MITOCW | 11. Renormalons

IAIN STEWART: [INAUDIBLE] effective theory workshop. So last time we were talking about renormalons, and we're going to continue to do that today. So we said that typically in quantum field theory you have perturbation theories, but actually they're asymptotic. And the kind of growth that you might expect as some power. I think I use two different conventions for this power, whether it's plus n or minus n, but that's just a goes to 1 over a, and there's a factorial growth.

So we said we could characterize those series by thinking about doing something called a Borel transform, and that makes them more convergent. Because, basically, we take the coefficients and we divide the coefficients by an n factorial. So we get more conversion series. And it may be that this thing exists, or it may be that this thing exists and has poles. And if it has poles, as I've indicated here on the real axis, then the inverse transform might not exist.

So even though this thing is something that you can define, you can't get back to f. And if that's the case, you can characterize how the original series was diverging based on those poles. And those poles are called renormalons.

So if we could calculate to all orders in perturbation theory and some in quantum field theory, like a gauge theory, then we could just calculate these f's. We would find that actually there is this factorial growth, and we'd be able to do this. But we can't do that, because we don't know how to do all the diagrams.

So in order to give you an explicit example of what we talked about last time, instead of looking at all the diagrams, we pick a particular subset that we can identify. And that's what we're going to do today, at least start by doing today.

Part of our discussion of motivating why one might want to think about renormalons was thinking about the pole mass versus the MS bar mass and showing you that phenomenology starts to work better in terms of the MS bar mass. And then we even further than that and said we should talk about this thing called the 1S mass, and we'll come back to that discussion today. But I'd like to consider just the explicit calculation of the pole versus the MS bar mass and show you that you can see this renormalon.

So if we want to think about the definition of these masses, then we have to implement those definitions order by order in perturbation theory. And order by order in perturbation theory, we could calculate the f's. But we need to know something about the entire series, and that is too hard to get the complete series.

So we'll pick out an infinite subset of that complete series that we could identify which is unique. And of course we should pick an infinite subset that's the easiest to calculate. And the one that we can pick out that's easy to calculate is the following-- we look at so-called bubble sum. So graphs with a bubble, and then more and more of them.

OK. So the reason that these graphs are unique is because if you look at any order in perturbation theory, they're the grass with the most factors of the number of fermions running around in these bubbles. So if you call the number of fermions running around in these bubbles nf, this graph has the most powers of nf. So at each order in perturbation theory, it's unique. It's a unique contribution. And it's pretty easy to calculate, because there's only one type of diagram contributing.

AUDIENCE: What does unique mean? [INAUDIBLE].

IAIN STEWART: Unique means that there's no other diagram. So you might think at each other in perturbation theory there's lots and lots of diagrams. There's only this diagram that contributes to the highest power nf. It's also gauge invariant, but it's really a unique diagram that contributes to that color slash flavor structure.

OK. So obviously an ingredient in this is the bubble itself. So if we go back to the bubble plus it's counter term, and we say there's a momentum p going through it, it's diagonal in color, transverse in momentum. It has a factor of an ns nf, some other numerical factors. There's a logarithm, there' constant, and then there's some divergence, but we cancelled the divergence with a counter. So the graph would look like that.

And I'm going to absorb all these things into a single logarithm just for convenience. So I define the constant c bar, which is this 5/3, and then I just put it into the logarithm. OK. So then this is a geometric series, so it's easy to deal with. So G mu nu, which is as a function of p and alpha s which sums up this chain plus counterterm diagrams where there's n bubbles here, and we start with no bubbles. This n is a projector.

OK. So this is explicitly writing out the sum. And what I've done here is I've used the fact that even though I wrote up nf up there, I can convert it to beta 0. So beta 0 was 11/3 C a minus 2/3 nf. I can solve that equation for nf and substitute it back in. And that is a convenient thing to do, just because you can think that these terms are then related to the beta 0, which has to do with the running coupling.

In some sense, what we're doing is we're saying that even though we just calculated these nf contributions, we know there's going to be some non-Abelian contribution that comes along with it that makes this really into a beta 0. Or another way of saying it is that if you think about characterizing color structures, you could write nf C a, nf f squared. Or you could write beta 0 C a, beta 0 squared, and these are equivalent ways of representing the different kinds of color structures that can come in at higher order. And we're just choosing to do it this way with the beta 0, rather than the nf's.

So we convert our nf to a beta 0. And it's still a unique subset of the entire set of terms. The terms we're not including our terms like this-- nf C a or C a squared. And we're just keeping these guys, which is equivalent to this guy. Foes that makes sense? Everybody happy?

AUDIENCE: [INAUDIBLE] photon [INAUDIBLE].

IAIN STEWART: Sorry.

AUDIENCE: Where are the photon propagators?

IAIN STEWART: The photon propagators. So the photon propagators all cancel each other out. So you get a piece greater than 1 over p squared, p squared 1 over p squared. So the whole thing looks like a photon propagator, just with a bunch of logs. Any other questions?

All right. So this is already of the structure where we could think about doing the Borel transform. Because we have a series in alpha s, and what the Borel transport meant was taking that series and converting it to a series in the Borel variable, just dividing by n factorial and then doing a sum.

So previously, the way I wrote that, for the conjugate variables, we wrote that we had alpha to the n plus 1. And if you think about what we do, we just went to b to the n over n factorial. So basically, to convert from this guy to this guy, that's what we did. And we had to take special care at the first term. But other than the first term, we just had this replacement rule.

Now, it's a little convenient actually not to do that with exactly the same variable but a slightly different variable that's just rescaled. So I'm going to use a slightly different variable here, which is typical. So what we'll do is we'll say that instead of just taking along the alpha s, we'll take along with it the beta 0.

And we'll say that the conjugate variable of that is u instead of b, but it's just rescaled. We're just taking something along with the coupling constant. Which is always we're always free to do. It's just a multiplicative factor.

Now, if we look at this diagram here, the factors of the coupling that we've included are all the ones that are attached to the bubbles. But there's two more-- there's one here and one there. And if we're going to make this replacement rule, we have to identify all the factors of alpha s, because they're all going to disappear and become u's. So we have to take these guys into account too.

We have an extra G squared, which I'm going to write in terms of this variable. So there's an extra factor of G squared from the ends. And if you think about what that means, it means that every diagram that we're considering here starts with n equals 1, so we don't have to worry about this delta function term. It starts with n equals 0. There's no term with no couplings here. So then it is literally what I said, that we just take this factor and replace it by that factor.

So a Borel transform, remember, is a transform of the variable alpha s. Alpha s goes over to a different space. So it's not a kinematic variable. It's not like what we're used to transforming in, but it's perfectly fine. For those people that came in after I said this, there's no lecture on Thursday.

So let me use the following notation, taking that G squared along with our geometric sum, and now expressing that whole thing as a function of momentum still, and u instead of alpha s. So I replaced the alpha s's, and I get that series, which we can see is just an exponential series. So I can just calculate the sum.

Borel variable u log-- which if you have an exponential of a log, it's just a power. So let's organize it that way. So let me write it this way So just writing the exponential of a log is that thing raised to the power u, taking the mu squared to the c bar u, that's a constant.

And then taking all the factors of p squared, there is one from here, and then there's u of them from there. but I took an extra one from here, multiplied on top and bottom by an extra p squared. Hopefully I got the sign right. Looks like it should be plus.

OK. So this thing is kind of nice for computations, because when you stick it back into our bubble sum-- when you stick this bubble sum back into the diagram we want to compute-- we just have effectively a modified gluon propagator. So think of it as some kind of modified gluon propagator. Maybe I should write it in a different color. So it's just like this, where we use a different Feynman rule for that propagator.

So it's not actually-- doing these types of bubble sum calculations is not any more hard than doing a one loop calculation, you just have to use a different power for the propagator. But your Feynman integral tricks can handle that very easily. OK. So it's about as hard as computing this one with graph.

So let's call this thing sigma bubble, and let's think about calculating it in terms of the MS bar mass. And then we also cancel any one of our epsilon poles, which we're not really interested in here. The 1 over epsilon poles actually come from-- if we think about what happened with the variables-- the case u equal 0, which is the log divergent piece.

And if you think about what we want to do in order to look for poles, u equals 0 is actually related to the ultraviolet divergence, while values u greater than 0 and u less than 0, those are probing these poles that are off the axis in the Borel plane. And it's those power law divergences that are probing for renormalons. And we'll see that they lead to poles in the Borel plane through this modified propagator.

So let's calculate-- the goal here is going to be to calculate an expression that relates to the MS bar mass and the pole mass. And that'll be an infinite series, and it will be a function in Borel space. The alpha s space, that would be an infinite series in alpha s, because we've included an infinite number of alpha s's in our diagrams. And if we go over to Borel space, there'll be some functional relationship between those two masses.

OK, so what's the definition of a pole mass. If you go back to the propagator and you write it in terms of the MS bar mass-- so you formulated your whole quantum field theory in terms of the MS bar mass, you calculate that bubble some, called it sigma, wrote it in terms of MS bar mass-- the general structure of sigma would be that there is some constant piece, which I'll call sigma 1.

And then there's some sigma 2 piece. And the pole mass is the place where you demand-- you just have p slash minus m pole. So another way of saying that is that if p squared is equal to m pole squared, you set that to be 0, which I can write as an equation like that. And then I can square it.

So m pole squared, which is p slash squared, m bar squared, this thing squared. And if you just think about what that gives, this is giving some equation that looks like m pole is m bar, m 1 plus sigma 1 plus the extra stuff. So let's do that and figure out what we get from this diagram for the sigma.

So it's a usual one loop type calculation, except instead of the gluon propagator, I have this G mu nu thing, which at this point is in alpha space, let's say. And then I have the fermion propagator. And I've taken into account all the g squareds and the alpha s's in here. And so to go to the Borel space, I can do that under the integrand.

So I just take g squared G mu nu bubble k and alpha s, k being the loop momenta, and just send that over to this thing that I'm calling g squared G mu nu bubble, which is not a function of g squared anymore, but just a function of k and u.

And as long as you're staying away from u equals 0, there's no issues related to the order of doing those operations. So at u equals 0, you have to be actually careful if you really want to go figure out what's happening at u equals 0. But that's actually not what we're interested in, so we'll just ignore what's happening at u equals 0.

OK. So combine-- stick in the expression that we had over here for this guy, and it's just some p squared to a power. So it becomes a k squared to a power. So you have k squared to a power, you have this guy to a power-- use the usual tricks. Combine denominators, do algebra, and you get some expression for sigma 1 in Borel space. And then, from that, you can construct the relation between the pole mass and the MS bar mass.

And it might not surprise, having done some one loop calculations that you're getting some gamma functions, after the dust settles, this is what it looks like. 6, 1 minus u, gamma of u, gamma of 1 minus 2u over gamma of 3 minus u. So the piece that we just calculated, or sketched how you would calculate, is this.

And then there's the tree level relation between these two things, which in alpha s space was just that m pole was equal to m bar. But when you transform one to the Borel space, you get a delta function. So this comes from-- just that equality at lowest order becomes a delta function in the Borel space.

So now, if you look at this and you ignore u equals 0, then the closest pole to u equals 0 is equals 1/2 coming from this gamma function. And remember that we had a kind of notation for the strength of a renormalon. Those closest to the origin were the strongest. So the strongest renormalon here is a u equals 1/2 renormalon.

That strength was related to the fact that basically you got something that would be, in this case, 2 to the n, n factorial. So 1.2 to the minus n. And so the guy that's closest gets the strongest prefactor through the n factorial growth. That's where that naming came from. Yeah.

AUDIENCE: So there weren't [? mu poles ?] in-- the only pole in the modified gluon propagator, was it-- Oh, like, the one that made the denominator vanish. But the rest of the poles came from actually doing the one with the diagram.

IAIN STEWART: That's right.

[INTERPOSING VOICES]

AUDIENCE: --in the series.

IAIN STEWART: Yeah. So what you're doing is you're actually thinking of probing the infrared structure in this loop. That's what the gluon is doing. And the modified propagator p squared to the u is just giving you a kind of a regulator if you like, which is the u that's allowing you to probe what's the infrastructure in powers in this loop diagram, not just kind of logarithmic divergences which we usually focus on but also in power law sensitivity to the infrared.

So it's like probing low momentum gluons with this modified gluon propagator. And that's what's resulting in these poles. You're probing the infrared structure of the loop.

So there would be some other terms that I haven't written. So let me just comment that there's actually a term that's proportional to 1 over u that would make it finite at u equals 0 if we're more careful, when we carry out the renormalization properly. And I also wasn't so worried about anything analytic in u. So I only was looking at these gamma functions.

OK. So if we actually just specify ourselves to that u equals 1/2, we can simplify things even further. So stick in u equals 1/2. And if I stick in u to the 1/2, this becomes mu over m bar e to the c bar over 2. If I just put this 1/2 in there, all this stuff with the gamma functions becomes a minus 2 over u minus 1/2.

So there's a pole. It's sitting on the real axis, and we can't do the transform.

AUDIENCE: At this point can you drop the delta u? Because you know it's not--

IAIN STEWART: Yeah, I don't really need it. But just sort of for completeness I kept it. Yeah. If I just wanted to think about this term, I could drop it. Yeah.

So let's think about the inverse Borel. So we wanted to do an integral which in this u space would look like this-- so u is the conjugate variable to 4 pi over beta 0 alpha s of mu. That was what we transformed. So that's the difference between our old notation and our new one. We just have a conjugate variable to this thing.

And so this has an ambiguity, because on the real axis we have a pole at u equals 1/2. So there's that pole at 1/2, and you have to decide do we want to go around that above or below.

So I mentioned this last time, we could think about going past the pole that way or going past the pole this way. And the fact that we have to make a choice means there's an ambiguity. And if you like, you can say, well, it's the average above and below or that the ambiguity would then be something like 1/2 going around the pole.

So let's actually look at what the ambiguity is. So the ambiguity, well, if we go around the pole, we get a 2 pi i. There's 1/2, integral around the pole, this factor.

So mu c bar over 2, bar. And if we put everything together, the rest of the prefactor, there's a C f over 3 pi beta 0, and there's also another m bar, which is this m bar that's out front. And I combined the minus 2 with the six together, 3.

So the m bars actually are cancelling. We close on the pole, stick that value in everywhere else, and we get that. Does anybody recognize this?

AUDIENCE: It's like an instanton.

IAIN STEWART: It's lambda QCD. Yeah. Yeah, it looks kind of like an instanton with the exponential, but this particular thing is actually exactly lambda QCD. So if we solve this, and we invert it, and we solve for alpha s, that would be the usual formula for lambda QCD-- alpha in terms of lambda QCD.

So we've just shown that the ambiguity up to some constant prefactor is exactly lambda QCD. And that is exactly our physical idea that we said last time, that the pole mass has a delta m of order lambda QCD ambiguity. And we've just seen that explicitly arise from calculations with this Borel space. So that's kind of neat Yeah.

AUDIENCE: So Is there a way to be careful about the pi epsilon description when we do these Borel transforms?

IAIN STEWART: Yeah.

AUDIENCE: Because it seems like you just multiplied a bunch of p squareds plus i epsilons together, and then just ignored the i epsilon.

IAIN STEWART: Yeah. I didn't ignore the i epsilon. The reason I wrote it as a minus p squared to the power was because of that i epsilon. I didn't write the i epsilons on the board. But if I had, it wouldn't change anything that I wrote. So the one way I can say it is that when I actually did these loop calculations, I was actually careful about the i epsilon.

AUDIENCE: I'm not worried about getting the sign--

[INTERPOSING VOICES]

IAIN STEWART: Yeah.

AUDIENCE: --like of logs. I'm worried about actually defining products of distributions right. Like when you interchange the order of integration, presumably you're doing something [INAUDIBLE]--

IAIN STEWART: You mean the order of the summation and the integration?

AUDIENCE: Right.

IAIN STEWART: Yeah. When you interchange that order, the only thing that you-- so you can do it in the other order. The only thing that actually goes wrong-- and it's much easier to present it in this order. This is why I did. If you do do it in the other order, the only thing that will change is what happens at u equals 0.

So you'll actually see everything that we saw, except that u equals zero, it'll be slightly different. And that 1 over u term that I was talking about, if you want to get that right, you have to think about the issue of changing the order of integration more carefully than I did.

AUDIENCE: But if it's near u equals 1/2.

IAIN STEWART: Near u equals 1/2, it's exactly-- this is the right result. And near any other u except u equals 0 we're OK.

AUDIENCE: Iain?

IAIN STEWART: Yeah.

AUDIENCE: So which [INAUDIBLE]?

IAIN STEWART: Sorry.

AUDIENCE: Which [INAUDIBLE] would it be to define--

IAIN STEWART: We don't know. So the ambiguity-- you have to make a choice, right? If you make this choice, you've picked one definition. If you make this choice, you've picked another. And I said, we can quantify the ambiguity in our choice by thinking about it as 1/2 going around the pole. That's kind of a measure of the ambiguity that we have in which way to close. It's an ambiguity.

AUDIENCE: I know. But when [INAUDIBLE] on how we calculate it, right? So when we can m pole, what do we mean by--

IAIN STEWART: Different people mean different things. I mean, you're not going to see this, right, unless you got all orders in perturbation theory. So order by order in perturbation theory, you will never see this pole.

So unless you're working to all orders-- as soon as you work to all orders, you have to make a choice because of what we just did. If you work order by order, there's no choice to make, and you just order by order define it with the usual definition. So the moral is not how do we define m pole, the moral is we should use something else other than m pole. Because m pole has an ambiguity.

So we use the MS bar mass m bar here for our calculation, but actually the ambiguity wasn't depending on that. It really will show up-- even if we thought of some other short distance mass beside the MS bar mass, we would still see the same ambiguity if we tried to associated it to the m pole. And actually, this thing is a unique property of the pole mass. It's the pole mass that has this problem, not the MS bar mass.

Another thing you'll notice is that when we went back and looked at the ambiguity back in the original space, in the coupling space, which is where we live, then the mu dependence went away. We just got something that was a constant times lambda QCD. There was no mu dependence.

So the ambiguity is mu independent. It's proportional to lambda QCD. If we looked, on the other hand, in the Bore space, at the residue of the pole, it was mu dependent. So before we did the integral, if we just look at even the pole as u minus 1/2, the residue has mu power in it in the numerator. And that just went along for the ride and became this mu. But when we transformed back, we had alpha mu there, and the mu dependence was cancelling.

And this actually is-- there's one important fact about this. So when we express something like a decay rate, which we might originally calculate in terms of the pole mass since maybe that's simpler, and we get some series, if we want to express that, say, in terms of the MS bar mass, then what we're after doing is we're after actually canceling a pole in the Borel plane that occurs in kind of the series that relates this guy to the MS bar mass. And then there's a corresponding pole in this series in the decay rate. And those have to cancel against each other.

So you should think that there's a 1 over u minus 1/2 pole in the relation of this thing to the MS bar mass that we just computed. And one could also look at, for example, bubble chains for this particular observable. People have done that. And if we did that, you'd also find exactly the same, u equals 1/2 pole in that series. And these are cancelling if we make that change of variable here.

I guess I've already said that. When you do that-- in practice of course, what you do is you don't have the whole series. So what you do is you want the cancellation to take place at the order you've worked. So say I know alpha squared in this decay rate-- which is true, people do know that-- and the alpha squared relation. What you then want to do is you want to make sure that the terms, any large numerical factors in front of the series, are canceling out to order alpha squared.

And there's only one little caveat which you have to be careful about, and that is that you have to expand in the same coupling. So when we do this cancellation order by order, you should use the same coupling constant with the same scale. So as long as both series-- so you can't, for example, make a conversion with the M b pole with alpha s evaluated at the b scale or some other scale. You have to really do it at the same scale that whatever this series is expressed in terms of.

And you can also think of that as being related to the fact that the Borel variable u itself was defined as the conjugate variable to something that was mu dependent. So even the definition of u involves alpha at it's particular scale mu. OK. So I said that in a bit of a convoluted way. But hopefully what I'm saying is clear, even if what I'm writing is not so clear.

OK. So same coupling in this series and the conversion series. And this, what I've just said, is also a general fact, that these poles are artifacts of splitting up physics at different scales. And they will always be canceling out observable things. They're remnants of splitting up physics in particular ways. They always cancel out of observables.

AUDIENCE: When you say physics at different scales, is that a proxy for taking a certain infinite subset of diagrams--

IAIN STEWART: Yeah. So if you like, we introduced a cutoff in our diagrams when we did them. Right.

AUDIENCE: Like the number of gluon propagators [INAUDIBLE]?

IAIN STEWART: Yeah. I mean-- right. Effectively, you could think of it like that or the number of bubbles. Yeah. And that was a proxy that became this u variable. And that was becoming kind of a probe for the cutoff dependence. So we'll talk more about that right now.

So what is this? How should you think about this cutoff? So when we remove the ambiguity, we actually introduce a scale. And the way you can think of that is as follows-- you should think that, in general, the relation between the pole mass and some other mass is some infinite series which has the general structure of a bunch of logarithms perhaps, and coupling constants, and some coefficients.

So it's a double series in n and k. It starts at order alpha. So that's why n is starting at order one. And just because of dimensions-- if this is dimension 1, that's dimension 1-- there has to be something of a dimension 1 that we stick there. And let's be flexible about we put what we put there and call it R.

So this is a general scheme conversion formula. We can put any kind of conversion into that form. So if we call this thing here delta m, then I can also move it to the other side and write like this. And then, what you can say is that if I pick my delta m such that it's infrared structure, it's asymptotic structured alpha, is the same as the pole mass, I can set up something where I get a renormalon-free mass m R.

So the pole mass has this renormalon that we've been talking about. If I set it up so that my subtractions also have the same kind of structure-- so for example, with the example that we did, you would need the nonlogarithmic terms to be n factorial 2 to the n, beta 0 to the n-- as long as we have the right terms in the delta m, we can construct something that's free of that renormalon. Now, in MS bar, which is what we did, this R scale is just the MS bar mass itself. Which is the running quantity, but it's evaluated at m bar.

So the formula that we get would have m pole is equal to m bar plus m bar times a series in alpha. That was the conversion. So the scale R was fixed to be m bar.

In this scheme that we talked about last time, which is called the 1S scheme, where we talked about defining a scheme by thinking about two heavy quarks calculating corrections between calculating perturbatively the potential between those heavy quarks and defining a mass that way, that's called the 1S scheme. In that particular case, you get an extra alpha that we talked about. And the corresponding R would actually be the m 1S mass times an alpha, which is effectively the inverse Bohr radius for this potential.

But we can even be more general than that. We can make this R into just a free parameter. We don't have to fix it in the way that these two schemes do. We can think of it as a floating cutoff.

And for any R that we pick, the ambiguity delta m will be R independent. Just like the m bar was cancelling out when we picked the MS bar scheme, when you pick the 1S scheme, that R cancels out. And generically, the ambiguity, which is just lambda QCD, is independent of R.

So R is really setting a scale. So really what you're doing is you're cancelling something that's happening if you like it in the deep infrared. But whenever you cancel something, you have to sort of decide how much of the leftover do you take. You're canceling something at 0, but then you're taking something from 0 to something. And that "to something" is exactly what this R is.

Just like in dim reg-- my spelling is atrocious. Just like in dim reg, when you cancel the UV pole that's at infinity. You have 1 over epsilon UV, but the mu comes in, and the mu is setting kind of a soft cutoff on your integrals. And you think of that as where you've moved down to. In the same way, this R is kind of saying, you're canceling something at 0, and we have to take along with it some fluctuations up to some scale. And R us providing that choice.

So we take some infrared fluctuations together with the pole mass-- these are the infrared fluctuations that were causing the [? stability. ?] They are [? dressing ?] the pole mass, if you like, which is like a point particle mass. They're making it into a kind of [? dressed ?] mass, and they're yielding a well-defined mass parameter that then depends on a scale. So here's the origin. I think that you include all the fluctuations in some sphere around the origin node to R. That's the right physical picture.

So the right physical picture here is that when we're using the MS bar scheme, we're ignoring powers. And there was actually an improper separation between certain power law terms. Those power law terms that you didn't really see very well in MS bar can be seen when you look at the asymptotics of the perturbation series. They come back to haunt you.

You can look at the asymptotics of the perturbation series using these renormalon Borel-style techniques. And you can figure out how to define masses that are sensitive to these problems, like the MS bar mass. The MS bar mass has too large an r scale to be used for b physics.

The 1S mass is something that has a small enough scale that you can use it for b physics, although it would have potential problems for top quark physics. But you can actually make the scale where you're absorbing the fluctuations up to a free parameter, and then you're good to go with any physical problem, because you can choose it to be an appropriate scale, much like we choose mu to be an appropriate scale.

So let me give one very definite example of how you could define a scheme with this R. So we'll build something that's where we don't have to do any additional calculations. What we're going to do is we're going to use the MS bar mass, the MS bar pole scheme conversion, and from that we define the a's that we need. We can take mu equal to R so that the log of mu over R is 0, and then we don't need any of the other a's.

So then we have a formula that's a definite thing that looks as follows-- so we're tweaking the MS bar result, if you like, so that instead of having a fixed scale, it has a floating scale. So these are the MS bar values. And we don't have the MS bar front, we have an R in front. And we have R's in these couplings.

And that's a well-defined scheme called the MSR scheme. And in this scheme, you have a cut off R that you can just pick as a parameter and adjust it. It's just a parameter like mu. It's part of the definition of the scheme.

And so this kind of change here would be good for doing physics. This is a good scheme for doing physics where you want to absorb things up to that cutoff. So you can sort of-- well--

So if you're thinking about HQET, you can pick the R to be say a GV freely, and it wouldn't care about whether you're talking about a top quark or a bottom quark or a charm quark. It's just a free parameter. You can pick it to be something fixed, and it wouldn't depend on what kind of mass you're starting over here with. So you're just kind of decoupling this cutoff from the mass itself.

All right. Any questions?

AUDIENCE: So when you wrote the series in k that you not got rid of with this choice, that was just you being careful about including--

IAIN STEWART: Everything.

AUDIENCE: [INAUDIBLE].

IAIN STEWART: Yeah. I mean, I was just being careful about including all the kind of things that in principle I could see in any arbitrary choice of scheme change. And in any arbitrary choice, there's both mu, which we defined-- that was related to the alpha-- and there was this R showing up. And we got logs of mu over R. And in MS bar, the R was in bar, but you still had mu showing up as well. And mu is related to the running of the MS bar mass.

But in general, the fact that you have both mu and R is kind of-- they were separate. Because you define mu for the log cutoffs. And you defined R here for the power log cutoffs-- the power log divergences. At the end of the day, you might as well just take those cutoffs to be the same. And that's what I just did-- mu equals R. That gets rid of the logs.

AUDIENCE: Do people worry about u equals 1?

IAIN STEWART: Yeah, you can worry about u equals 1. u equals 1/2 is the first thing you should worry about. If you want to go to u equals 1, then you need problems that have kind of more information in perturbation theory. So if you just have two loop information, u equals 1/2 is usually perfectly fine. If you have four loop information, you probably should start worrying about u equals 1. That's just kind of rough. So people don't often talk about u equals 1, though sometimes they do.

OK. So I need a little aside here in order to proceed with what I want to talk about next. And that is I just want to define for you how you would define lambda QCD at higher orders of perturbation theory. So the formula we used a minute ago was leading log, but we can define lambda QCD even if we go to higher orders. And I just want to do a little bit of algebra, set up some of the notation in order to define this thing at higher orders.

So MS bar beta function-- and we're going to just work to all orders formally. And we'll see that we can construct a solution where it's kind of obvious how you would work to whatever order you want to work. So let's just parameterize the series, even if we don't know all the terms. We only up to beta 4. But let's parameterize the series as a bunch of betas.

I'm going to call the cutoff R, because we've started calling it that. But I could call it mu. I just took mu equal to R. So I can rearrange that equation in this way-- I can write it as dR over R is equal to d alpha over beta. And then I can integrate on both sides between two values. And on this integral, it's just giving me a log. This integral, it's giving me an integral over alpha.

So what I'm going to want to do here is make a change of variables. And we'll see the following change of intervals convenient. I'm going to make a change of variables in the integration to this t. So I have some dummy variable that I'm integrating over. Let's change the dummy variable t. And basically that's going to simplify the lowest order result.

Because the lowest order result d alpha over beta of alpha, well, the beta goes like alpha squared at lowest order. So I'm basically making a change of variable where this d alpha over beta at lowest order will just be DT. That's what I'm doing. I have to change variables in the limits as well. So I call those t1 and t0 and change variables consistently everywhere.

And if I do that, this is the kind of form of the result. So this is just a Laurent series in 1 over t. And the first term, we had d alpha over minus beta 0 over 2 pi alpha squared. And that was just minus dt. So the reason that this thing here is 1 is because my change of variable with dt was exactly taking into account all the factors to basically just make it into a 1. And there's a sign change, but I flipped the limits.

So we made the first term trivial, and then the higher terms of these terms. And these b hats we could figure out what they are. And they're just combinations of the beta functions. So once we expand the 1 over beta, we get various combinations of things. I think I won't write b3.

We could also do the integral-- pretty easy integral to do. And so if you want to write a solution for what the integral is, you could say, well, it's some function that you get from doing the indefinite integral evaluated at the upper limit minus that function evaluated at the lower limit. And G is just that function. So doing the integral of 1 gives t. Doing the integral of the other time, and being careful about what's positive and what's negative, I can write as log of minus t.

So t was negative. And if you look at my definition of t, it had a minus sign in it. So this log is a logical positive number.

So we have this G of t and G prime of t is just b hat. And if we write a formula-- which is not what we're actually interested in here, but we might as well write it anyway-- it's this way. Just taking that formula and writing it this way, then if you think about what this formula is, it's an all orders relation between the coupling at one scale and the coupling at another.

It's a formal relation, because you need to know the coefficients. But if someone tells you the coefficients, you can plug them into this formula. And this is the extension of the usual running formula that you would write down for relating alpha at one scale to alpha at another scale. This is a generalization that includes higher order terms in the beta function.

So the other thing that we can do is we can rearrange the formula and take account of the fact that we have R1's and R0's, and we can make it such that we have only R1's on one side of the formula and R0's on the other side of the formula by splitting the log into two pieces. And that's what you do to define lambda QCD. So we can rearrange it as follows-- taking an exponent.

So now we only have our R0's in this part, R1's in that part. This thing has to be independent of the choice of the R, and that's just this constant lambda QCD. And that's the generalization of this definition of lambda QCD that we had before, where it was R0 e to the minus 2 pi over beta 0 alpha. Now we have the higher terms that I just decoded in this G function.

So if you didn't remember that formula for the lambda QCD, now we've derived it. And we included the higher order terms as well. So the next term would be this. [INAUDIBLE] higher order terms. And usually you call this the leading log expression for lambda QCD, with the next leading log once you include that term, et cetera.

But the whole combination is mu independent. The mu independence of this term is only cancelling to leading log. Once you include this term, it's cancelling to next leading log. The whole thing is mu independent, getting this from lambda QCD. So that's end of the aside.

So what I actually want to do is I want to come back to our definition of this MS bar mass, and I want to treat it as if we have-- I want to think about it like an RG. We have this cutoff. Let's see if we can write down an renormalization group equation for that mass. It has a cutoff-- there should be some equation that tells us how to flow in that cutoff. Let's see what we can learn from that.

So that's end of aside. So we have a cutoff, and we already said that we can treat R as a variable that parameterizes the mass scheme. And so we can vary R in this MSR scheme, just like we varied u in the MS bar scheme.

So, if you like, what you're doing when you're vary mu in the MS bar scheme is you're properly taking into account kind of what fluctuations you want to put into your mass in the ultraviolet. And here it's related to this kind of physics that's going on in the deep infrared that you're also being careful about. So we'll have instead of a mu RGE we'll have an R RGE. The difference between the mu RGE and the R RGE is this time we're talking about powers. We'll see what happens.

So M pole didn't depend on R. So you get 0 for that. And if we just look at our scheme change formula, then we can figure out what the RGE is. This delta M was a function of R. And we can define this as R, to get the dimensions right, times some kind of anomalous dimension that's just a function of alpha.

So what this guy would look like is again just a series-- some perturbative series. RGE is writing out our equation. The equation we've come to is this. So R d by dR of the mass, because of the fact that we're dealing with a power, we get an R on the right-hand side of this equation. And that's really the only thing that's causing a difference from this kind of standard RGE's that we've solved so far.

And the fact that we got this power, which was 1, is exactly related to the fact that it was a u equals 1/2 renormalon. If we looked at the renormalons that are further out, like u equals 1 would give R squared. We haven't seen really enough to identify that, but that's true.

So let's solve this guy in the usual way. Integrate on both sides. From what we just talked about, R is equal to lambda QCD e to the minus G of t. And we can also therefore write that d log R is, from this formula. dt minus G prime of t. If we follow that through as a change of variable, that's the change of variable we get.

So let's switch variable from R to dt. So d log R is the product of these things, and I can write it actually kind of a compact way if I write it is lambda QCD dt d by dt of e to the minus G of t. So we've got a G prime, and then we've got an e to the minus G of t. Let's just write that as a total derivative of that.

So if I write it in terms of t, then that's the solution. That's the formal solution in terms of-- and I switched variable also in the anomalous dimension. And that's the formal solution to the RGE. And this is a well-defined integral. There's no problems with doing it.

The place where the integral could start causing problems is t equals 0. But we're never getting to t equals 0. We have cutoffs that are keeping us away from t equals 0.

So it shouldn't surprise you that it's a well-defined integral, because we said that these things are supposed to be well defined in masses that are not sensitive to these renormalon problems. And taking the difference of two of them should again give us something that's not sensitive to renormalon problems. But it's interesting that you see lambda QCD popping out. And that's because these two masses and the difference of these cutoffs is related to absorbing a different amount of these fluctuations which were related to lambda QCD.

So the evolution would just yield a new well-defined mass of R1, which absorbs a different amount of IR fluctuations-- a shell out to R1 instead of just out to R0 if you like. So you're absorbing those fluctuations together with them M pole to get something well defined. And how much you absorb is related to this cutoff.

So we could do that integral at whatever order we decide. Let's just do it at leading log. So at leading log, we take this guy and we just take the first term, some constant, alpha over 4 pi. And so gamma R of t, switching variables, becomes that constant over 2 beta 0 if we keep all the factors. And then a 1 over t-- alpha comes 1 over t.

So if we're at leading log, we should think about having the leading log-- we'll have the QCD. So that's what the 0 means-- there's this factor. And the integral we'd have to do would be this one, e to the minus t over t. So you see that t equals 0 is where there could be a problem. But away from t equals 0, there's no problem.

If you do that integral, it gives you an incomplete gamma function. That's just a way of saying that there's a special function that's defined to be that integral. Mathematica knows all about it. It's very useful.

And actually, the incomplete gamma function is an example of a function that has an asymptotic expansion. So if we look at what happens if I expand this incomplete gamma function in alpha, t, Remember, is 1 over alpha. It can expand about alpha. Then it has a series that has factorial growth. Let me just write that.

So expanding about alpha equals 0 is expanding about t equals infinity. And the gamma function actually has an asymptotic expansion. And if you just work it out, plug it into Mathematica-- it's a Taylor series-- look at the terms, and you exactly see this-- 2 to the n n factorial, which is a u equals 1/2 renormalon.

And so what's happening is when you take the difference of two of these gammas, you're converting this asymptotic series. If you take the difference of two of these asymptotic series, you're converting it into a convergent series. And we'll talk a little bit more about that next time, but I'll stop there.

So we can see, from this renormalization group, again the renormalon, and we can see in what way it's sort of solved by thinking about making it shift from here to here. Because it's the difference of these two gammas, not one of them individually. So we can use normalization group techniques to think about these renormalon effects. And we'll talk a little bit more about that next time.

AUDIENCE: So is this a convergent series now?

IAIN STEWART: The difference of these is convergent series. Yeah. I'll write that down next time and show you it.