MITOCW | 4. Matching and Decoupling

IAIN STEWART: So last time, we were talking about regularization and power counting. And in particular, we were-- we argued that there some things that are nice about dimensional regularization when you're doing dimensional power counting, which is what we've been discussing so far, power counting and ratios of mass scales. So I want to continue along that theme today, and in particular move towards really talking about matching calculations in the context of mass of particles.

So just continuing with this discussion of dimensional regularization, we also have to pick a scheme. And the scheme that is a nice scheme for dimensional regularization is this MS bar scheme. There are some things that are good about it. Well, it's good because it works under the context of dimensional regularization and preserves-- and it doesn't mess up any of the nice things about that regulator. So it preserves symmetries-- gauge, symmetry, Lorentz symmetry, the things we mentioned last time.

In terms of doing calculations, it makes them technically easy, or easier. So a lot of-- if you look at the literature and you look at multi-loop calculations, they're all done on the MS bar scheme. And there's a reason for that, because you've only-- you haven't introduced these extra scales that would make your loop calculations more complicated.

And finally, in the context of our discussion of effective field theory, it's nice because it often gives what I would call manifest power counting, where we can power count both the regulator diagrams, the renormalized diagrams. We don't have to worry about whether we've added the counter terms or not. We can just do power counting.

So if there's some good, there should be some bad. So what's the bad? Nothing is free in life. So one thing that we kind of lose with dimensional regularization is the physical picture. When we had this cutoff that we introduced, or if we have a Wilsonian picture, it's very clear what you're doing as far as removing degrees of freedom.

And in MS bar, that's a little less clear. So if it's just less clear but it's still fine, that would be-- that would just be a technical aspect and we would just pretty much ignore it. So what's one-- I'll give you one example of something in which the picture being less clear could actually mislead you.

So if you calculate some renormalized quantities in MS bar, you could have some a-priori knowledge that these renormalized quantities should be positive. Maybe they're supposed to be kinetic energy or some operator, higher-dimension operator, that gives a kind of higher-order kinetic energy term, some kind of physical intuition that it should be positive.

Well, that might not be true in MS bar because MS bar loses this physical picture. You're removing just the 1 over epsilons. You're not really having any control over the constants, or the constants are kind of predefined. If those constants are kind of too negative, then you can end up with renormalized matrix elements in MS bar that could be negative, your physical intuition tells you they should be positive.

So that's something to be aware of. Where a physical picture could guide you and tell you what to expect, but you might lose that by using MS bar. There's another thing that's kind of technical but is worth knowing. And we will talk about it a little bit more later.

You could ask the question, is the Wilsonian picture and the Wilsonian scale separation really equivalent to the scale separation that you do in dimensional regularization? And the answer is, almost but not quite.

It certainly is for all the log terms. And I mentioned last time that the log terms that we saw in the two ways of doing the calculation, you could just see a direct correspondence between them. But there's a leftover residual effect called renormalons that does show up in the MS bar scheme and is related to power divergences.

And so this effect is carefully hidden. It's actually hidden in the asymptotics of the expansion in alpha s. And it goes under the rubric of something that people call renormalons. So there is a leftover physical effect from kind of having the freedom to drop all the power divergent terms. But it carefully hides itself in a strange place in the gauge theory. And we'll talk more about this later.

It's not so important if you're working to order alpha s. But if you start working toward alpha squared or higher, then-- or if you start going to higher orders in perturbation theory, then this could come in and be important. And it does come in in QCD at fairly low orders. At order alpha squared, it can cause numerical effects if you ignore this. And there are ways of taking it into account without losing the nice things here. So we'll talk about that later on.

And then there's a final thing-- we had three good points, we should have three bad points-- the final thing we'll deal with right now. And that is the fact that MS bar does not satisfy something that's a theorem called the decoupling theorem. So what is the decoupling theorem?

So this goes back to Appelquist and Carrazone. And it says the following. So you're thinking about an effective field theory, you're thinking about deriving a low-energy effect of field theory by integrating out some mass of particle.

And the decoupling theorem says that if the remaining low-energy theory is renormalizable, and we use a physical renormalization scheme, then you kind of get what expect, that all the effects of the heavy particles turn up in couplings or effects, they're suppressed.

So this seems very physical from what we've described about what effective field theories do. So in that sense, it's-- I'm not going to try to prove it to you or anything, but there is this caveat that you need to use a physical renormalization scheme, of which MS bar is not one. So we decided that we liked MS bar, but then we found some problems. And this problem here, this last problem, we're going to deal with right away.

So MS bar is not physical. And hence, it doesn't satisfy this theorem. It's mass-independent. And that is something that we like, but it also is causing exactly this problem. That's how it-- it's why it violates the decoupling theorem. Because it's mass-independent, it doesn't know enough about the mass to know that the effect of massive particles should always decouple.

So in MS bar, what you have to do is you have to implement that decoupling by hand. And we've gotten actually so used to this logic that it's often even not mentioned that this is something that we're doing. So go over that a bit.

So I'll just go over this in the context of a particular example, which is the most common example, which is QCD with massive particles in MS bar. And I'll show you how it works. So QCD has a beta function of the renormalized coupling mu d by d mu of g of mu. And just to establish some notation, I'm going to remind you of some facts that you've seen before.

So this is the beta function at lowest order. I'll call this factor that depends on the group that you're dealing with and how many light fermions you have, I'll call it b0. And this is less than 0. And this is only the first-order term in a series, but it'll be enough for our discussion right now. So this is g to the fifth terms, et cetera. That's the beta function for g.

We can also think about using-- since g likes to come squared, we can switch to alpha. QCD is asymptotically free. When we solve this equation for alpha, it looks as follows.

And the coupling constant decreases as you go to higher energy. So this is alpha at md, and this is alpha at mW. Alpha at mW is less than alpha at md. This is the lowest-order solution.

So just a little bit more in the way of setting the stage, you can also associate to this solution here an intrinsic mass scale dimensional transmutation. If you form the combination mu exponential of minus 2 pi over beta 0-- b0 alpha of mu, and you use this equation over here, then what you'll find is that this combination is also equal to mu 0, same thing alpha of mu 0. So it's independent of what choice of scale you use, mu or mu 0. Therefore, you define it to be a constant, and that constant is called lambda QCD.

Once you do that, you can also just take this equation and write it like this. Write that, rewrite that solution like this. So it's another way of specifying a boundary condition, if you like, for the differential equation. One way is to pick a value of the coupling somewhere, another way is to fix this constant.

So this guy is independent of mu, as I said. And it's the scale where QCD becomes nonperturbative. And the reason I have to mention this is because we're going to be talking about anomalous dimensions, and I have to tell you when the things that I write down are valid. And it's all going to be valid as long as we're not near this scale. Because if we're near this scale, the coupling gets too large for us to do the perturbation theory that we're doing when we calculate anomalous dimensions.

So you can ask the question, what does this thing depend on? And if we look at the right-hand side here, we can already see some things it depends on, because it depends on b0. And if we looked back at what b0 is, b0 depended on the fact that we were at su3 if it's QCD. It also depended on the number of light flavors.

So this scale that you fix here, well, it depends on the order of the loop expansion. We wrote down a formula that was valid at first order, but we could extend that formula at higher orders as well. And the formula would change, and the value would change if we went to higher orders.

Depends on the number of light flavors. And it actually also starts to depend on the scheme but only at two loops. So it'll depend on MS bar versus MS, for example, but only beyond two loops.

Now, the issue with this is that a priori, nothing tells us whether if you have a top quark or an up quark or a bottom quark. In MS bar, nothing a prior tells us what to do with these formulas, what to include in the b0. Just seems like we should include everything that exists, all the known light fermions that couple to the gauge field. But there might be some we don't know about. Should we include those, too?

MS bar, with the logic we've presented so far, doesn't tell us what to include there in that nf. So let me phrase it this way. But the top quark and the up quark, which have very different masses, both contribute to b0. And it seems like they do that for any mu.

Even though-- even if I'm at a very low-energy scale and the top is very heavy, MS bar is not smart enough to decouple the top quark. If you were to work in a physical scheme, then this decoupling theorem guarantees, actually, that the top quark would decouple, and it would drop out of the beta function in that scheme. But in this MS bar scheme, it doesn't happen.

So the solution to this is to build back in the thing that happened in the physical schemes into our MS bar scheme. And we do that by implementing the decoupling by hand. And we just say that when we get to a mass threshold, we're going to integrate out the heavy fermion in this case. So at some scale of order of the mass of that fermion, we're going to integrate out the heavy fermion.

And that's an example of during matching, actually. You're moving from one theory to another, because you're changing the field content. You're removing some fields, and that's an example of doing matching.

So what this means is that we'll define different b0's depending on what scale we're at. So nf would be 6 if we're at scales above the top quark. But then we drop that nf down to 5 once we're below the top quark mass, et cetera. So it's a discrete jump in what the b0 is.

So in some sense, what that means is that this kind of matching is forced upon you in order to preserve physics in MS bar. If you were to do some other scheme like a physical scheme, it's not to say that you couldn't take the same logic. You could do matching to integrate our heavy particles in those schemes as well. And in fact, some sometimes people have.

But in MS bar It's really forced upon you. There's no way out, because you otherwise wouldn't get the physics right. So we've got to be careful. If you gain something, sometimes you lose something. And if it's physics that you're losing, you have to build it back in.

OK. So how do we actually do this matching? Well, we use what are called matching conditions. So what is a matching condition? At some scale which I'll call mu m-- that could be equal to m, but it could also be equal to twice m or 1/2 m, some scale that's of order m-- we're going to demand something about the theory above and below. And that is that S matrix elements in the two theories are going to agree.

Of course, we should be careful to make sure that we work within the realm of things that we can calculate in the low-energy theory. So we should consider only S matrix elements with light external particles. If we're getting rid of the heavy ones, we don't want them on the external lines. So the basic idea is that we set up matching conditions which are conditions that say that S matrix elements should agree between theories 1 and 2, what we called theories 1 and 2 last time.

So if we do that for QCD, the example we've been talking about, and we do it at lowest order, then the conditions are actually very easy. So here's the picture. Let's imagine that the top quark is here, bottom quark is here. My picture is not to scale. Charm quark, et cetera.

Then what we do when we want to say that the theory above and below, that that threshold is the same, if we just match QCD above and below, it just gives a continuity condition on alpha. So let me write it down at this scale. So here we would have alpha 6. Alpha depends on the number of flavors. Here we have alpha 5. A different definition of the coupling a different number of active flavors, a different value of b0. 3.

And what these arrows are representing are just kind of renormalization group evolution, which I'll just call running. And then what the lines are representing is doing some matching. So every time we reach a threshold, we do a matching, we switch the field contact, and we get a new effective field theory with a new coupling constant. So the coupling just depends on what theory content we have.

And the matching condition is, say, at this b quark scale that alpha s at 5 at some scale that's of order mb is equal to alpha s at 4. That's the leading order matching condition here. And similarly, the same condition at all scales. And this is mu b, just to be clear, is something that's of order mb, could be equal to mb, could be equal to mb/2, could be twice mb. And sometimes people use these different choices to get uncertainties.

OK. So it seems fairly straightforward, just continuity of the coupling. But that has some caveats. But go ahead and ask the question.

AUDIENCE: Yeah. So how do you know that that's the physical observable that you care about? What if I was trying to measure the [INAUDIBLE] alpha s?

IAIN STEWART: So this is the-- yeah. So if you're going-- so you have to think about it as not the derivative of alpha s, but you have to think about it as an S matrix element-- something two-to-two scattering, right? If you do two-to-two scattering, then it's just going to be proportional to alpha of mu.

But the mu dependence in your leading order prediction is not really fixed. You need to go to higher order to do that. So if you're just ensuring continuity at leading order of S matrix elements, then this is all you need. And if you want to construct something that's like a derivative of alpha, you have to think of the derivatives of the S matrix.

But you can't take derivatives of mu, because that's a higher-order question. So this is all you need at leading order.

AUDIENCE: OK. So maybe my derivative example is [INAUDIBLE] necessarily know the alpha s, you'd get only--

IAIN STEWART: So in general-- right.

AUDIENCE: --but I don't necessarily know that--

IAIN STEWART: Yeah. No, in general, you have to-- this is like one particular example. In general, you have to ensure continuity of all S matrix elements. And that will-- in this case, we'll give you conditions also and what's happening with masses and stuff. So it's all the parameters of the theory. There should be conditions for all of them.

AUDIENCE: I guess these words seem more complicated than just asking for [INAUDIBLE] some different equation. Does it ever boil down to anything-- do you ever have to do anything more complicated than just like--

IAIN STEWART: I'm just-- yeah, so--

AUDIENCE: --change the beta function [INAUDIBLE] continuity and--

IAIN STEWART: It's really-- it's not complicated. It's just continuity of S matrix elements. But you have to just know that it's S matrix elements and not the Lagrangian. Because those are two different things.

So let me give you an example why it's not the Lagrangian. So this kind of thing seems simple, right? Just say, well, the Lagrangian is continuous. The alpha-- the g that appears in the Lagrangian is continuous. But that's not true once you go to higher orders. That's only a leading order statement.

If I write down the analog of this condition at higher orders, it looks as follows. So the coupling is actually not continuous at MC bar at higher orders. So demanding continuity of the S matrix elements does not lead to continuity of the coupling. And the matching condition at some scale looks like this.

So the words are carefully crafted to be correct. And they sound a little more complicated than they need to be, because I want them to be correct, even if I want-- even if I were to do matching at higher orders in perturbation theory. And it's really at this alpha squared level that you start to see things a little more interesting.

OK. So now we've gone two others beyond what I just told you before. So continuity at lowest order-- at the next order, you can retain continuity as long as you pick the particular point where mu b is equal to mb. And you'd have-- then this log would go away, and you'd still have continuity.

But even that doesn't work out once you go to one higher order, there's this constant. You could get rid of the logs by picking a scale choice, but there's no scale choice which will make it continuous once you get to alpha squared in this matching condition. So this is the condition that's necessary to ensure continuity of S matrix elements once you get to that level of perturbation theory.

OK, so that is why I said S matrix and not just continuity of couplings.

AUDIENCE: So this condition is for 2-by-2 scattering, And then you get other conditions--

IAIN STEWART: This is the only condition you need for all the S matrix-- you can use different S matrix elements, and they'll lead to the same condition.

AUDIENCE: How did you-- where does it come from?

IAIN STEWART: So this, it comes from ensuring that I calculate S matrix elements, say, 2-to-2 scattering, in the theory with five flavors and four flavors.

AUDIENCE: At two loops?

IAIN STEWART: And I demand that-- yeah, up to two loops. So where would this guy here come from? This guy here would come from the graph with an explicit b quark, which is in the theory with the b quark but not in the theory without. Once you get to this level, then there's more complicated diagrams. Just think of generalizations of this. And they involve constants as well as logs. This guy just involves a log.

And really, what you're ensuring is that in the theory with the b quark, which is this five-flavor theory, you get the same S matrix elements as in the theory with four flavors. Since the diagrams differ in those two theories, they're kind of the same at lowest order because you're just doing two graphs. But once you have the b quark and it can go around in the loop, then they differ, and the conditions become more complicated.

So any other questions about that? OK. So one other thing that we see from this is related to these logarithms. We also see from these conditions here that there is going to be no large logarithms as long as we pick the scale where we do this matching to be of order mb.

So you don't want to pick mu to be the gut scale or something you want to pick it to be some scale so mu is equal to mu b, which is of order mb. Because you don't want, for example, to make this log so large that it starts to overcome the coupling. You want it to be-- you want this to really be a perturbative thing so it makes sense to think about this, and then this is a correction, and that's another correction.

OK. So the general procedure is the same idea. So if I just kind of adopt a more general procedure for massive particles [INAUDIBLE] a more general notation. So this is with any operators and couplings. It doesn't have to be gauge theory.

And if we have a hierarchy of particles-- let's say we have n of them-- when we want to pass from an L1 to an L2, an L3, to an Ln-- and we're doing the same type of thing we just did over there. So the steps are the following.

Consider S matrix elements in theory 1. Do so at a scale which I'll call mu 1 that's of order m1, and match that onto in the same way of assuring the continuity and do-- using these matching conditions, match that onto L2.

So this is the technical step by which I said that you could do-- you could in a top-down approach take theory 1 and determine the parameters of theory 2. The matching conditions are determining the parameters. Alpha s 4 is of parameter of theory 2 to alpha s 5. This is theory 1. And we're just determining what this alpha s 4 should be, and this is the condition that relates them.

So after you do that step, if you want to go through this picture here, or the analog of that picture for this case over here, then you need to compute the beta functions in anomalous dimensions in this theory. So it's the theory that doesn't have particle 1. And then you evolve/run the couplings down, which just means using the evolution equation for the couplings, whatever they may be.

So we had the evolution equation for alpha S a minute ago on the board. And I would just use that to go from a scale of order mb down to a scale of order mc. And then I repeat. And this is the general kind of paradigm of matching and running that you hear about in effective theory all the time.

So we just keep going. And at the end of the day, we're going to stop somewhere. And the place we're going to stop is the place we want to do lower-energy physics. So let's say we stop at the n-th level, since that's the last level I wrote. So if you're interested in dynamics at that scale, that's where you stop.

And that's where you compute your final matrix elements. So everything up until that stage is just to determine the theory Ln and what are the values of decoupling in that theory-- determined from knowing information at the high scale. Knowing information in theory 1, how do I propagate that knowledge all the way down to low energies consistently without losing information I need? And then once I'm at the lowest scale, I just do my computations of observables.

Yeah.

AUDIENCE: Professor.

IAIN STEWART: Sure.

AUDIENCE: [INAUDIBLE] mention at the scale where the particle is there-- actually, I was a bit confused because if we write the full theory, then, say, 2-to-2 scattering in that mass scale, isn't there any like [INAUDIBLE] say resonance effect, and then you want to match that to a theory without the particle?

I thought we should match the s--

IAIN STEWART: So there is no resonance effect. And the reason is because the momentum on the external lines that you're taking-- you're thinking of it as small, right? The scale is chosen to be the mass of the particle, the cutoff, mu, the soft cutoff. But not the momentum of the particles. Good question. Any other questions?

OK. So we're not really done talking about subtleties here. And we're not really done talking about some of the key ideas that come into these calculations. I've given you a very simple example just the coupling. That's a little bit too simple.

So I want to do something a little more sophisticated but still fairly simple and widely used. And that is in the standard model just to take the heaviest particles, which are the top, the Higgs, the W, and the Z, and remove them. So we'll spend a bit of time talking about what happens when you do that.

That's not the only thing you could do. And in particular, before people do how heavy the Higgs was, you could think of other possibilities. And before people knew how heavy the top was, people did think of other possibilities. This is actually a reasonable thing to do, though. let me give you one example of another possibility.

Or you could say the top is heavier than the W and the Z, so why don't I do the top first, and then I'll just follow your diagram over there. I erased it, but it'd follow the picture. And then I'll do some running in the theory without the top quark, and then I'll get down to the W and the Z scales, then I'll remove the W and the Z.

OK. That would be a valid thing to do. But it introduces complications that are actually not worth the effort. And benefit is actually not that great. So I want to emphasize that.

When should you think of things as being comparably heavy versus when should you think of things being hierarchically heavy? Well, one complication of removing the top quark is that it breaks su2 cross u1 gauge invariance, because the top quark was in a doublet with the b, and you're trying to keep the b and remove the top, which doesn't sound good. You're trying to keep the gauge particles, the W and the Z, in your theory. So you should still have that gauge symmetry, even if it's spontaneously broken. But you're trying to remove one member of a doublet.

And that leads to [INAUDIBLE] terms that you'd have to clue to the effective theory so you can deal with it. And it's just a little bit annoying. But, you know, it's something you can deal with.

The real crux of the matter is that if you compare mZ over m top, or mW over m top, that's about a half. So if you think about what you're expanding in when you take external particles that have momentum of order mZ, integrate out particles of order mt, you're expanding in a half, which is not such a great expansion parameter.

Usually in effective field theory, you want to expand in something that's at least a third, hopefully a quarter. And a tenth if you really want to have a good expansion. So half is not really that good. So that's the real reason not to do it.

And you could ask, well, why do we lose by not doing that? And that, of course, is the real question. Well, you miss some running, right? Because you don't have a theory that has no top quark but still has a W and a Z, and that theory could have anomalous dimensions, and you'd miss any running in that theory. You're kind of collapsing two of the lines in my picture with the arrow between, you're collapsing them down to just a single line.

So you're missing the running that would go between mt to mW. Well, of course, that's not a very big hierarchy, and it's logs of two. And that's why we don't care that much.

What it boils down to is that you're treating alpha s at mW. And if we want to be generous, we can say it's mW squared over m top squared. So they're logs of four. But you're treating this perturbatively.

If you remove both the top and the W at the same time, then this is going to show up in your matching conditions, and you're not thinking of that is something as large. Thinking of it just as order 1.

So that's the cost, which is not a big cost. Especially when you give, say, that the cost of going the other way would be expanding in the half. Better to take the cost in the logarithms than expansion in the half.

So that's a general kind of rule, that if you're thinking about removing massive particles, you should ask, how close are they to each other? Should I integrate out a slew of them at the same time, or one at a time? That'll depend on exactly the scales in the problem. And there's-- and it means you're organizing the theory differently if you do it, the two different approaches.

All right. So we'll do an example of this. We'll take a very simple example, although not completely trivial. So just b quarks changing to charm quarks u bar and d. So on the standard model, which is where we start, let's say we have a W boson.

We have an up quark that's left-handed. We have a CKM matrix connecting flavors together. And we have down quarks. So this is-- there's a matrix space here in the flavor, which includes bottom charm up and down.

And tree level matching is easy. And some of the complications I want to talk about will come into the loops. So let's first get through the tree level and set up some notation. And then we'll talk about what happens with the loop.

So tree level, this is the diagram in the theory 1. Just calculate it at tree level. W propagator, unitary gauge. And then there's some spinners.

So we have an antiquark. So we have a v spinner for the antiquark. That's an up. Everything else is a u spinner. I've put in explicitly the P lefts to denote the fact that it's left-handed.

So there's some momentum in this diagram, which is momentum transfer, which is the be momentum minus the c momentum and the kind of obvious notation. And of course, by momentum conservation, that's also the d momentum plus the u momentum. And that's the momentum that's going through the propagator.

We're going to count momenta as being of order masses. And the heaviest mass in this case is the b quark mass. And we'll take it to be the b quark mass. We can use the equations of motion. So Pb slash on ub, spinner for the b quark is mb ub, et cetera. And we can simplify the diagram by doing that.

And of course, the key thing is that we can expand the propagator since the momentum is smaller than the mass of the W for integrating out the W. So the leading term when we do that is this term. And any other terms that we drop are down by that much.

So for example, this term here, we just said that the k's are of order mb so it's down. And then we could drop the k squared here, I'll get the mW squared, so we just have that.

And then we put that together, we can calculate the Feynman rule in the effective theory. The Feynman rule in the effective theory, we can determine the coefficient of it. So let's start out with just saying it's some four-quark operator that couples together those flavors, conveniently chosen so that they're all different to avoid [INAUDIBLE] factors.

And in some conventional normalization for this Wilson coefficient, we call it GF, which is for Fermi. And the Feynman rule would again give spinners, the same spinners as before, with the same Lagrangians for these light quarks. Haven't changed anything about that. We're just removing the heavy particle. And it's conventional also to pull out the CKM factor [INAUDIBLE].

OK. So if you like, you can say that there's a coefficient here, and that coefficient has been fixed to be 1. And what this G Fermi is, then, would just be the leftover factors of the gauge coupling and the mass of the W. And that's the usual convention.

So G Fermi is just fixed in this case, not as the thing that is going to get corrected at higher orders in QCD but would get corrected at higher orders in electroweak, if you like. And then you sort of put in with the QCD corrections into some other coefficient that we can call C. And we'll just say that it's 1 at the level of this. And this line agrees with the expansion of that line, and that's the matching of something that's measurable, which is this four-point function. So we can call it an S matrix element.

So this theory is very popular. It's called the electroweak Hamiltonian. It's used all over the place. And it involves more than just doing what I did, because I chose a particular set of flavors for a particular channel, and you have to do it for the whole-- for all sorts of other channels as well.

So we'll study some aspects of this theory. We won't study every possible aspect. But I've given you a handout that studies a lot more. It's 250 pages long. I'm not asking you to even read all of that. I've pointed you at some pages of that in the reading, which are relevant to the discussion we're having here. If you really want to dig deeper, you can.

So people often call this an electroweak Hamiltonian, which is just minus the Lagrangian. So then that's 4 GF over 2. And as a Hamiltonian, we write fields instead of spinners. So we have this four-quark operator. And we would have determined the Hamiltonian from tree level matching, from what we've done. And this would be the result. Standard stuff.

How do we want to go further than that? Well, if we want to think about theory 2, which is this theory, and we want to think about it in more detail, we should worry about whether we're-- with just this operator, whether we've got a complete set of structures that could possibly occur. So we should think about the symmetries.

And it's useful to, therefore, construct the most general basis of operators in theory 2. And we'll just do that. So if you read this 250-page review, then it's done for the full electroweak Hamiltonian, and we'll just stick with the flavors we're talking about here.

So what are the most general bases of operators? How should we think about that? Well, we're constructing some theory that's going to-- that we're matching onto it mu equals mW. And at that scale, when we're determining the theory and determining the coefficients of the theory, we can treat the bottom the charm the down and the up respectively as if they're massless.

And really, what I mean by that is that the masses of these particles are only going to show up in the operators and not in the coefficients, which I've-- you see that I talked about over here. The thing that shows up in the coefficients are the mass scales we're removing. And we're not removing the mass scales of these things.

So that means, if we can think of them as massless, that we can think about the matching in terms of using something like chirality. And we can use the fact that QCD for massless quarks does not change chirality. And we can use that in constructing our basis.

So that means even though the b quark and the charm quark really have masses, for the purpose of constructing the operator basis, we can think of them as massless. And we only have left-handed guys to worry about here.

Chirality means more than that. It also-- well, it means effectively that. But it means also that we will only get one gamma matrix here. And we can think about why that is. Chirality means that you get an odd number. You need an odd number to have left on both sides.

But you can reduce 3 to 1. So there's an identity which I won't write out for you, but you can reduce 3 back down to 1. So any higher odd number can be reduced back down to 1.

So once we use chirality and that fact, then we have a fairly restrictive basis in terms of Dirac structures. You could ask, what are the most general possible ways of contracting spinner indices?

Who said I had to put this charm quirk with the bottom quark? I could have put the up quark over here. And maybe in higher orders that happens. Well, at higher orders, you could think about that happening.

But you can always get back to the form that I wrote over there using what's called the spin Fierz. So if I write it for fields, it means that there's an identity that I can rearrange this guy the other way. And so these are equivalent operators.

And sometimes, you have to use these relations when you're doing matching calculations, because you construct a complete basis, which is the minimal one. And then maybe when you do your calculation, you get this operator, so you have to turn it back into this one.

There's two minus signs in this relation. One is from the Fierz identity, and one is because when you do the manipulations you end up anticommuting to fields. So from the statistics. So if you were to use the same relation for spinners, then there would be one less minus sign. Something to be careful, though.

So what this tells you, once you put that information together, is that you know that you can write the operators this way, and you also know that this gamma, capital gamma, has that gamma u p left form in terms of the Dirac structure.

So the only really thing that can happen is color. And you can contract the color of these 3's-- so you have four 3's, you can contract the color of them in different ways. Well, I have two 3 bars and two 3's. Think about having different color contractions. And that will expand our operator basis by one more operator.

There is also a color Fierz. And I can use that color Fierz to get rid of having explicit TA's. So color is a lot like spin in the sense that if I had more TA's, then I can always reduce a higher number of TA's back down to a lower number. So I've had two, I can reduce it down to one or zero.

OK. So I only really have to think about having a TA here and a TA there. If I have a TA there and a TA there, there's a spinner-- or there's a color Fierz relation for this guy that you could use. And there's a handout that I posted on the web that has a summary of all these relations. I'm not going to bother writing them down in lecture here.

So once I take that into account, then it ends up in two operators. And if I were to write them in a kind of renormalized notation where I introduce the scale mu, then I would write them as follows. Let's call them O1 and O2. And the difference between O1 and O2 is just simply color.

So in O1, the color index alpha is contracted between charm bottom and up down and up. And then in O2 it's the the colors contracted the other way. So I don't write explicit TA's, but I do have to consider the other possible way of contracting the indices. Up, down, doesn't matter.

OK. So these operators, these are the two operators they have once I satisfy all the symmetries and I have some coefficients. And if you ask really what the coefficients can depend on here, well, they can depend on mass scales like top and the other particles that I'm integrating out, Z. I'm not going to make that explicit in my notation. If I tried to make every possible mass scale like the Higgs or the top, it would just be too much.

So let me just denote two things that it can depend on. Mu over mW can show up, and alpha of mu can show up, as well as some ratios of other particles which we will suppress.

So then what tree level matching is saying is that you've got one of these operators, and you didn't get the other one. So it says that at tree level, what we did before-- and so say we do matching at mu equals mW and C1 at 1, which is mu over mW equals 1 and alpha mW is 1. And then corrections to that would be suppressed, no large logs, just by alpha. And C2 of 1 alpha mW is 0 plus something of this order. So that's what we determined by doing our tree level matching.

Now, when we did our tree level matching we did something. We used external states which were quarks. We matched up the S matrix elements in that way. If you were really interested in this process b goes to c u bar d, you'd not really be interested in measuring quarks. We don't see them. You'd be interested in thinking about the process for mesons-- B meson changes to a D meson, and a pion, for example.

And who's to say that the matching that we did for quarks is also the matching that's valid for hadrons? Well, it is. So there's a key fact about matching which is important. One of the things I wanted to emphasize to you is that it doesn't depend on what states you pick. It's independent of the choice of states. And it's independent of the IR regulators that you pick. So when you do matching, you often pick some IR regulators to regulate IR divergences.

And the only thing you have to do is pick the same states and same IR regulators in the two theories. And once you do that, your results for your coefficients will be independent of the choice you made. So this is intuitively something I think that's, once you think about it, fairly obvious.

What this sentence is saying is that the matching, which is supposed to be a high-energy property, is independent of the low-energy physics. The choice of the states and the choice of IR regulators, that's parameterizing something about low-energy physics-- lower energy than the scale we're trying to determine. These things should only depend on higher-energy physics. So the outcome for them will be independent of this choice.

So just to emphasize, even when we use hadronic states, the result that we obtain from quark states is valid. That's just different choice of state.

And what we're doing when we do the matching is we're picking a convenient choice of state, something that makes it easy to calculate. If we tried to calculate the matrix elements of B and D and pi mesons, then I'm saying if you're a strong enough, you'd get the same matching coefficient. Of course, you should pick the result that makes the calculation easy, and that would be to use quarks.

Any questions about that? OK. So this is something that you should keep in mind. And you should keep in mind that it is-- you do have to make sure that you're using the same states and IR regulator in the two theories.

In this case, that's pretty easy. The states are really the same states, because you're defining the states with the Lagrangian machine for the b quarks, say, and the b quark Lagrangian hasn't really changed. So there's no change in the state. And you can set things up so that you have the same IR regulator. And that'll become important in the example that we're just about to do.

So let's do an example of carrying out this matching for C1 and C2 in a little more detail. And before we carry out matching, we actually have to renormalize the two theories. Well, we've been talking as if the theory above is the standard model. So imagine that we've already renormalized that. And so we only have to renormalize the effective theory in MS bar.

So we'll talk about doing that, remind you of doing that-- maybe something you've seen before. So there's going to be wave function renormalization. So you have this 4/3. I'm going to leave out the prefactor, this prefactor. It's always going to be there. I'm going to stop writing it and start just focusing on the thing in square brackets here.

We'll do calculations with Feynman gauge. I'm not going to really do the calculations. I'm just going to quote results to you. And in order to simplify what we have to write down, let me define the set of spinners that you get from taking the tree level matrix element of O1 to be S1 and of O2 to be S2.

So this is just the spinners that we wrote down before for the case of S1, exactly what we had before. And then for S2, just a slightly different contraction of color indices. OK. So that's just trying to make the lowest-error result simple to write down so that when I write down the higher-error result, we can focus on the things that are changing and mattering, and not on the complications that come in from what we're talking about.

So if we think about diagrams here, we have four-quark operator. And we just have to draw all the loop graphs Right. We have to regulate them in some way, because they are IR divergent.

So let's regulate them with-- if we really want to talk about the matching, which is what I really want to do eventually, we should regulate them in some way. And so let's regulate them with off-shell momenta on the external lines. And I'll take it to be a common off-shell momenta p. And I'll just set the masses of the external lines to 0, since they're not going to matter.

OK. So then what does it look like? So the matrix element of O1-- the 0 means Bayer-- is going to have divergences. And it has the following structure, times spinner 1. And then there's another piece that involves spinner 1. And then there's a piece that involves spinner 2 that shows up in from these loop graphs.

So even though we're calculating the Bayer matrix [INAUDIBLE] of operator 1, the spinner combination 2 shows up, because this is supposedly a gluon, and the gluon moves the color around. And I'm only writing the divergent terms, although later on we'll be interested in the constant terms as well. So the dots are the constant terms underneath logarithms, and all these little round brackets have constant terms in them.

Part of the reason for introducing the basis in the way I did is that when I want to quote you the result for O2, it's the same, just switching 1 and 2. And so the statement that you would have from this-- these are ultraviolet divergences, the infrared divergences are regulated. The statement that you would have from this is that you can look at these various terms, and you can ask, what's going on with those divergences?

Well, this one here is actually cancelled by wave function renormalization, which I haven't put in yet. And this one here is O1 mixing into O1. So this one here is a counterterm, if you like, for O1. And this one here is a counterterm for O2.

And the language you use as you say that O1, which we're calculating, has mixed into O2. So there's two different methods we could use to carry out the renormalization here that are actually equivalent.

So method one is called composite operator renormalization. It's useful to know what things are equivalent so that you know what you can get away with ignoring. So what is composite operator renormalization?

Well, you think about having a Bayer operator. And you need to introduce some constants to renormalize. These are not related to wave function renormalization. It's actually operator renormalization. You have multiple fields at the same spacetime point, you need additional renormalization constants, that's what these Z's are.

When you go to take the matrix element, you have to include the wave function renormalization as well. So the Bayer matrix element has a wave function renormalization as well as this Z. And this is the relation between the renormalized matrix element and the Bayer one. So this guy here is renormalized and amputated.

OK. So that's the relation you can calculate O0. That's what we were doing. We just calculated O0. We could remove Z psi. And I just told you if you remove Z psi, it's going to get rid of this. And then you use the remainder of this stuff to get Zij.

I'm going to need it the other way around. So let me write the-- so I can write this relation the other way around. And then it looks like this. OK, so that's one method. The other one is related more to how we usually think about things in terms of gauge theory. And that is to have counterterm coefficients for the various operators.

So start with the Hamiltonian written in terms of Bayer coefficients and operators that have Bayer fields-- Bayer operators, if you like. And now switch over to renormalized quantities, so you get Z psi squared.

We have to switch over from the Bayer coefficient to a renormalized one. So let me do that first. So we get some Z for the coefficient, some renormalized coefficients. And then we get a Z psi squared from the four fermions that are in this operator. Then we get an O.

And then we can write this in terms of Ci Oi plus Z psi squared Zij C minus delta ij Cj Oi. And then this would be our counterterms. So we would stick in this guy, that operator we haven't renormalized-- we haven't done our operator renormalization. We're not doing method 1. So this, the matrix element here will still diverge, but we cancel them off with the counterterms.

So that's the logic of method 2. And once you do that, the divergences cancel between these two terms, and you're left, if you like-- or even if you don't-- with the coefficients times the operators, which are renormalized, both renormalized, even renormalized separately.

And these two are equivalent. These are equivalent ways of thinking about the same thing. And we can even go further and derive the equivalence of them. So if we take this theory and we take the matrix element of the Hamiltonian in this second way of doing things, what do we get? We get this.

So even though as Oi, I said, we haven't done operator renormalization, so we still have the divergences there, that's Cj Oj. And if we write the OJ using the relation at the top of the board, then we get Cj Zji inverse Z psi squared Oi 0.

And then we can just look at the two sides. They both have the Z psi squared. They both have a Cj, they both have Oi 0.

The only thing that's different is this. So we find the Zij for the coefficient is the transpose of the inverse of the Z's for the operators. So that's a more definite way of saying the two would lead to the same results. And this is how you would actually get the relation between the two ways of doing it.

So it's not completely trivial. You have to know you have to take the inverse and transpose. But it's the same information. OK. So in our example, the Z is a matrix. It starts out as the unit matrix. And in MS bar, we collect the divergences that are not cancelled by wave function renormalization. And that gives us this little 2-by-2 matrix.

And from that little 2-by-2 matrix, we can construct anomalous dimension for the operators. So let's do that. So how do we do that? Well, if we take the renormalization of the Bayer operators, remember those dependent on the regulator, but they didn't depend on the scale mu. So if we take mu d by d mu of Oi 0, which is the Bayer operator at 0. That's what I've written.

And then we can write O in terms of the Z, in the equation I raised. And both Z and the renormalized operator depend on the scale mu. And so that gives us this equation. So I'm doing method 1 here, if you like, using the notations from method 1.

If I take that equation and rearrange it, I can write it as anomalous dimension equation for the operator. So it's mu d by d mu is one of the terms. I isolate this, and I move that over by taking an inverse of it. And I just call everything that I get there the anomalous dimension of the operator.

So gamma ji is all the other stuff, Zjk inverse. And I put the explicit minus sign here so there's no minus sign in that equation. Anomalous dimension is determined by the Z factors. And we determine the Z factors, and which we can stick this equation into that equation and get the anomalous dimension.

When we do that, we have to be careful about the fact that alpha s in d dimensions has a little extra piece that's actually important for this discussion, which is this piece. So when we take mu d by d mu, you ask, what depends on mu? And it's the alpha here that depends on mu.

And it's still in-- depends on epsilon as well in that equation. And so this term is the term that matters at one loop. And this matters at two loops, and so does that. But at one loop, we could just replace this by the identity, and we can drop this.

OK. Put those things together. Anomalous dimension, of course, is something that doesn't depend on epsilon. And it's, in this case, a 2-by-2 matrix. And we can solve this problem.

If we want to solve this matrix, then what do you do? Well, you could diagonalize it. That's how you would solve it. So if you want to run the operators in this theory just for completeness, you would diagonalize by forming O1 plus or minus O2, call the coefficients of those new operators, C plus or minus, and an obvious notation.

And when you write down an anomalous dimension equations for these guys, there's no mixing anymore at one loop. In general, this is the procedure if you do this diagonalization that you have to carry out again when you get to two loops. It's not like the basis that you pick at one loop will be fine for two loops. We're not so lucky.

But at one loop, this is a perfectly valid basis for the problem. And in an hopefully self-evident notation, I either take all pluses or all minuses.

OK. And if I write my Hamiltonian, I could write it in the original basis. Then if I switch to the other basis, I'll set up my convention so that it's simple. And that means I picked C plus or minus to be C1 plus or minus C2/2.

AUDIENCE: [INAUDIBLE] minus?

IAIN STEWART: Whoops, thank you. Yeah, they have the opposite sign. First one's positive, second one's negative.

And in this new basis tree level matching, which is your boundary condition for your revolution, is that both coefficients are 1/2. OK. So then we could solve the differential equation the same way we were doing for QCD-- write down a result that would sum logs below the scale of the W mass.

OK. So we'll continue along these lines next time, say a few more words about renormalization, which is really just a lead-up of doing the matching, which is what I want to-- kind of the thing I want to spend a little more time on. So we'll finish that up also next time and move on to some other things.

So there's a few things to emphasize when we do the matching. And I want to write down the matrix elements and the full theory, which are box diagrams, I'll write down the results for those. We'll compare those to the results of the effective theory, and we'll draw some lessons from that.

And we'll carry out the matching, figure out what the next order coefficients would be here. What would those alpha s terms be? What's the procedure I would go through to get those coefficients?

OK. So we'll stop there for today.