PROFESSOR: All right, so last time we were talking about symmetries of SCET. We talked in detail about gauge symmetry, and we started to talk about reparameterization invariance, and we are going to continue with that today. So reparameterization invariance in SCET is kind of rich, because we have lots of things that are messing up Lorentz invariance. But we started out talking about three different types here. One where we rotate the n, the physical n vector that was the direction of our collinear particles to some other vector that's equally good, which I call n prime in the figure.

So a type one, transformation. Or we could change this auxiliary vector, n bar, by a large amount. That was type two. Or we could change both at n and n bar in a way that dot products remain invariant. That's type three. So all three of these preserved these relations, n squared equals 0, n bar squared equals 0. n dot n bar equals 2. And really, these transformations are just giving us a different basis for our decomposition, which satisfies all the things that we wanted that basis to satisfy. Doesn't change any of our power counting, and so is an equally good description for the effective theory. And therefore, we want to have invariance under these transformations.

So we're restoring Lorentz symmetry in this way. We break it by introducing these vectors, but we're restoring it in a way by having transformations on these factors. But it's not, you're not restoring, if you like, it's still different than Lorentz symmetry, for example, because you're not making huge transformations of the vector n here for any arbitrary transformation. And it is a reparameterization not a Lorentz symmetry in general.

So we'll talk more about how this connects to the HQET one in a minute. But if we have a vector, then obviously the vector, p mu-- I don't write the right hand side-- is invariant to what choice of basis I use. So that means that this p mu should not change. And that means if I change n here, let's say I do a type one, then I'll change n here and I'll change n here. But I must compensate those changes by changing what-- perp also depended on the meaning of n, so there will be a compensating change for the perp So we can figure out from statements like this what the transformation laws are.

So this guy is invariant to the decomposition.

So the transformation of p perp mu compensates for n in type one, or n bar in type two. Type 3 is already invariant because we have an n and an n bar here, an n and an n bar here. So that's type three invariant already. So if we go through that, we can make a table of all the different transformations, and we can derive the transformations. We also have to do the same thing that we did in HQET, because we also-- the other important fact is that we have a projection relation out here.

So we also have this, and you'll remember that when we talked about reparameterization invariance and HQET, the projection relation was part of the discussion, and so that's true too. But let me just quote to you the results, that you could kind of get an impression for where these various terms are coming from. So let's summarize, first, type one. So type one was n goes to n plus delta perp. So if you have something like n dot D, That goes to n dot D plus delta perp dot D, there's a transformation of p perp or D perp. Let me write it as D perp. This is really for any vector which has to compensate for the transformation of the n in the way I described.

There's a transformation like that. The n bar component under type one does not transform. And if you go through the spinner, or you go through the field, the fermion field, it does get-- because the projection relation has a transformation, there's a transformation of this guy. The Wilson line, which we talked about, was only a function of M bar dot A, and so it doesn't get transformed. So that's what type one transformations would look like.

And type two-- Summarize. They start out looking the same, and then there's some differences.

So everything so far looks exactly the same as type one, just with a sort of suitable replacement. But the fermion ends up looking a little more complicated here.

And the Wilson line also transforms because the Wilson line was built out of n bar dot D field, so. Working the first order in that transformation, there's an additional term. Epsilon perp dot D perp. OK, so I didn't want to take you through the derivation of all these equations. I have provided references. There's a reference list at the beginning of the notes, and there's a reference with a paper that talks about how to derive these equations. We won't be spending too much time talking about it. So I won't go through the derivation, but it's very analogous to what you would do in HQET.

So how do we use these results? Let's take them as given. We can use these results to close out our discussion of what we were talking about last time, and show what the leading order Lagrangian is, imposing all the symmetries. And so the first thing that we can note is that if we do a type one transformation of this guy, and you simplify the result, then at the end of the day that boils down to just the following.

Well, it can be simplified to a single term. And if one does the same thing on this guy, then that gives us the same turn, with the opposite sign. So these two operators are connected by reparameterization invariance. The sum of them is reparameterization invariant, and that looks good. So the sum is 0, connected by RPI. And that means, for example, that you couldn't have any non-trivial Wilson coefficient between them or anything like that.

Now, if you go through the same argument with the other operator that we talked about last time, where it was D perp mu, D perp mu. So delta 1. So same thing as above, but just having D perp mu, D perp mu. D n perp mu. You actually find that it gives the same result as this. OK, so if you just have reparameterization of type one, then you can't really-- the sum of this and this would be invariant. Or the sum of this and this is invariant, or the sum of any combination of 1 minus a times that, and a times this would be invariant. But type two does distinguish between this operator and this operator.

If you do a type two transformation, you get terms from this that you can't possibly cancel by any other term that you could write down. And those terms are such that the power counting, there's not a subleading operative that could compensate either. OK, so going through all the type two transformations is kind of messy. So again, I don't want to do that. But I would refer you to some reading for that.

OK, but if you take these two things together, then you do rule out that additional operator. And so then, taking everything together, we have a unique leading order Lagrangian. It's not so easy, actually. It requires, it requires looking at the transformations carefully. Yeah. So it is-- so even though I made it sound like it's simple, but it's not. There's a reason why I don't want to spend the rest of the class on it. OK.

So if we put all the things that we've talked about last time and this time together, what we find is that the Lagrangian that we've been discussing is unique for the CN field. So it's the unique leading order Lagrangian for the CN field. And the things that we've used are power counting in order to say that the terms here should be lambda to the fourth, that's what leading order meant. Gauge invariance, make these covariant derivatives of appropriate types. And now also reparameterization of symmetries.

So as we kind of discussed last time, the reparameterization of symmetries here can do different things. They can connect leading order operators to leading order operators and hence, constrain the form of leading order operators that you write down, and that's what we just did. And in particular, the type two, remember the epsilon perp was order of lambda to the 0, so maybe it's not surprising that that's what it's doing.

The type one transformation, the delta perp took you down a power in lambda, but if you had some subleading operator, then it would also take you down a power in lambda, so that's one of the things that you have to be careful about when you're sort of counting powers of lambda. But there is a sense, also, in which reparameterization symmetry does the same thing as HQET, which is it connects. So it can do two different things. It can constrain the form of operators you write down at lowest order, or it could connect the Wilson coefficients, for example, of leading order operators to subleading operators. And that's what we saw on the problem set in the HQET example it can do that here too.

And so if we want to think about that freedom, then we should note that there's actually more kind of reparameterization than we've talked about just with this example. So far, what we've talked about is changing the basis. But we could also change the amount of the momentum that we stick in the label versus the residual. That was also a choice that we made in doing things.

And we should be careful to think about that freedom as well. And that's also a reparameterization freedom. And this one looks, in some sense, more like what we talked about in HQET. In some sense, what HQET it is doing is a combination of these two things. Because of the decomposition of momentum there is a bit simpler. So we had this split of label and residual momenta, and we thought about the labels as discrete. But still, even if they're discrete there's a freedom in how we make this split.

And so we could, if you like, take our label momentum or our label momentum operator, and transform it by adding some beta, which has a power counting of order lambda squared. And if we make a compensating transformation in the residual momentum, or the residual derivative, then that would be a symmetry of the decomposition. Wouldn't change anything, end up beta being 0. Projects onto these two cases.

So if you wrote that as a change to the field, then you'd be saying that the field picks up a phase. That's the transformation for the derivatives. And then, you could shift the label by beta, that's the transformation for the beta, for the label. OK, so this is a different transformation than the ones we were talking about over there, and it's a transformation that has to do with this freedom.

But basically, what this freedom does is it connects the derivatives to each other. So it connects this combination, it does something very simple, so after you know what it does it's simpler just to say what it does and forget about thinking about it as a transformation. It simply connects the label operator to be label operator plus the derivative operator. OK. But these two have different orders in the power counting. And so what this will do, is it'll connect things that are leading order to things that are subleading order, for example.

So one way of saying that is it that it connects leaving and subleading Wilson coefficients, because you can always think of the coefficient as the thing being fixed. And then it would do it in both something like a leading order to subleading order Lagrangian, or if you also wrote down operators. If you wrote down a series of some external operator, maybe it's a weak interaction, then it could connect as we saw on the problems set. It could connect C1 to C0 like it did in HQET. OK, so this one clearly does that.

So after you see that, then you should think about the following thing. Well, this is reparameterization symmetry. What did reparameterization symmetry do here? It connected label and residuals to each other. It said, well, they were both part of the same thing in the beginning, so there should be some connection later on in the effective theory, and that remains true. It's encoded in this reparameterization symmetry.

But you can also now try to think about, given that we have these derivatives, what about gauging that formula? Gauge symmetry basically was telling us how to turn p mu into a covariant derivative and partial mu into a convenient derivative. So can we think about combining what we know from gauge symmetry together with reparameterization? And we actually can, if we're careful. And where this is leading is, we'll find that there's basically in the end of the day, there's very simple-- I don't know if we'll get there today, but we'll try. There's very simple building blocks that in the end of the day you can build all the SCET operators out of. And we're kind of moving our way in that direction.

OK, so let's try to gauge this. n dot label operator is 0. So for that guy, you just have i n dot partial, there's no label operator piece. And gauging it just takes you to i n dot D, which is the full D that had both an ultra soft and collinear piece to it. And if you look back at how our transformations were defined for U c and U ultra soft, then this guy is basically just transforming in the way that you would want a covariant derivative to transform, and it does that under both types of symmetries, because of the way we set it up like a background field for the n dot a ultra soft.

OK, but that's actually not really related to this story, because there is no split in the n dot p component. So it's really the other components, the and the n bar, where we have to pay more attention to what's going on. So if we look at what the gauge transformations mean for those components, it's the same story but I'm going to write it out anyway.

OK, so this isn't quite how we wrote it before. We wrote it before as a transformation on the A, but if you put the transformation on the A together with the derivative then you can write it like this. For this piece here, for example, remember what this thing is. This thing here is a label operator p bar plus g n bar dot A n. Label operator doesn't hit this guy, there's no label momentum in the ultra soft U, so it just goes through that guy or gives 0, and UU dagger is 1. So this piece doesn't contribute there, and this piece transformed in this way already. These ones are slightly more involved, but what I'm saying is just a summary of what we already learned earlier or talked about earlier.

And then finally, there's also the ultra soft transformation. And the ultra soft didn't transform under the collinear. And of course it did transform under the ultra soft. So this is a summary of what we said earlier. So the simplest idea that you would say, well, is just like, take these guys and replace this by a covariant derivative and that by a covariant derivative, right?

Always useful to think about the simplest thing first. So this would be the simplest thing, and that doesn't work. The reason that that doesn't work is that if you look at these two pieces, they don't transform in the same way. So if I were to build operators out of this guy that are gauge invariant, then when I looked at the collinear transformation, because this guy doesn't transform and this guy does, this whole thing is not transforming. So if I just stuck this guy in, if this guy was invariant, this guy wouldn't be, and vise versa. OK, so this doesn't work.

AUDIENCE: Can you remind me how the ultra soft field transforms under collinear with its label summing conventions?

PROFESSOR: Ultra soft under collinear? So this guy didn't transform.

AUDIENCE: I'm thinking of the top equation, in the middle. i n D goes under--

PROFESSOR: Oh, yeah. So that one's a little, you know, yeah. So, you want to know which one? I can look it up for you.

AUDIENCE: Well the one you were pointing at, so that's how the D and the A n part that transforms, but the A U soft--

PROFESSOR: Right.

AUDIENCE: Is there something, like,

PROFESSOR: Me remind myself, then I'll remind you.

So the A ultra slot never transformed under the collinear, right?

AUDIENCE: Yeah. I guess I don't know how--

PROFESSOR: And then, so then, yeah. So right. So how does this work out? The reason that this works out is because there was an n dot A ultra soft field in the gauge transformation of the calendar n dot a field. That's why it works out. Good, because it would take me forever to find it.

All right, so what are we going to do? So we need some kind of compensating object to make these things transform in the same way. And we have such a compensating object, that's our Wilson line. So the Wilson line transformed on one side. And so we can use it to modify those formulas and get a result that, where both terms transform in the same way.

I just stick to the W in where I need it in these two terms. And then I have set up things so that the transformation of the two terms are the same. And you could call this, if you wanted to define some kind of full D, then this is the kind of closest that you can get to defining a sort of full D. OK, so these W's ensure both terms-- it would ensure that under collinear transformation, both terms transform at the U c n, on the outside. OK, so we could just stick anywhere that we have i n bar dot D perp, we could just replace it by this and then expand out. I.e. we could get a subleading term by just using this operator here.

And we know that that is going to be there because of reparameterization symmetry, which was connecting the derivatives. And therefore connects these two. And once we put in gauge cemetery, then it's-- this is the formula. So if you go back to what we talked about earlier, we said earlier, I said A mu, for the full theory, that you could just sort of think of it as splitting between a collinear and an ultra soft field. And then I said plus dot dot dot, OK? And I said, we'll talk about what plus dot dot dot is later. So now is later.

What plus dot dot dot is, is terms that you would generate from, for example, the derivatives here hitting the Wilson line, or expanding out these Wilson lines, or the gauge field. And so those are making it have terms with more fields than just one field. And so, if you thought about the A in here as sort of a full theory A, and the A's in here as the effective theory, you could drive a formula like this one but it would involve more complicated stuff. It's not good to think about it this way, it's better to think about it in terms of the covariant derivatives, because then you're already thinking about gauge invariance, so this formula was useful earlier, but it's actually these formulas here that are the more useful way of thinking about taking full theory derivatives and turning them into effective theory results.

All right, so we can dispense with thinking about the plus dot dot dot there because it's better just to think about it in this form, where you can write a nice closed expression. OK, so let's do one example of a subleading term. So we had an operator that involved covariate derivatives that were collinear. This one. And if we just use the top formula there, then we know-- so this is a term that was in L0. Then we know that there's a term in L1 that looks as follows. And I want to write it out so you sort of see how also the gauge symmetry works out nicely.

OK, so using the fact that I can write this guy as a Wilson line. 1 over p bar times the Wilson line. And the fact that I get some Wilson lines when I look at the subleading term. So some of the Wilson lines, I get W-W dagger is 1. And then the remaining Wilson lines are sitting next to collinear fields in such a way that this guy here is collinear gauge invariant, this guy here is collinear gauge invariant, and then there's an ultra soft transformation that connects up all the pieces, OK? If we didn't have these W's here, we wouldn't have got those W's in the right spots there, and this operator that I wrote down here would not be collinear gauge invariant. So this operator is both collinear and ultra soft gauge invariant.

But most importantly, it's collinear gauge invariant because of what we did there. And there's no Wilson coefficient to this operator. It's got the same coefficient as the leading order term, so there's nothing non-trivial. We could just use it to get subleading stuff. OK. And you could also do similar things for currents. I'm not going to give examples of that, but it's also a powerful tool for connecting coefficients of leading and subleading operators and currents. Not every operator in the subleading Lagrangian is connected. So there are some that could not be connected in some order, but there's actually more connections than there are in HQET because we have more symmetries.

OK, so any questions about that? OK. So so far, all of our discussions have been about one collinear field. When we SCET, we actually in general want to talk about more than one collinear field. So how would we generalize everything we've discussed to more than one?

So we would have, in this case, more than one energetic hadron, more than one energetic jet. So far we've been talking about one energetic hadron, or one energetic jet. What if we have two jets? Then we would need two types of collinear field, one for each of those jets. And basically, what we have to do is take our collinear Lagrangian-- well let me call it L n 0, which is the fermion piece plus the gluon piece. And we sum over n. We have to sum over all n's which are corresponding to individual distinguishable collinear fields.

So the question is, what does it mean, sum over n? What is the distinction between collinear fields? And so here's the words, and then we'll explain what they mean. This sum is over inequivalent n's, though that should be obvious that they have to be equivalent. But what makes them inequivalent is the fact that they are RPI equivalence classes. So that's a funny sentence. Inequivalent RPI equivalence classes.

So, two ends are the same if they belong in an equivalence class that could be connected to-- where they could be connected by reparameterization invariance. So you should think of the n's that I'm summing over here as just members, one of each class. They're kind of just picking out what that class is, just labeling it by one member. And then I sum over an inequivalent set.

So let's just imagine that we have some n's. n1, n2, n3. And we can ask the question, what makes them equivalent or inequivalent?

So let me call them, that the n i collinear modes for any i are distinct. And there's a condition that if I dot two of these ends together, they should not be close together. And in fact that they should be some value that's much bigger than lambda squared. Obviously, if i was equal to j, we would get 0. But for any i not equal to j, we will say the n's are equivalent if the dot product is much bigger than lambda squared.

So let's see why it's lambda squared by doing an example. So that's, if you like, how you can define the equivalence classes. Let's imagine you have some momentum, p2, which is a large piece times n2. And then you dot n1 into it. So n1 dot p2 is Q n1 dot n2. And that would be of order lambda squared if n1 dot n2 were of order lambda squared. Right? But if n1 dot n2 are order lambda squared, and therefore n1 dot p2 is order lambda squared, you would say p2 is an n1 collinear particle, because this is the right power counting for an n1 collinear particle.

So it's both n1 collinear and n2 collinear, and that just means n1 n2 are just two members of the same equivalence class. So if this is true, that the dot products of order lambda squared, then n1 and n2 are within the same equivalence class. Which if you wanted some notation, you could say n2 is in the class defined by n1. So you could really think of this as sum over classes but its-- usually people just write sum over n.

OK, so that is in some sense clear, that you want to be summing over things that are distinct. In this case of back to back jets that we talked about, they're pretty distinct. One's going this way, one's going that way. The n's dotted into each other are 2, so that's certainly much greater the lambda squared. But in general, this is what you have to have in order to make them distinct fields. So then, everything basically that we've talked about kind of goes through again, and I'm not going to dwell on it, but I will just kind of repeat some things.

So collinear gauge transformations, for example. You would have now a new type of scaling that you can have for different fields, and you could have two different types of collinear gauge transformations. One for your n1 collinear field, one for your n2. If n1 and n2 are distinct, then those will have distinct scalings for the corresponding momenta, so they'll be distinct transformations. And fields won't transform under the other guys'-- n1 collinear fields won't transform under the n2 gauge transformation because, again, that would spoil the power counting for the momenta.

So at some level it's very intuitive to figure out how the results are. I'm not going to go through it. But suffice it to say that we would have collinear gauge transformations for each collinear guy. Reparameterization, same story. We have separate invariances for each pair of n's. So n1 and n1 bar, for the n1 sector. We have a reparameterization symmetry. n2 and n2 bar for the n2 sector. Reparameterization symmetry, et cetera. And here is actually where you see that there's something that looks different than Lorentz invariance that's going on. Because the reparameterization are only acting within a sector.

So if you do an n1 type reparameterization, there's no transformation of an n2 type Wilson line, or an n2 type gauge field. So an n1 transformation affects the n1 collinear fields and objects, n2 type, which could affect that. And it's more like a Lorentz transformation that sort of acts within the sector all by itself. But it's not really a Lorentz transformation, it's just reparameterization symmetry. OK? But it's exactly the transformations we wrote down, just you don't transform n2 type fields when you do an n1.

And, just like we had before, if you do matching calculations you get Wilson lines, but now there can be more than one type. So we had this W Wilson line that showed up, and we did matching calculations. And I want to give you here one example which we'll come back and talk about more later on, and which we've already mentioned. So consider our example of e plus e minus producing two jets. So in the full theory, you would just have a vector current from the photon. And if you want to match that onto the two jet operator, you can go through the same type of thing that we did when we were doing the B to S gamma example.

And the difference is here that we get two different types of Wilson lines. So this n1 Wilson line, W n1. My notation here is that the subscript is supposed to indicate to you that it's n1 bar of A n1 that shows up. And then likewise, W n2 is a function of n2 bar dot A n2. So you have to decide whether you're going to call it W sub n2 bar or n2, but anyway this is my notation. So you get Wilson lines that are built out of the n1 bar dot A n1 field, which are order of lambda 0, or the n2 bar dot A n2 fields, which were order lambda 0. So this is lambda 0, and this is lambda 0. By power counting we can certainly get objects like that. And when you go through the process of integrating our off shell particles, just like we did for B to S gamma where we attach gluons and we found that some lines were off shell, so we had to integrate them out. If you do that for this process, you get this operator.

So when we construct the effective theory, we have to integrate out off shell particles and doing so generates this Wilson mine operator. It's a little more complicated in this case because we get these two Wilson lines. And I'll talk a little bit more later on in a different context about what type of diagrams are involved in getting these two different Wilson lines. But the result is, in some sense, more intuitive than a way of getting there.

What's happening is, you're getting this W n1 Wilson line next to the C n1 field, and then this form of combination here is gauge invariant under the n1 collinear gauge transformations. And the same thing here. This guy doesn't transform under the n1 collinear gauge transformations. This guy does, this guy's invariant. This guy's invariant under n2 collinear gauge transformations. They both transform under ultra soft gauge transformation, so they get connected in that way. And again, you have-- if you just think about gauge symmetry and how it should come out, then you would have guessed that it should be this. But you can also derive it this way. Yeah.

AUDIENCE: So for the one collinear receptor, it's been very top-down. Is this, when you start to throw in [INAUDIBLE] would you--

PROFESSOR: So this is the top-down way of thinking, that you just generate it by integrating out. And you can do that. Do It to all orders of the tree level diagrams, that's possible. Or you can-- but we're starting to see a picture emerge from the bottom up, right?

AUDIENCE: Right. I'm talking about the top order. So that's pretty bottom-up only, is there a way of, because you're saying let's now state that the effective theory has many copies of the collinear gauge.

PROFESSOR: So. Right. So I mean, you could try to think of writing a formula, right? You could start, try to think of it like, let's take A, and let's write A. Let's just do two. Right. You could start trying to do that. But at some point, it just loses its friendliness. It's not really buying you anything. So, starting to think from the bottom up is actually a good way of going at this point. You could still do it from the top down, but it just gets more and more cumbersome.

AUDIENCE: OK.

PROFESSOR: Any other questions? All right.

So let's come back and study are our leading order Lagrangian. And actually, we've already learned something about factorization although we don't know it yet. So what is this word, factorization? One way of thinking about what factorization is, is it's how different degrees of freedom talk to each other. And given that we have a leading order Lagrangian for the collinear and ultra soft fields, we should know something about how collinear and ultra soft fields talk to each other.

So let's come back and study Lcc 0. So the propagator, if we read out what the propagator is, if we're careful about signs of i epsilons-- We had both particles and antiparticles. You can think of the antiparticles as the guys that have the minus i0. And if you combine these two things together, then that's just giving you the thing that we got when we expanded QCD. So when you think about deriving it from the [? SCET ?] Lagrangian, you think about getting this piece. But then you have to also sum over the other pieces with the other side of the n bar dot p, and you get this piece and they come back to that thing.

So this is the particles. If I want to split out the particles and antiparticles I can do this. This is the particles that have n bar dot p greater than 0, and this is the antiparticles. In this notation where n bar dot p falls, the fermion line flow, we have n bar dot p less than 0. OK, so that, in some sense, we already alluded to, that the propagator works out correctly. This is showing it explicitly. What about interactions?

Well, I want to be interested in a minute about ultra soft interactions because they're kind of special. For the ultra soft gluons, only n dot A ultra soft showed up at leading order. In the LCC 0. So if we look at the Feynman rule there, we have this-- let's give this guy an index mu. And this guy is ultra soft. This guy is collinear. Then the Feynman rule just has n mu in it, not gamma mu. OK?

So that's an observation that we know from our LCC 0 and that just comes from-- remember this just comes from the i n dot D term because that's the only term that had the ultra soft gauge field in it. But there's another fact that our Lagrangian told us. And that is that only the n dot k ultra soft momentum, so if I call this momentum k, only the component n dot k was also showing up in the propagators.

So there's a gauge field statement as well as a statement about the ultra soft momenta. And that was due to the multi-pole expansion that we did. So if we do that at the level of thinking about some kind of diagram-- so let's think about some type of diagram like this, and let's look at this propagator here. That propagator-- we brought this up before but let me write it out again. So if I say p is this guy, and k is this guy, then this guy is p plus k, but k being ultra soft, these guys are supposed to be ultra soft, let's say. k being ultra soft, only the n dot k shows up in that propagator.

And if we work on shell, for p squared of 0, so let me first rewrite this so you can see where I'm going. I can form a full p squared in the denominator from the terms that depend on p. That's just the full p squared of the particle. And if this guy is an external particle as it is in my figure, then p squared would be 0. OK, so if I go to the on shell case, p squared equals zero, then this just becomes n bar dot p, n bar dot p, n dot k, plus i0, which is looking like just 1 over n dot k. So it's becoming very simple.

For cleaner gluons it would, of course, not be-- it would be no simplification like that possible, but for the ultra soft particles there is. And so what has just happened is that the propagator is becoming eikonal. So our collinear propagator reduces to the eikonal approximation, which is just 1 over n dot k, when appropriate. And when appropriate means when it's interacting with ultra soft fields. So we can kind of summarize that for all the different possible cases in the following way.

So you could have, if you want to be careful about signs, you can think about attaching ultra soft gluons to a fermion coming in or a fermion going out. Or likewise, to a antiparticle coming in or going out. We always take k going up. And if I work with the external particle on shell, then combining together the Feynman rule for the vertex and the rule for the propagator, I'm getting these what are called sometimes eikonal vertices, or propagator vertices.

So these are eikonal. Eikonal in both the interactions and in the vertex. And that's what should happen for having a very soft particle talking to something very energetic, that's called the eikonal approximation and it just falls out of our effective field theory. So this can actually lead us to something deeper, which is called ultra soft collinear factorization.

So let's consider more than one gluon. He'll have one collinear fermion and we'll just touch a bunch of ultra soft gluons to it. Kind of obvious notation. Let's call it m. So we could sum up those diagrams, and if we sum up those diagrams we get something that looks familiar. So sum over m, sum over permutation. Factors of g. Gauge fields. OK. Be careful about ordering.

And I'm working here on shell. So that external collinear particle has p squared equals 0. So if you think about what this is, it's just our Wilson line. It's not the same Wilson line that we were talking about before. This Wilson line is built from ultra soft fields, not collinear fields. And it doesn't even point in the same direction, it points in the n direction, not the n bar direction. But it is a Wilson line.

AUDIENCE: Are these incoming?

PROFESSOR: Yeah, I think so.

AUDIENCE: I think you put a minus sign in the top.

PROFESSOR: So for this guy, there's no minus sign there because, well, I didn't tell you what the-- there could be a minus g or a plus g here, right? Yeah. Yeah. That's minus g, probably. Yeah.

OK. So actually, what this motivates us to do from the effective theory point of view is to consider the following. It motivates us to think about making a field redefinition because what we just did is we iterated the leading order L 0 Lagrangian over and over again to get these vertices as well as the propagators. And we ended up with something that was just a Wilson line. Could we capture that somehow, in a simpler way? And the answer is yes, if we make the following field redefinition.

We take our original CNP field and we pull out a Wilson line, Y. And we can do a similar thing for the gauge field. This is just the adjoint version of the same field redefinition, where Y is a Wilson line, which corresponds to that in momentum space and in position space. Pathway to exponential.

So when I say that the Wilson line's at x, that means I've shifted the whole line by x. And I could just denote that by putting an x here. And then from that point in space time, there's a line going out in the n direction. That's what this formula is saying. So at S equals zero, you just sit at x. This should be a picture. Oh, this is minus infinity so it's not a very good picture.

And so this Wilson line, like any Wilson line, has an equation of motion, which is this. It satisfies Y dagger Y is one. And if you go through with our transformation for the collinear gluon, which I could have also motivated in this way. If I'd gone through the calculation where instead of having a collinear quark here I had a collinear gluon, I would have found exactly the same thing, except all these A's would be in the adjoint representation. I didn't write the t's anyway, so. All right. So either the fundamental representation or the adjoint representation, and that would motivate you to make the same type of field redefinition to try to capture what's going on there. OK.

So let's see what that does to our Lagrangian. So our original Lagrangian-- I'm not going to write out all the terms. Just enough terms to get you the idea. If I make this field redefinition, it goes to this guy with a superscript 0 that I'm writing. And then-- let me write out here. Let me split this derivative into two pieces. So I'm still not writing in the terms there in the dot.

So what I did is I took the Y from this guy and the Y from that guy and I put them inside the square bracket. This n dot D, remember, has two pieces. It's i n dot D is sort of an i n dot D ultra soft, which just has the ultra soft gauge field, but then there's a piece that involves the collinear gauge field. But then the collinear gauge field also transforms, as I wrote up there. So I get this Y, Y dagger for that guy. So for the collinear gauge field, these Y's are cancelling. And this guy here, if you push the derivative through the Y using this formula, then the Y's are also canceling and this is just becoming i n dot partial. So that's becoming i n dot partial, and this is becoming n dot a n of 0.

So what's happening is that the ultra soft gauge field is dropping out. Now there's terms in the plus dot dot dot, those terms involve the D perp slash and the 1 over n bar dot D. And if I had written out those terms, too, then the same thing would happen, actually. All the Y's would cancel out in those terms as well. So what I'm getting here is something that I might call an n dot Dn. It's a covariant derivative that only involves the A n0 field. And there's actually no soft fields at all left in the Lagrangian after I make this field redefinition.

So that's obviously giving us a very much simpler Lagrangian. OK, so it looks like a good thing to do. And that, if you go through it, is actually true also for the gluon action. Gluon action was also built of n dot D type fields, type covariant derivatives, that's where the ultra soft gluon showed up, and basically the same thing happens. This same relation right there means that the ultra soft gluon would decouple from that Lagrangian as well. So making this field redefinition, we actually decoupled in the Lagrangians the ultra soft and the collinear fields.

So even though the effective theory allowed these modes to couple to each other, they coupled in a very simple way. And so what we're saying is that it's convenient because they couple in such a simple way. If they didn't couple in such a simple way, we would stop and we'd just use that Lagrangian, we'd do physics. Since they do a couple in such a simple way, we can do physics in terms of some reparameterized variables, or redefined variables, which are these new variables.

And that's convenient because now, if I think about our original Lagrangian, it goes over to something that looks simpler because it's going over to something that has no-- the L0 has no ultra soft fields. This wouldn't be, of course, true for the L1 or higher order Lagrangians, it actually would still simplify all those Lagrangians a certain way. But for the leading order one, it seems to make a dramatic simplification because we no longer have this coupling.

So you might think, well, the interactions have all disappeared. But they haven't disappeared, because when you make the field redefinition, you can't just make it to the Lagrangian. You also have to make it on operators in their theory. And what this does is it moves some interactions out of the Lagrangian and into currents.

So let's do some examples about it. Since it's for the Lagrangian, it's always the same. That's one of the reasons why this is powerful, because if we do it once and for all, we just did it for the Lagrangian. And then we can see what happens for a bunch of different currents, so I'll do three examples. So one example that we had was this current for B to S gamma, which had a heavy quark field and a one light collinear up quark, and a Wilson line.

I should have said another thing that I'm using, which is important. It's that if you go through what-- let me make sure I got my Y, or my Y dagger's on the right side. If you go through what happens for the collinear Wilson line after the field redefinition, this guy remember, is written in terms of these fields. And this guy here is written in terms of those fields. And it also gets Y's on the outside. That's important when you start considering, for example these dots or-- OK. But I need that here.

So given that that's also true, if I want to look at this guy, I just make the field redefinition. So I got this. I got a Y, Y dagger on both sides of the Wilson line. These guys cancel each other. This is a Wilson line 0. I have C bar 0, Wilson line zero. And I have gamma. And I have Y dagger, hv. So now I have a Wilson line sitting next to the heavy quark and another Wilson line sitting next to the collinear quark.

So all the, if you think in the diagram that we drew, the interactions that were those gluons attaching to the collinear quark, they're now just all represented by this Wilson line here. But it's even more than that, because we also transformed the Wilson line. So even if we had attached ultra gluons to that Wilson line, they all magically simplify into a simple, one simple Wilson line Y dagger. So from a diagrammatic point of view, if we had actually tried to carry out the calculation that would give this formula, it would be kind of horrendous because there's an enormous amount of calculations going on to give this. It just looks complicated in the diagrams, but here it looks very simple. And the fact that the leading order Lagrangian gave us 0 there tells us that it really is the sum of the diagrams.

Let's have a couple more examples. So you could take our example for two jets, and you could ask what would happen in that case. What's the generalization if I have more than one collinear direction? And the generalization is that you just make the same field redefinition, but it's a different component of the ultra soft field that couples to different collinear fields. So you're making a different-- you're making a field redefinition that's appropriate to each of the different collinear sectors.

But other than that it goes through in the same way. And so here, this Y n1 dagger involves n1 dot A ultra soft fields, and this guy here involves n2 dot A ultra soft fields. And they don't cancel, but they do simplify all the interactions, in this case simplify to the simple Y dagger Y combination. OK, so having more than one clear direction is not-- I mean, it means that you have more than one type of Y showing up. But again, it's just the leading order Lagrangian for each of those sectors that tell you what's going to show up.

Type of Y. Let's do one more example. Let's do an example where we have collinear fields that are in the following form. Where both directions are n. And we have an operator like that. So it's the same operator I was writing down here, but in this case, I had n1 n2 for two jets. I haven't really told you about an example that would involve this operator here, but it turns out this operator here is something that shows up, for example, in a part time distribution function and other places. So this operator also does make an appearance in physics.

And if you look at what happens for this operator, all the Y's cancel. So when you go through the transformation, you have Y, Y dagger, but then the Y's exactly cancel. So in that case, it's really true that the ultra soft gluons are dropping out effectively when you add up diagrams that there's no ultra soft gluons left. In these cases, the leftover is these Wilson lines that are showing up in the operator. In this case, there's no left over.

AUDIENCE: Are those both values in the same position?

PROFESSOR: Yeah.

AUDIENCE: Y to the w?

PROFESSOR: Because, yeah. So they would if I just wrote this. In general, what you could have inserted in here is some kind of something that picks out, like, the momentum of one of those w's. So let me just throw that delta function in there so they don't cancel. We'll talk about that in a minute or two. Yeah. But in general, I could cancel them in this particular formula. But there's reasons why, actually, we won't want to cancel them, because we will be putting other things in between that won't change what I just said. So imagine that you had something that measured the momentum of one of these products of fields. It would still be true that these Y's, which are not having labels, would cancel in the way that we said, but the w's wouldn't cancel.

OK, so this is called the BPS field redefinition, and S is me. And this thing sums up an infinite class of diagrams. What it does in example one, if you want to think about what the diagrams would look like, let me draw kind of an example for you. So let's have some collinear particles. And just to make it look nontrivial, let me draw something it seems kind of nontrivial. So there's a bunch of collinear particles, and then we could add ultra soft gluons to them. And the ultra soft gluons can couple to all those collinear particles, the gluons, the quarks, everybody.

If we consider all the ultra soft gluons coupling to the collinear particles everywhere, and we add up all those attachments, then it just becomes a single Wilson line. So this thing becomes a single Wilson line. And then times exactly the same collinear structure. So what I'm saying doesn't rely, since we did it at the level of Lagrangian, it doesn't rely on whether or not they're loops, or tree level, or anything like that. So this diagram there, if we add up all possible attachments, will be equal to that one. Where this is a Y. In this case, it's a Y dagger. OK, so that's the simplicity encoded in this formula right here. This is an example of J1.

In example three, then, it's a little simpler, even. Because the ultra soft gluons are decoupling at lowest order from any graph that you might consider. So you go through the same exercise, but now that the Y's are all canceling out. And this has a name that sometimes people use, called color transparency. So one place that it shows up is the following. Let's take our current J3 and imagine that we produce from that current an energetic pion. That was one of the examples we talked about when we were talking about BDD pi.

So there's a collinear pion, and these are collinear quarks. collinear quark and anti quark. Supposed to be inside the pion, and there's collinear gluons. If we attach all the ultra soft gluons to this object, then the Wilson lines cancel. The ultra soft gluons are decoupling from energetic particles. And the reason that they're decoupling is because the partons here are in a color singlet state, which is this pion.

OK, so here we're producing a color neutral pion out of color contracted collinear quark fields, and there's these Wilson lines for reasons we'll discuss in a minute. How they kind of play a role in physics here. But the reason that Wilson lines are canceling is because the collinear things were already contracted. All the collinear things in n direction were contracted in a color singlet, global color singlet.

So the words that go along with this phrase color transparency is that you have these very soft gluons, and they're trying to come in and see this thing. But they can't really, all they can see is sort of the overall color charge of the whole thing. They couple, of the multi-pole expansion, they're only coupling to a single component, and you can think of that as if they're only seeing an overall color charge. And therefore, since it's overall color charge is neutral, you don't see, they just cancel out. OK.

In my notes, I also have a page talking about how you could think of gauge transformations after making this field redefinition, but it's kind of an aside so I'm not going to talk about it in lecture. But I will post it. OK, so what about these? What about this additional kind of thing that I was alluding to here? How does that come in to our story? So, so far in our story, we haven't really talked about Wilson coefficients except to say that they could be constrained by reparameterization invariance to be absent.

So let's think about Wilson coefficients now.

Obviously that's something important. And the way that Wilson coefficients can come in is the following way. They can depend on the large momenta that we're at order 1. And one way of denoting that is by saying that they depend on label operators. OK, so nothing stops that.

But if we want the momentum that's picked up by this label operator to be gauge invariant, then we should act on products of fields that are collinear gauge invariant. So the way that we should set it up is to have an operator. Here's a kind of notation that's sometimes used. Where the operator acts on both fields, the C bar and the W.

Because of our formulas for the label operator, we could also write this as a label operator acting to the left if we wanted. So, what this Wilson coefficient is the function. It's not just simply a number. And it picks out the momentum of this product of fields. And that's because that product is collinear gauge invariant. So it's a well-defined thing to talk about. OK, so that's actually the general structure, that whenever we have these products of fields that are collinear gauge invariant, if we ask what the Wilson coefficient could be a function of, it can be a function of the momentum of those products.

So one way of writing this in a sort of more elegant fashion as follows. So take this guy, and write it as a delta function in the following way. So if I do the integral over this omega, then I would just get back that I stick the p bar dagger inside the Wilson coefficient, and that it acts on this product of fields.

But if I write it this way, what I get is that my Wilson coefficients are just functions of a number, not functions of an operator. And my operators have these delta functions in them. But then could depend on some variables that are distinguishing those delta functions. OK.

So in general, products of fields like this, we have to think about their momenta as being something that we could label. If you like, we could label it by omega. Because it's a linear gauge invariant concept, the Wilson coefficients can depend on those momenta, and then we have Wilson coefficients that depend on those momenta and operators that are just labeled by those momenta. And this is the convolution formula that I sort of promised you at the beginning of the discussion of SCET that was going to show up, and now it's shown up.

OK, so Wilson coefficients can depend on those large order 1 momenta. And traditionally they're written as integrals, even though you could think of them as sums at this point and it wouldn't make much difference. So this here is what's called hard collinear factorization. In the traditional QCD literature. Because it's telling you how hard degrees of freedom, which are encoded in our Wilson coefficients, can talk to collinear degrees of freedom, which are encoded in our operators. In SCET, that's just come out of the formalism in a very sort of simple way. In QCD, you'd have to use word identities and work hard to get what I just derived for you in a couple of lines.

So we're kind of done for today, and we've seen kind of two examples of mode factorization in the effective theory. collinear and ultra soft fields and collinear and hard degrees of freedom, and how it simplifies the discussion of factorization which is kind of traditional, in a more traditional language which I haven't taught you, but you can believe me is more complicated than what we've discussed.

So we'll talk a little bit more about this next time. And then we'll talk about how the ideas that we have here lead us just to define a set of objects that we build operators from. And once we know what those objects are, then we can kind of dispense with a lot of the steps that we've done and just jump right to building operators out of those objects. But the steps are necessary to understand why it's those objects that we want to build operators from. OK. So any questions? So we'll talk a little bit about how this generalizes to other operators next time. So that's just the one time I did here.