Lagrangian are what we need to cancel off the divergences. So the counter-terms that we have that renormalize these loop graphs come from this Lagrangian.

So the theory is renormalizable in a EFT sense, order by order in its power counting expansion. When you go to order p to the fourth, you have to consistently put in everything that's order p to the fourth. That includes the loops, which are p to the fourth, as well as new local Lagrangian interactions.

AUDIENCE: Is that a choice to not renormalize the f bare and [? d ?] bare?

IAIN STEWART: They just don't get renormalized.

AUDIENCE: But why couldn't I just multiply by 1 plus--

IAIN STEWART: Oh--

AUDIENCE: --order p squared by times delta [INAUDIBLE].

IAIN STEWART: Oh, you want to try to make a different scheme for them?

AUDIENCE: I'd screw up the power counting [INAUDIBLE].

IAIN STEWART: So effectively, you could screw up the power counting by doing that. So let me tell you what you can do. You could put a 2 here, right? But you don't really have much more freedom than just multiplying by a number.

AUDIENCE: So I can't multiply by 1 plus p squared over s squared.

IAIN STEWART: Yeah. That would be a screwing up the power counting. And effectively, you wouldn't be multiplying by p squared. You'd be putting derivatives in, right? Because there's no p. So p is not something that you're allowed to multiply by.

AUDIENCE: So that's the only thing that's stopping me-- is what is derivatives [INAUDIBLE]?

IAIN STEWART: You could put derivatives in, but then that's equivalent to something here. So you have to ask what you're doing because you're mixing up things.

AUDIENCE: OK.

IAIN STEWART: Yeah.

AUDIENCE: [INAUDIBLE] definitely a better way to do. I was just wondering if there was sort of freedom.

IAIN STEWART: Yeah, there's no freedom really.

AUDIENCE: Good.

AUDIENCE: Why you don't get the renormalization [INAUDIBLE], like from [INAUDIBLE] [? loop? ?]

IAIN STEWART: Because all the loops end up being suppressed by p squared or M pi squared. So if you write down any diagram and you think it might give something here, even if it's a two-point function, it just doesn't.

AUDIENCE: [INAUDIBLE] to 4 [INAUDIBLE].

IAIN STEWART: That's right. Oh, so if you want, you could think that some terms here are kind of corrections to the kinetic term if you derive the equation of motion and you include L and that L and this L. But this Lagrangian, there's no loop corrections to this Lagrangian.

There's no corrections to this lowest order Lagrangian. And that's because the loops are all suppressed. Totally different then gauge theory, loops are suppressed by p squared or M pi squared.

So they only renormalize some high order things. And we don't have to worry about really thinking about f. I mean, the way I've talked about f here and the factors of mu is just in order to see where these factors and mu come out here.

But what actually happens is that you get 1 over epsilons that are cancelled like the counter-terms here. And there's no need from a renormalization perspective to sort of change your definitions here. People play with different definitions, but it's like they use f squared over 4 instead of f squared over 8. That's the extent of what people do with playing with the leading order Lagrangian.

AUDIENCE: So you have [INAUDIBLE] the same thing with the [INAUDIBLE] that [INAUDIBLE] [? is not ?] [? renormalizing ?] [INAUDIBLE].

IAIN STEWART: Both of them are not, yeah. Yeah. So loops that you build out of these interactions end up renormalizing these terms, which is kind of neat.

AUDIENCE: That's just because [INAUDIBLE]. That's just because there's [INAUDIBLE] [? sort of ?] [INAUDIBLE].

IAIN STEWART: Yeah. Well, so we'll prove in a minute a general power counting formula that tells us how to organize all this. So for now, you can think of it as an observation, but we'll build it into a general formula that tells you sort of how you would organize the power counting and renormalization of this theory to all orders in its expansion in a minute.

AUDIENCE: So when you write down all these [INAUDIBLE] at [? power ?] [? p4, ?] does that include all the [INAUDIBLE] you need?

IAIN STEWART: Yeah. That's right. Once I've enumerated all the terms, which I will do for you with SU(3)-- so it includes things like this and things like this as well-- then those are all the possible counter-terms that you could actually need to renormalize any of the one-loop diagrams that you could generate with the leading order Lagrangian.

And you know that by power counting. Once you know the loops are that order and you've included all local interactions, then you're done. So there's things I didn't write down there, so just comment about that.

So what is the equation of motion here? It's a little more complicated because we have the sigma field, but you can work out that the equation of motion is the following.

And I've given you some reading, so you can read about a derivation of this equation which takes a little effort. So your leading order equation of motion basically allows you to get rid of partial squareds on the sigma. And that's why I've always written one partial on each sigma. So that's what you should think of this guy doing.

There's also some SU(2) identities that have been used, even in what I've written, because you could think of having other types of traces like this one, for example. Instead of having trace squared, I could have the following. Just put everything in one trace.

I always have to alternate sigma sigma dagger, sigma sigma dagger because del mu sigma sigma dagger is 0. Because sigma sigma dagger is 1. And that provides an identity that I'm also able to use.

And so I want to have sigmas and sigma daggers next to each other. Sorry, chiral symmetry means that sigmas and sigma daggers should be next to each other. This is an identity that I can also use to simplify things. So that allows me to move things around.

And this operator, though, in SU(2) is actually related to these two. It's not unique. Because of the structure of Pauli matrices, you have a formula which you can work out, which says that this guy is actually half of the trace squared. And there's another one if you did.

This guy, he also can be related to those two guys over there. So there's a bunch of things that went into even writing down the level of information that I told you. You could think there's more things, but then, when you enumerate all the possible things that you can do, you find that those more things are related to these things. So you really want to construct a minimal basis. And there's some things that go into doing that.

So at order p to the fourth, as I said, we include both loops. And those loops will have terms like p to the fourth log mu squared over p squared. And we include terms like the L1 and L2, which are p of the fourth type interactions.

Once I take out the counter-term, the renormalized coupling is mu dependent just like the gauge coupling. This just comes from a four-point interaction with the Li. And the mu dependence, by construction, was canceled between these two things.

The divergence from the loop is cancelled by the counter-term. And correspondingly, the corresponding statement, if you want to make it in terms of mu, is that the mu dependence that is tightly tied to that 1 over epsilon is cancelled in the renormalized quantities.

So there's these two contributions. They're both mu dependent. And that mu dependence cancels.

So the way that you should think of this is you should think mu is a cut off. It's not a hard cut off, but it is a cut off. And what the cut off is doing is dividing up infrared and ultraviolet physics.

In this case, the low energy physics is in the matrix elements in the loops where we have our propagating low energy degrees of freedom, which are the pions. And the high energy physics is in the coefficients, as usual. And they're both mu dependent.

And you can think of that mu dependent as a cut off that divides up how much of the physics goes into those low energy loops, how much of the physics goes into the couplings. So the difference between this and integrating on a massive particle is not the physics of where things go because the low energy physics always goes in the matrix elements. The high energy physics goes in the couplings.

The difference is that, here, the way that you should think of it is that we can calculate the matrix elements explicitly because our theory is in terms of the right degrees of freedom to describe long distance physics, which are the pions. And then the coefficients are the unknowns. That's the way in which it's different. And that's really kind of the difference between bottom-up and top-down.

So with that in mind, if you want to think of what these couplings are, if we just divide by f squared, then you can think about dimensionally what they are. In order for them to sort of match up with what we got from the loops is that they have some dependence that looks like this. You can think of it as parameterized by some coefficients a and b. If I want to match up with the 4 pi f squared that comes from the loops, then I have to sort of say that there's a 4 pi hiding inside the Li's.

So I can do that. And then there's some coefficients here. There's some mu dependence. And the scales that are in the coefficients are the high scales, like lambda chi and higher and rho, those types of things. So let me just call it lambda chi. And then there's some numbers here a and b, which encode the high mass physics.

Now, the point of thinking about power counting is that thinking about the fact that the loops and these coefficients are the same size actually tells you that these ai's and bi's should be order 1. And this actually goes under the rubric of something called naive dimensional analysis.

So what naive dimensional analysis says is that this cut off that we've put between the low energy physics and the high energy physics is arbitrary. And we could change it. We could change it by a factor of 2.

And what changing it does is it moves pieces back and forth. But if we could move pieces back and forth, then you wouldn't expect that the two things that you're talking about would be different in magnitude because we're allowed to move pieces back and forth. So you expect that the size of contributions from the coefficients are about the same size as the loops.

And that's this naive dimensional analysis. So changing mu moves pieces back and forth. The sum is mu independent, but each individual thing is not.

And because we're able to move things back and forth, we expect them to be the same order of magnitude. And that is this statement here, that the ai's and bi's, once I account for the 4 pis-- so with this argument, I can figure out where there's 4 pis hiding in the coefficients. Because I can identify 4 pis in loops. And then if they're supposed to be the same size, I can also identify 4 pis in coefficients.

So if you had figured this out, I don't know, 20 years ago, then you could have got a PhD thesis like Aneesh Manohar did, which was 25 pages long. Short PhD theses do exist. OK.

So what do we do in practice? In practice, we have to pick a value of mu. Just like when we were talking about gauge theory, we need to pick a value of mu. There we were picking things like mu equals Mb.

Here, what people do is they typically pick mus that are high. So maybe they would pick the rho mass. Maybe they would pick lambda chi. Or they pick just something in between, like a GeV. These are typical values that people use.

So what that means is that you've removed all large logs from your coefficients. And that's because you want dimensional analysis to hold for your coefficients because you don't know them. So you would like to have some power counting estimate for them.

Then you'd like to go out and fit them to data. And you'd like the result that you get from fitting to the data to agree with your power counting estimate, so that you're happy. It turns out that it does, so you are happy. And so to avoid large logs in the story, you put the large logs into the matrix elements.

So that's different than our story in gauge theory, where we were thinking that we would re-sum all the large logarithms in the coefficients. And then our matrix elements would also have no large logarithms. Here, we're just saying, well, let's allow for large logs in the matrix element.

But the story is also different from gauge theory in another way, which is that there's not an infinite series here of large logarithms that you need to re-sum. And that's related to the fact that the kinetic term didn't get renormalized.

There's no mu dependence in the coefficients of the kinetic term. So when the loop graphs aren't having coefficient squared depending on mu, that doesn't happen. We simply have one log in our loop graphs, one log from our counter-term diagrams.

They explicitly cancel. There's no high order terms. The renormalization group here is completely trivial. If you integrate, you just get one log. So there's not really even a reason to talk about it.

OK, so there's certainly some differences in this theory than there are in gauge theory.

And so typically, the paradigm that you have is you can calculate matrix elements. You're going to fit the coefficients. You think of enough experiments such that you can get all that information about those coefficients. And then you can think of other experiments and make predictions.

Now, as you go to higher orders in the theory, you get more coefficients because you keep having to go to higher orders and construct higher dimension operators. And so the paradigm of figuring out how to fit the coefficients gets harder and harder. And at some point, you effectively lose predictive power because you can't think of enough observables that can be measured in order to fit all your coefficients.

But certainly, at order p to the fourth, people can think of many more observables. And actually, people have worked out p to the sixth as well here. So two-loop chiral perturbation theory is kind of the state of the art.

AUDIENCE: I have a question.

IAIN STEWART: Yeah.

AUDIENCE: So why allow log in this case [? is not ?] screwing up the [INAUDIBLE]?

IAIN STEWART: Yeah. It doesn't screw it up because it only happens once. So basically, what you could say is you can say, I have this matrix element. It's got a large log and a coefficient.

The coupling, it's not like large log times the coupling is something that you could count as order 1 because that's not going to reappear in the higher order when you go to the higher order. So you still have a large log. And you are allowed to say that large log is the most important piece of my matrix element, but it doesn't repeat itself in the higher orders.

Because once you go to higher loops, you're actually getting something that's power suppressed, not loop suppressed. So the loops are giving power suppression. And that changes a lot of things.

OK. What would happen if we used a hard cut off? Maybe some of this story would be a bit more transparent if we'd done that because we'd see that we were explicitly dividing up low energy and high energy physics. But there's a price to pay for that transparency.

And here, effectively everybody decides it's too high a price to pay. And so they use dim reg. So what type of prices would you pay?

So if you did that, this loop here would actually have terms that are cut-off to the fourth over lambda chi to the fourth. And that guy breaks chiral symmetry. I mean, it's giving a constant term. And you're not seeing the derivative coupling. And that means, effectively, that there's no counter-term to absorb this dependence in our L chi because our L chi respected chiral symmetry. So that's kind of bad.

If you had a cut-off, you'd also get terms that went cut-off squared times momentum squared where two powers of the p are replaced by cut-off. And that, in this language that we used earlier, would break the power counting in the sense that we have a power counting that says that the loops should be suppressed by d to the fourth. And it's broken by this.

And then you would need to absorb that and renormalize you're leading order coupling. But effectively, all the renormalization would be doing is restoring your power counting. It's not doing something that you should interpret as really a physical thing. So it's just something to be avoided.

And then finally, the physics that you actually want, which is the logs of the cut-off, would also be there. And you would do the same thing as you do in dim reg. You would absorb that in higher dimension operators, OK? So we won't use a cut-off because we don't want to think about things that are breaking power counting or messing up chiral symmetry.

Another thing that's different about chiral theories versus gauge theories is the structure of infrared divergences. And that's because you have derivative couplings. So you have many fewer infrared singularities than you have in gauge theory.

And in fact, you usually don't have any. And one way of describing that is that you usually have a good M pi squared goes to 0 or p squared goes to 0 limit of your results. So you can just explicitly take these limits and talk about the results in those limits.

And if we look back at what we were talking about, you get p to the fourth log p squared. So there's a p to the fourth multiplying the log p squared. So you're not seeing log p squared blow up because it's got so many powers of p multiplying it.

Although our focus is on formalism, I have to at least give you one example of something predictive and phenomenological. So pi pi scattering is a nice example of phenomenology. And it's particularly nice if you look at it below an elastic threshold. So you just have elastic scattering.

So you could look at pi pi goes to 4 pi. And that's something you could actually look at in chiral perturbation theory. But if we look at just pi pi goes to pi pi where we have not enough energy to produce 4 pions, then the scattering is particularly simple. It's just described by an S matrix, which we can enumerate channel by channel. And it's just a phase where this L is a partial wave phase. And the i is an isospin.

So for each isospin and for each angular momentum state, we get different phases. But that's encoding all the scattering. That's like doing non-relativistic scattering or very simple quantum mechanical scattering.

And when you have an elastic scattering like that, there's something called the effective range expansion, which is a derivative expansion for the phase shift delta. And if you do it for an arbitrary partial wave, this is how it looks. So this is, again, something that you would find in the discussion of scattering theory in quantum mechanics.

And you have a derivative expansion of this quantity p to the 2L plus 1 cotan of that phase shift. And the a's here and the r0 depend on what channel you're talking about. And if we actually just take the fact that we can-- so this is a description from quantum mechanics. This is true irrespective of what theory you're talking about as long as you don't have other channels you can produce.

If you talk about it from chiral perturbation theory, though, we can just calculate pi pi goes to pi pi. And if you do that, which [? Weinberg ?] did, then you just get results for these coefficients. So I'll just quote a couple of them to you.

So you just get parameter-free results, parameter-free in the sense that they just involve M pi and this thing f. This thing f is actually measured by pi on decay, so it's not an unknown. And then from that, at lowest order in chiral perturbation theory, you have no parameters once you fixed M pi and f pi.

And you can just make predictions for these scattering lengths, and those are the predictions. And they're parameter-free in the sense that I just said. OK, so that gives you some of the idea of what chiral perturbation theory can do for you.

OK, so let's talk about back to formalism. There's many more phenomenological examples you could do, but that's not our focus. Back to formalism and back to this general power counting discussion, so let's consider an arbitrary diagram in this theory, this chiral perturbation theory.

And let's enumerate some pieces of that diagram. So we'll say that it has some number of vertices. We just count how many times I've inserted vertices from the Lagrangians, and I'll call that NV. Some number of internal lines-- call NI. Some number of external lines-- the external lines are all pions. That's what our theory is describing. It could be kaons and etas if we're doing SU(3)-- and. then some number of loops.

So we have an integer associated with each of these things. And when I talk about vertices, I don't want to restrict myself just to the leading order vertices. I want to talk about also these Li's and higher order vertices as well. So let me have a notation for that, where I take this integer NV, which counts all vertices, and split it into pieces that count the vertices at each of those different orders in the expansion in p.

So Nn is the number of vertices that are order p to the n or M pi to the n or combinations of p and M pi to the n, but n of them, all right? So hopefully, that's clear. So if I have two insertions of the leading order Lagrangian and one insertion of the sub-leading, then n sub 0 would be 2.

And N sub 1 would be 1. And the total would be 3. That's what this notation means.

So we'll assume we're using a regulator like dimensional regularization. So we don't have to worry about the regulator messing up power counting. And basically, that means we could ignore the regulator as far as this discussion is concerned. And we just count.

And effectively, we can just count mass dimension. So we'll count lambda chi factors. Let's think about it that way.

And we'll think about counting lambda chi factors for a matrix element that I'll call curly M, which has NE external pions. So NE lines are poking out of it. So then if we look at the vertices, we can count how many factors of lambda chi there is. And I'm counting all dimensionful and turning all dimensionful things into lambda chi.

And so each different order in N gets a number of lambda chis because we even saw that already in the examples we treated where we got f squared from the leading order. But from the sub-leading, we got Li. And that was dimensionless.

So for n equals 2, which is the leading order here, leading order was p squared. I said that not quite right a minute ago. So lowest order is p squared n equals 2. That's L0. That had an f squared.

And that comes out, if I just have 4 minus 2, that's 2. n equals 4, that was giving our Li's, which were dimensionless, OK? So you can see the formula working.

And if we went to higher orders in the derivative and chiral expansion, then we'd start getting lambda chis in the denominator. So this is just counting from the vertices, which you should think of as counting just from the pre-factors in the Lagrangian. There's also f's that come with the pions because every factor of the pion field comes with a factor of f.

You always have pi over f. And I'm turning f into 4 pi f for this discussion. I'm not worrying about the 4 pis. You could do a more fancy version of this where you worry about the 4 pis, but let's just focus on the dimensions.

So if you have an internal line, that's a contraction of 2 pion fields. So that gets 2f's. And the external line is just 1 pion field, so that just gets 1.

Topologically, these different things that we enumerated are not unrelated. So the order identity tells us that the number of internal lines is the number of loops plus the number of vertices minus 1. And so we use that to get rid of NI. So then we can just put these things together.

So if I get rid of NI here and I replace it by NL-- so that's that term-- I replace it by the sum over the Nn's which are the vertices. And then this gives me a plus 2. OK. Now, that's not the dimension of the left-hand side.

That's just the dimension of the ingredients. There's also things that are coming from factors of M pi or factors of p. And let me just call that something, E to the D where D is just some number, integer, and then some function of logarithms of p over mu or M pi over mu.

So E could be M pi or p. And for the purpose of power counting, I'm not distinguishing. So let's just call it E just to have a notation that could be either M pi or p.

And then there's one more thing we can do, which is we can look at the left-hand side. And we can say, just by dimensional analysis, the matrix element, what should be its dimension? And depending on how many bosons you have sticking out, your matrix element should have a certain dimension, which you could figure out.

And that dimension is just 4 minus NE. Two-point function would scale like p squared, et cetera. So now, I have different things that are giving mass dimensions, the lambda chis and the E's. But because I know what the answer has to be, 4 minus NE, I can solve for D.

And that's what I got. OK. So the answer for D, in order to get the dimensions right, we have to compensate for the number of lambda chis by factors of M pi and p. And when we do that, then we need this many of them in order to get the dimensions right.

So one thing you see from this formula is that D is greater than or equal to 2. That's because these things are positive. The Lagrangian starts with n equals 2 and then goes higher.

And when I add either of these terms, I cause suppression, or I stay the same. So this could be 0. This could be 0, or it could be bigger. But it can be smaller.

So you always get more E's, which are M pis or p's, by adding vertices or adding loops. So when you looked at the loop graphs that were built out of the leading order Lagrangian, those terms had this be 0 because n was 2. But then you got suppression because we built loop graphs. So you got suppression from this term.

And when we looked at the higher dimensional operators, there was no loop. So we didn't have this term, but then we had a contribution from this guy. But those were trading off.

So having this is, effectively, part of what we need in order to make sure the theory is well-defined because it tells us how to organize the theory and what parts of the theory we need to worry about if we want a certain accuracy and what parts we can ignore. We need to know that we don't need to think about two-loop diagrams. And this tells us that we don't. Yeah.

AUDIENCE: I didn't understand [? where you ?] [INAUDIBLE].

IAIN STEWART: Yeah, let me say it again. So we figured out the lambda chis by this stuff up here. Then I said, let there be an arbitrary E to the D, some parameter which we haven't figured out anything yet. But I know by just what possibly this could depend on that it could also depend on an M pis or pis. So let me put some polynomial power in and then some function that could be non-polynomial.

AUDIENCE: But where would that-- can you give an example of a calculation where I would see exactly what that is?

IAIN STEWART: Yeah, so if you did this loop calculation we did a minute ago, we got a p to the fourth. And then D would be 4.

AUDIENCE: And those p's came from where?

IAIN STEWART: Oh, they came from the momentum going-- so if you looked at this diagram, there's p coming in here. And then it goes into the loop, right? And this is derivatively coupled, so you get p's in the numerator.

AUDIENCE: But that wasn't taken care of with [INAUDIBLE].

IAIN STEWART: Right. Because NV is just counting lambda chis, which are constants, not the p's. It's just counting the constants, the f's.

AUDIENCE: OK.

IAIN STEWART: Yeah.

AUDIENCE: OK.

IAIN STEWART: Yeah. And then you equate it to this, and then you get the D. Yeah. So you could have tried to set things up by thinking about counting p's instead of counting lambdas. But it's-- yeah, anyway.

All right, so what people often refer to this is they say it's p counting because you're counting momenta. And that includes p or M pi, but sometimes people call it p counting.

And just to do some examples, when we had the lowest order Lagrangian, this guy comes out as 2 powers of p because there's two derivatives. And that, in our formula, is just the fact that D is 2 for that. When we thought about this loop with leading order Lagrangians, which are scaling like p squared, we ended up having D equals 4.

And the way that that comes out of the formula is because of this loop term, which is 1. And then there's 2 plus 2 is 4. And this guy is 4 because of the explicit suppression.

OK. So the theory is organized as an expansion of this p. All right, so I want to come back to SU(3) partly because the problem that I gave you is in SU(3). So we'll do some discussion of the SU(3) case.

I'll go into a few things in a little more detail than we did for SU(2), where we were focusing on more formal things. So SU(3) would have gamma matrices instead of the Pauli matrices. There's two bases that you can use.

You can either use this basis, which is like enumerated 1 to 8, or you could use the charged basis. And if you use the charge basis-- and often you write it out as a matrix like this. On your problem set, you're free to pick which basis you want to use.

It may be that one or the other is easier, but I can't even tell you which one is easier since I don't remember. So sometimes one or the other is easier to use. And you have a freedom of what basis to pick.

If you expand, in this case, the trace of sigma Mq dagger plus Mq sigma dagger, which is that term that had a V0, then that gives mass to the mesons, as it did for the pions. Because the symmetry group is bigger, you get more predictions. Here, you get predictions for the kaon masses.

And you get things like the fact that the neutral kaons have massless order Md plus Ms. And you get things like eta pi 0 mixing, where there's a mixing matrix. So if you look at the masses of eta and pi 0, they're actually non-diagonal.

So for the eta in the pi 0 system, you actually get a matrix. So M squared is a matrix. So for example, for the pi 0, it was just M up plus Md. But then once you're in SU(3), there's actually a mixing term. And there's like an M up minus M down term here that is mixing between, and then same thing over here.

So the etas and the pi 0s actually are mixing. And there's something in this n tree. And the mixing is isospin violating in the sense that it's M up minus M down. So it's a small effect, but this is something that you can predict from chiral Lagrangian, something about because is describes isospin violating effects from the quark masses.

So the reason that I mention that is because, often when you do calculations, keeping track of Mu's, Md's, and Ms's all as separate independent parameters is a little much. And so you want to make an approximation. And so if you make an approximation that ignores isospin violation-- so we often ignore isospin violation. Isospin violation is very small.

And if you remember, for the SU(3) case, you're expanding in Ms over lambda QCD or Mk over lambda chi. So it's not a great expansion. You have something like a third.

So ignoring isospin is perfectly valid if you're expanding in a third. So basically, because there is a hierarchy between the strange quark mass and the down and the up, you want to focus on places where you get the largest corrections. And one way of making an approximation that allows you to do that is to ignore isospin and take Mu equal to Md.

So if we take Mu at Md to be some M hat-- which if you want an exact definition, you could say it's the average. And you'd drop the difference. And then you can think of the strange quark mass as being somewhat bigger than M hat.

That's an approximation that you can use on your problem set, for example. So let me write out here with all the terms in the chiral Lagrangian is. And I'll do it for a case where we include in our chiral Lagrangian one other type of coupling, which is this left-handed current that we talked about last time.

So we talked about a spurion analysis for the chi term. And I said you could do something similar to a couple in a left-handed current. And we had this, where D mu sigma was partial mu sigma times the left-handed current sigma. So we thought of putting it together into a coherent derivative. And if I do that, it modifies the leading order Lagrangian and just makes these partials into covariant derivatives.

Power counting, for counting purposes, you count sigmas of order 1. You count D mu sigma as order p, which means you count L mu, the source, as order p. This is a left-handed source. And you count chis and Mq's as of order p squared. That's just repeating what we've already said.

And these higher order terms, which where L's, we can then enumerate. And I now just use D's, covariant D's when I write them down. I'm now in SU(3).

And it turns out that one of the relations that we used in SU(2) doesn't carry over to SU(3). So if I just think of guys with four covariant derivatives, it turns out that there's one more operator there. So there's those two, which we talked about, L1 and L2. And then there's also L3.

So we can't get rid of this guy. I'm always writing sigmas next to sigma daggers because of the chiral transformation. I'm also imposing parity, though I'm not going to spend much time talking about that.

And really I want to enumerate, also, for you some of the terms that involve the quark mass, the chi guy. So you could have a guy that's a cross-term. This would take care of the renormalization of things like p squared and pi squared times 1 of epsilon. That's L4, L5.

We could also have the quark mass type term just squared. So take the trace and square it. That's L6.

It turns out that we could also build something with the right parity by just having the difference instead of the sum. And that's a different operator. So that's L7.

We're going to go up to 9. Don't be afraid. You can have something with 2 sigma daggers and 2 chis, which is another way of building a chiral invariant.

And then for parity, you need the other way. And then finally-- something called L9, which involves a trace that involves L mu nu, where L mu nu is built out of this external current. So we can take two of our covariant derivatives and take a commutator.

That's giving the final operator. So there's a complete basis for SU(3) for a left-handed current in the chi.

AUDIENCE: [? So does ?] [? L mu ?] [? nu ?] [? have a field ?] [INAUDIBLE] [? associated with ?] [INAUDIBLE]?

IAIN STEWART: Yeah, I'll write it down.

AUDIENCE: Oh.

IAIN STEWART: So if L mu is something that has SU(3) indices, has an SU(3) matrix hiding inside it, then there's a commutator term as well. And then it's just this combination. I also use the equation of motion.

And I didn't talk about it, but I use the equation of motion. And I used SU(3) relations. Much as I talked about for SU(2), there's some SU(3) relations that survive. And I got rid of some operators doing that.

Now, you could say, well, I have this SU(3) and SU(2), so why don't I try to relate them? One of them has a kaon. The other one doesn't.

The one without the kaon thinks about the kaon as a heavy particle. The one with the kaon thinks about the kaon as a light particle. Those are two theories. I could try to make them match up with each other. And that's something that you can do, actually.

So there's a little bit of a lesson there. So that was why I want to mention it. So SU(2) and SU(3) seem like they're describing similar physics.

They both could describe pions. They both have pions in them. But the SU(3) has more. It's got the kaon.

That means that, in the SU(2) theory, the kaon is in the coefficients, OK? So if you do a correspondence, you get relations that are like this. So the 2 here means SU(2). And where I don't put any subscript, it's SU(3).

And something that-- this is a 96. And if you really do that, what I said, compare observables, you get relations like this one where you actually see the kaon is showing up on this side and is being encoded in coefficients in the SU(2) theory.

So what you think of as the coefficients in your chiral theory depends on the matter that you've put in, including things like what particles like the kaon, what group you're talking about. This is an explicit example of the kaon being the coefficients if we use SU(2).

OK, so just like in SU(2), we have to go through a renormalization of the Li's. And you can think of that by writing a formula like this one. Bare Li is equal to some renormalized one, which I put a bar on top of and then a counter-term. And the counter-term will be some coefficient, some number of 4 pis, and then some epsilons.

And just like in gauge theory, we get rid of the Euler gamma and the log 4 pi. We have an Ms bar type definition. And in chiral perturbation theory, people often take an extra 1 along with the ride just because they're allowed to. Because it tends to show up in the kind of loops that you encounter. So sometimes that's included, sometimes it's not.

So for example, as an example of a diagram, you could think about this one. And that does cause mass renormalization for the physical boson. So if you think about our lowest order relation as where we started, let me write that as M0 squared is 4V0 over f squared Mu plus Md.

Sometimes people call this-- well, making up some more notation, yeah, which I'll use in a minute. And so if you actually calculate this loop, then you get a correction to that formula. This would have be M pi squared, but here we get a correction.

If you did it in SU(2), just to keep the formula a little simpler, then it would look like this. So M0 is like the pi n mass, but the pi n mass at lowest order in the chiral expansion. So it's not the physical pi n mass.

So 2L 4 bar in theory two with two flavors with two SU(2) plus an L5 bar minus 4 n L6 bar-- so these various coefficients are coming in with some numbers in front, some combination of them. And then there's some contribution from a chiral loop. And if I take away that 1, it's just a chiral logarithm.

So this is 4 pi f squared log M0 squared over mu squared. There would be an extra term of just plus 1 times this if I hadn't gotten rid of that, the minus 1 times that, OK? So this is from the loop. This is from the explicit dimension operators that were of the higher dimension, some combination of them.

The UV divergence is absorbed in some combination of them. And if you want to figure out exactly how UV divergences go in between this, then you've got to think of renormalizing more than just this diagram. And you can generically think of observables as having this kind of expansion in Mu and Md.

So if you like, you should think of M0 here as really just an up plus M down. This is saying, at lowest order chiral perturbation theory, there's a linear term. But then here there's an M0 to the fourth term, which is quadratic in the quark masses.

And there's also a quadratic term from the loops. So you have an expansion in the quark masses. Think of M0 as the quark masses and M pi as the meson mass.

I just gave you the example of the pion mass, but this is generically true for observables that you might calculate that they have this type of result where you have an expansion.

AUDIENCE: Is there any physical reason for why this mass square of the meson space [? would ?] [? be ?] [INAUDIBLE]?

IAIN STEWART: So one way of saying it is-- yeah. So you might think, well, why is it not M pi equals Mu and Md? And the glib way of saying it is, well, if you think about M pi squared and you think about it having an expansion, then it could have a constant plus linear term plus quadratic term. You don't have the constant because of the chiral symmetry. So the linear term is the first thing that's allowed.

That's one way of saying it. I mean, if you think about it in the bosonic theory, we have the Lagrangian is quadratic in masses, right, whereas in the fermionic theory, you have linear in masses. And that's also part of what it had to do with, but it's linear combination of symmetry breaking and that. It's allowed, so it happens is kind of the bottom line. But it's allowed by the symmetries.

OK. And so finally, this is the example I'm actually going to get you to do. So I'll have you look on the problem set at the k constants. And they have an analogous result to that one. You'll do the calculation in SU(3). And the kind of result that you should expect to get is something that looks like this.

So I'm using this B0 notation that I introduced over there just so I could write it all on one line. And these mus with the subscript i are just some shorthand for some contributions coming from loops.

And f is the result in L0. So f is the parameter in L0, whereas f pi is the physical decay constant of the pion. So one result which is encoded in this formula is that the Lagrangian parameter is actually equal to the decay constant at lowest order in chiral perturbation theory.

That's something I didn't cover. I cover it when I teach quantum field theory three. You can look at my notes to see that derivation. Those of you that have taken QFT3 from me, you've already seen that.

And basically, what I'm asking for in the problem set is I guide you with several parts, but how to think about these higher order terms. What are the loop graph contributions? How do you get these terms from the higher order Lagrangians? How you put it all together? It's a nice example of this use of chiral perturbation theory.

And again, we see that a physical observable, which is this decay constant, has a chiral expansion. And the thing that chiral perturbation theory is actually doing is allowing you to predict both the form of that expansion, as well as these things here, which are chiral logarithms. So if you ask about predictive power, the polynomial terms in terms of higher order in the quark masses that are polynomial, you end up having unknown coefficients.

But the terms with logarithms have coefficients which are fixed by your lower order Lagrangian. So those are things that you predict with the chiral perturbation theory. When people do lattice calculations, they need to do chiral extrapolations. And then they're using formulas like this one. OK, so any questions about that?

AUDIENCE: Did the M pi [INAUDIBLE]?

IAIN STEWART: Yeah, physical mass. This is the physical mass. This is the quark masses. M0 is just this combination with M up and M down.

AUDIENCE: So the mu dependence cancel between--

IAIN STEWART: And the mu dependence cancels because these guys all depend on mu. Yeah. And same thing here, these guys depend on mu. Here, I have just enough room to make that explicit.

OK, so that actually covers all the goals that we had for chiral perturbation theory, though it's a fun topic. And there's many more things we could discuss, but that's what we needed to discuss. And so we're going to move on.

So the next thing I want to talk about as an example of effective field theory is heavy quark effective theory. So again, what are our goals with this effective theory?

We will see some new features showing up that we haven't seen before. We will find out what it means to take a Lagrangian that has labeled fields. We'll spend a little bit of time on symmetry. Because in this heavy quark effective theory, there's actually something called heavy quark's symmetry, which is not apparent in QCD, but becomes apparent in this theory.

And there's a trick known as using covariate representations in order to encode symmetry predictions. And it's a very powerful thing that's not special to this theory, but you could use it in general. And I want to teach it to you. So that's one thing we'll do. It's kind of like a spurion analysis, but a little bit more powerful.

And finally, anomalous dimensions that are functions is something that we'll show up here, not just numbers. There's something called reparameterization invariance, which you can think of a kind of symmetry that we'll talk about. And finally, if that list is not long enough, we'll add one more thing which I hinted at.

So I said, when we talked about Ms bar, that it wasn't a perfect scheme for doing things, but there were some limitations. And we'll come, at the end of this chapter, to what those limitations are. The limitations come from power-like scale separation.

And that's related to something called renormalons. So we'll learn what a renormalon is and why it has something to do with the failure of Ms bar and how one can get around that failure and how one actually needs to get around that failure in some cases. OK, so that's the list of goals. It's kind of, in some ways, in order of importance, actually.

So when we think about heavy quark effective theory, you shouldn't think about it as just something that you would need to do if you wanted to do heavy work physics because the idea of what we're doing here is much more general than that. The idea of what we're doing here is we're saying, take a heavy particle and think about what happens if I tickle it. And that heavy particle could be a heavy source to some theory.

We'll talk about it in the context of them being heavy quarks, but you should really think of it as any heavy particle tickled by light particles.

And what we really mean is that we want to study the heavy particle. And since we want to study it, we better not remove it from the theory. So we want to tickle it with light degrees of freedom with small momentum transfer, but we don't want to integrate that particle out.

So another way of saying this, if you don't want to think of it as a heavy particle, is that you have some source. And that source can be tickled and could wiggle, but it's sort of mostly just a static source. And then it could wiggle. And that's what this is an effective field theory for.

So we'll talk about it in the context of heavy quark effective theory. And some of the things, like heavy quark symmetry, will be special to that theory. But more generally, the kind of approach we're taking is a more general thing.

So what's heavy quark effective theory? So you have a heavy quark. And you have it sitting in a bound state, which is a meson.

So that means it's surrounded by light degrees of freedom, one heavy quark, lots of light junk. In the quark model, you'd say it's a Q bar q, heavy cork capital Q and a light cork little q, like b bar d, which is the B0. But that's just a quark model approximation for what the degrees of freedom are.

And there's many more pairs of Qq bars and gluons that are really forming this hydronic state. So you'd like to be able to make model independent predictions for that without having to worry about the fact that you're not parameterizing properly that stuff that you can't calculate. And you can do that using effective field theory.

There's two scales in the problem. There's the size of this thing, the inverse size of order lambda QCD. That's the hadronization that scale.

And that's a scale that's much less than the quark mass. So you're really just expanding in lambda QCD divided by Mq. That's what this effective theory will be.

And what you want to describe are fluctuations of the heavy quark due to the light quarks. So you could think that this guy is so heavy he just sits in the middle of the state, and he's static. And then he gets tickled by the light stuff that's flying and whizzing around. And that's a reasonable picture.

So what we really want to do here is an example of top-down effective field theory again. So going back to our like integrating out heavy particles, but now we want to keep the heavy particles in the theory, not remove them. But still, we want to take a low energy limit of them.

So that means that we have to take a low energy limit of QCD, which is id slash minus Mq Q. And you can see that part of the problem here is that the Mq is upstairs, which makes taking the limit not completely obvious. In the case where we were doing the heavy bosons, we sort of saw that it was always in propagators.

And we could just think of, since it's always they're internal, we just expand. Here, we want to keep the Q in the theory. So we have to really think about how we take this limit.

So let's start slowly and consider the propagator for a heavy quark. And we'll come back to the Lagrangian. And we'll consider it with some on-shell momentum, which we'll consider an on-shell momentum to be parameterized in the following way.

So if I have an on-shell momentum, I'll say that p is equal to MqV, so that p squared is equal to Mq squared and this V is 1. So once I pull out the mass dimension Mq, the remaining thing I call V. And that's just some parameter which squares to 1 on-shell.

Now, if I have kicks, which are from light degrees of freedom, then I don't have exactly on-shell. So P mu is MqV, which is like on-shell piece plus something small, k mu. And k mu is of order lambda QCD. So this is like saying the on-shell piece plus the tickle.

And if I want to construct the propagator, then the propagator is encoding the optional off-shellness of the degree of freedom. That's 1 over the off-shellness. So it'll depend on the tickle.

OK, so we could just take QCD propagator, which is this, and just plug in that formula. There's Mq squared, which cancels this Mq squared. Then there's some cross-terms that don't get cancelled.

So there's one that comes from the dot product of the V dot M with the k term. That's this. And then there's the k squared term. So I have these. And then I can expand that.

M is a positive quantity, so it doesn't change the sign of the i0. And if I expand in M, then the leading term looks like that. It's M independent. And then there's some order 1 over M terms, which I could also work out what they are.

OK, so that we could expand even though we don't, a priori, know how to expand the Lagrangian. We could also think about vertices in this theory. This is the propagator. What about vertices?

And again, this is a top-down. So we can think about vertices in QCD. So if this is a heavy particle here, we could think about what would happen. How do we expand those?

And there doesn't look like there's really anything going on here because there's nothing to expand. But because these things are heavy particles here, you realize from the propagator formula here that you're going to have 1 plus V slashes on each side of this guy. So for a propagator on each side, you can make a simplification.

And that's because 1 plus V slash over 2 gamma mu 1 plus V slash over 2, after some Dirac algebra, is just V mu times 1 plus V slash over 2. So the gamma mu becomes a V mu. So the vertice, once you take that into account, is just minus igTA V mu.

Even if we don't think about starting with the QCD Lagrangian, if this is our Feynman rule and that's our propagator, we can write down an effective theory for that that gives those Feynman rules. And that actually is the HQET Lagrangian.

So sorry for going a little bit over, but I wanted to at least get a little bit into this. OK, so by construction this way, we can arrive at this Lagrangian. And next time, I'll come back and I'll show you how you can really properly take this limit of the QCD Lagrangian to get the same thing.

OK, but that is the HQET. The lowest order of the HQET Lagrangian is this. So it's a linear V dot D decoupling. V is a parameter, and so I've put it as a parameter on the field. We'll see more why that was done next time.

And there's a projection relation on the field. If I'm using a four-component field, then I have this projection relation. I could also read it in a two-component notation, and then I wouldn't need that. But for four-component notation, you do need that. We'll talk about that next time, too.