where

$$W_n = \sum_k \sum_{\rm perm} \frac{(-g)^k}{k!} \left(\frac{\bar{n}\cdot A_n(q_1)\cdots \bar{n}\cdot A_n(q_k)}{[\bar{n}\cdot q_1][\bar{n}\cdot(q_1+q_2)]\cdots[\bar{n}\cdot\sum_{i=1}^k q_i]} \right) . \tag{3.34}$$

Here W_n is the momentum space version of a Wilson line built from collinear A_n gluon fields. In position space the corresponding Wilson line is

$$W(0,-\infty) = {\rm P}\exp\left(ig\int_{-\infty}^0 ds\, \bar{n}\cdot A_n(\bar{n}s) \right) \tag{3.35}$$

Here P is the path ordering operator which is required for nonabelian fields and which puts fields with larger arguments to the left e.g. $\bar{n}\cdot A_n(\bar{n}s)\,\bar{n}\cdot A_n(\bar{n}s')$ for $s > s'$.

In summary, we see that we have traded the field $\bar{n}\cdot A_n$ for the Wilson line $W_n[\bar{n}\cdot A_n]$. Also, including this Wilson line in our current operator makes our current gauge invariant, as we will show below in the Gauge Symmetry section. For a situation with n and n' collinear fields the same type of Wilson lines $W_n[\bar{n}\cdot A_n]$ are also generated in a manner that yields gauge invariant operators for each collinear sector.

4 SCET$_{\rm I}$ Lagrangian

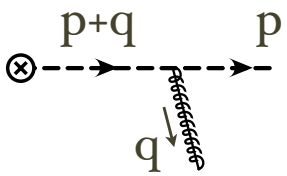
In this section, we derive the SCET quark Lagrangian by analyzing and separating the collinear and usoft gluons, and momentum degrees of freedom. On the way to our final result we introduce the label operator which provide a simple method to separate large (label) momenta from small (residual) momenta.

4.1 SCET Quark Lagrangian

Lets construct the leading order SCET collinear quark Lagrangian. This desired properties that this Lagrangian must satisfy include

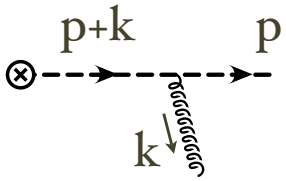
- Yielding the proper spin structure of the collinear propagator
- Contain both collinear quarks and collinear antiquarks
- Have interactions with both collinear gluons and ultrasoft gluons
- Yield the correct LO propagator for different situations without requiring additional expansions
- Should be setup so we do not have to revisit the LO result when formulating power corrections

To explain what is meant by the fourth point consider the propagator obtained when a collinear quark interacts with a collinear gluon



$$\propto \frac{\bar{n}\cdot(p+q)}{n\cdot(p+q)\,\bar{n}\cdot(p+q) + (p_\perp+q_\perp)^2 + i0} .$$

Here both the momentum p and q appear on equal footing, and no momenta are dropped in the denominator. This can be contrasted with the leading propagator obtained when a collinear quark interacts with an ultrasoft gluon



$$\propto \frac{\bar{n}\cdot p}{n\cdot(p+k)\,\bar{n}\cdot p + p_\perp^2 + i0}\,.$$

Here the ultrasoft k^μ momentum is dropped for all components except $n\cdot k$ where it is the same size as the collinear momentum $n\cdot p$. The dropping of $k_\perp \ll p_\perp$ and $\bar{n}\cdot k \ll \bar{n}\cdot p$ corresponds to carrying out a multipole expansion for the interaction of the ultrasoft gluon with the collinear quark. The LO collinear quark propagator must be smart enough to give the correct leading order result without further expansions, irrespective of whether it later emits a collinear gluon or ultrasoft gluon.

We will achieve the desired collinear Lagrangian in several steps.

4.1.1 Step 1: Lagrangian for the larger spinor components

In this section we construct a Lagrangian for the field $\hat{\xi}_n$. It will satisfy the first two requirements in our bullet list.

We begin with the standard QCD lagrangian for massless quarks.

$$\mathcal{L}_{QCD} = \overline{\psi} i \not{D} \psi \tag{4.1}$$

Expanding ψ and D in our collinear basis gives us

$$\mathcal{L} = (\overline{\varphi}_{\bar{n}} + \overline{\hat{\xi}}_n)\left(\frac{\bar{\not{n}}}{2} in\cdot D + \frac{\not{n}}{2} i\bar{n}\cdot D + i\not{D}_\perp\right)(\varphi_{\bar{n}} + \hat{\xi}_n)\,. \tag{4.2}$$

We can simplify this result by using the projection matrix identities for the collinear spinor found in section 3.1. In particular, various terms vanish such as

$$\frac{\not{n}}{2} i\bar{n}\cdot D\hat{\xi}_n = 0\,, \qquad\qquad \overline{\varphi}_{\bar{n}}\frac{\bar{\not{n}}}{2} in\cdot D = 0 \tag{4.3}$$

by virtue of the analog of (3.19) for $\overline{\varphi}_{\bar{n}}$. Lastly, terms like

$$\overline{\hat{\xi}}_n i\not{D}_\perp \hat{\xi}_n = \overline{\hat{\xi}}_n i\not{D}_\perp P_n \hat{\xi}_n = \overline{\hat{\xi}}_n P_n i\not{D}_\perp \hat{\xi}_n = 0\,, \qquad\qquad \overline{\varphi}_{\bar{n}} i\not{D}_\perp \varphi_n = 0\,, \tag{4.4}$$

since $\bar{\xi}_n P_n = 0$ and $\bar{\varphi}_{\bar{n}} P_{\bar{n}} = 0$. These simiplifications leave us with the Lagrangian

$$\mathcal{L} = \overline{\hat{\xi}}_n \frac{\bar{\not{n}}}{2} in\cdot D\,\hat{\xi}_n + \overline{\varphi}_{\bar{n}} i\not{D}_\perp \hat{\xi}_n + \overline{\hat{\xi}}_n\, i\not{D}_\perp \varphi_{\bar{n}} + \overline{\varphi}_{\bar{n}}\frac{\not{n}}{2} i\bar{n}\cdot D\varphi_{\bar{n}}\,. \tag{4.5}$$

So far this is just QCD written in terms of the $\hat{\xi}_n$ and $\varphi_{\bar{n}}$ fields. However, the field $\varphi_{\bar{n}}$ corresponds to the spinor components which were subleading in the collinear limit. These spinor components will not show up in operators that mediate hard interactions at leading order. Therefore we will not need to consider a source term for $\varphi_{\bar{n}}$ in the path integral.[3] This means that we can simply perform the quadratic fermionic

[3] At subleading order the coupling to the subleading components is introduced in operators via the combination involving ξ_n shown in the last line of Eq.(4.6), so there is still no reason to have a source term for $\varphi_{\bar{n}}$.

path integral over $\varphi_{\bar n}$. At tree level doing so is simply equivalent to imposing the full equation of motion for $\varphi_{\bar n}$. We find

$$0 = \frac{\delta\mathcal{L}}{\delta\varphi_{\bar n}} \;:\qquad \frac{\bar{n}\!\!\!/}{2}\, i\bar n\cdot D\varphi_{\bar n} + i\not{D}_\perp\xi_n = 0 \tag{4.6}$$
$$i\bar n\cdot D\varphi_{\bar n} + \frac{\bar{n}\!\!\!/}{2} i\not{D}_\perp\hat\xi_n = 0$$
$$\varphi_{\bar n} = \frac{1}{i\bar n\cdot D} i\not{D}_\perp\frac{\bar{n}\!\!\!/}{2}\hat\xi_n\,,$$

where the second line is obtained by multiplying the first by $\bar{n}\!\!\!//2$ on the left, and the plus sign in the last line comes from using $\bar{n}\!\!\!/ i\not{D}_\perp = -i\not{D}_\perp\bar{n}\!\!\!/$. Plugging this result back into our Lagrangian, two terms cancel, and the other two terms give the Lagrangian for the $\hat\xi_n$ field

$$\mathcal{L} = \bar{\hat\xi}_n\left(in\cdot D + i\not{D}_\perp\frac{1}{i\bar n\cdot D}i\not{D}_\perp\right)\frac{\bar{n}\!\!\!/}{2}\,\hat\xi_n\,. \tag{4.7}$$

The inverse derivative operator may look a little funny, but we can understand it in the same way we do for the operator $1/\hat r$ in quantum mechanics, namely by defining it through its eigenvalues, which in this case are in momentum space. Say we have the operator $\frac{1}{i\bar n\cdot\partial}$ acting on a field $\phi(x)$. Expressing this operation in momentum space gives

$$\frac{1}{i\bar n\cdot\partial}\phi(x) = \frac{1}{i\bar n\cdot\partial}\int d^4p e^{-ipx}\varphi(p) = \int d^4p e^{-ipx}\frac{1}{\bar n\cdot p}\varphi(p)\,, \tag{4.8}$$

and the eigenvalues $1/\bar n\cdot p$ define the inverse derivative operator.

Although we have a Lagrangian for $\hat\xi_n$ we are not yet done. In particular we have not yet separated the collinear and ultrasoft gauge fields, nor the corresponding momentum components. These remaining steps will be to

2. Separate the collinear and ultrasoft gauge fields.

3. Separate the collinear and usoft momentum components with a multipole expansion.

We then can expand in the fields and momenta and keep the leading pieces.

4.1.2 Step 2: Separate collinear and ultrasoft gauge fields

Recall that $A_n^\mu \sim (\lambda^2, 1, \lambda) \sim p_n^\mu$ and $A_n^\mu \sim (\lambda^2,\lambda^2,\lambda^2) \sim k_{us}^\mu$. Since $k_{us}^2 \ll p_n^2$ the ultrasoft gluons encode much longer wavelength fluctuations, so from the perspective of the collinear fields we can think of A_{us}^μ like a classical background field. In background field gauge we would write $A^\mu = Q^\mu + A_{cl}^\mu$ where Q^μ is the quantum gauge field and A_{cl}^μ is the classical background field that only appears on external lines. In general there is no need for a relationship between the full QCD gluon field A^μ and the SCET fields A_{us}^μ and A_n^μ, but if one exists then it does make matching computations much simpler. Based on the analogy with a background gauge field you might not be too surprised to learn that a relation exists which encodes basic tree level matching

$$A^\mu = A_n^\mu + A_{us}^\mu + \cdots\,. \tag{4.9}$$

Here the ellipsis stand for additional terms involving Wilson lines which only will become relevant when we formulate power corrections, and hence will be ignorded for our leading order analysis here (they are given below in Eq.()). The interpretation of A_{us}^μ as a background field to ξ_n and A_n^μ will also prove useful

when we derive the collinear gluon lagrangian and when we later consider gauge transformations in the theory.

Now, comparing the power counting between components of A_n^μ and A_{us}^μ, we find

$$\begin{aligned} \bar{n}\cdot A_n \sim \lambda^0 \gg \bar{n}\cdot A_{us} \sim \lambda^2 \\ A_{\perp\, n}^\mu \sim \lambda \gg A_{\perp\, us}^\mu \sim \lambda^2 \\ n\cdot A_n \sim \lambda^2 \sim n\cdot A_{us}. \end{aligned} \tag{4.10}$$

So we see that $A_{\perp\, us}^\mu$ and $\bar{n}\cdot A_{us}$ can be droped from our leading order analysis because in the combination $A_n^\mu + A_{us}^\mu$ they are always dominated by the collinear gluon term. Conversely, $n\cdot A_{us}$ cannot be dropped because it is of the same order as $n\cdot A_n$.

4.1.3 Step 3: The Multipole Expansion for Separating momenta

We want to find a way to isolate momenta that have different scaling with λ. Such a procedure is useful because it will allow us to formulate power corrections in a manner where operators give homogeneous contributions in λ order by order. For example, consider the denominator of the propagator of a quark with momentum $p_n + k_{us}$ expanded to keep the leading and first subleading terms

$$\begin{aligned} \frac{1}{(p_n+k_{us})^2} &= \frac{1}{(p_n^- + k_{us}^-)(p_n^+ + k_{us}^-) + (p_n^\perp + k_{us}^\perp)^2} \\ &= \frac{1}{p_n^-(p_n^+ + k_{us}^+) + p_n^{\perp\,2}} - \frac{2k_{us}^\perp\cdot p_n^\perp}{[p_n^-(p_n^+ + k_{us}^+) + p_n^{\perp\,2}]^2} + \dots . \end{aligned} \tag{4.11}$$

By power counting, we see that the first term scales as λ^{-2} and the second term scales as λ^{-1}. Although the first term dominates the second, we need to be able to reproduce the second term at the level of the Lagrangian when higher order corrections are needed. Expressed more formally, we would like a systematic multipole expansion. Our desired expansion is similar to the one found in $E\&M$ which gives corrections to the electrostatic potential for a given charge distribution.

In position space the multipole expansion corresponds to expanding the long wavelength field, $A_{us}(x) = A_{us}(0) + x\cdot i\partial A_{us}(0) + \dots$. To see what is going on here we can Fourier transform the operators (taking one-dimensional fields and ignoring indices for simplicity)

$$\begin{aligned} \int dx\ \bar{\psi}(x)A_{us}(0)\psi(x) &= \int dx \int dp_1\, dp_2\, dk\ e^{ip_1x}e^{-ik(0)}e^{-ip_2x}\ \bar{\psi}(p_1)A_{us}(k)\psi(p_2) \\ &= \int dp_1\, dp_2\, dk\ \delta(p_1-p_2)\ \bar{\psi}(p_1)A_{us}(k)\psi(p_2). \end{aligned} \tag{4.12}$$

We see immediately that this corresponds to a 3-point Feynman rule where the small momentum k is ignored relative to the large momenta p_1 and p_2, and that total momentum is not conserved at the vertex. For the next order term we get

$$\int dx\ \bar{\psi}(x)x(i\partial A_{us})(0)\psi(x) = \int dp_1\, dp_2\, dk\ \delta'(p_1-p_2)\ k\ \bar{\psi}(p_1)A_{us}(k)\psi(p_2). \tag{4.13}$$

Here the Feynman rule involves a $k\delta'(p_1-p_2)$ and we must integrate by parts to obtain the multipole momentum conservation expressed by $\delta(p_1-p_2)$. This integration by parts differentiates other parts of a diagram that carry this momentum, in particular the neighbouring propagators, which then would produce terms like the 2nd term in Eq. (4.11).

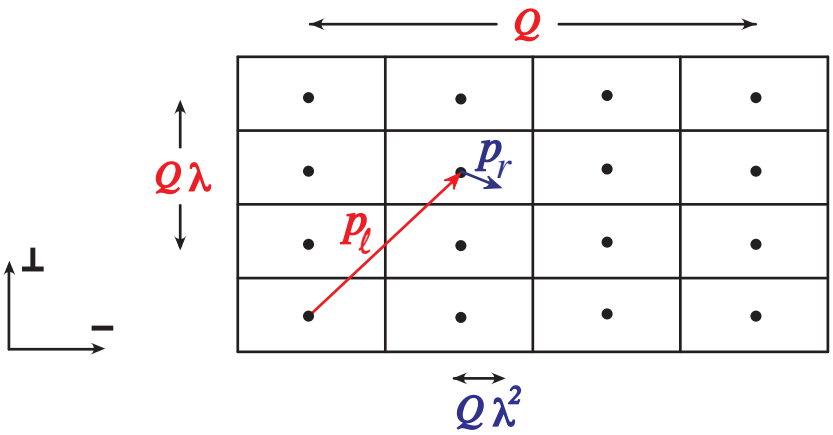


Figure 5: Grid to picture the separation of momenta into label and residual components.

Since Feynman diagrams are almost always evaluated in momentum space it would be more convenient to have a multipole expansion formalism that avoids the step of going through position space. In the remainder of this section we will set up a formalism to achieve this. It will allow us to 1) simply derive the corresponding momentum space Feynman rules, 2) simplify the formulation of gauge transformations in the effective theory, and 3) incorporate the multipole expansion into propagators rather than vertices.

For the moment we only consider the quark part of the field $\hat{\xi}_n(x)$. We will add the anti-quark part later on. Computing the Fourier transform $\tilde{\xi}_n(p)$ of the quark part of our field we have

$$\tilde{\xi}_n(p) = \int d^4x \, e^{ip\cdot x} \, \hat{\xi}_n(x). \tag{4.14}$$

Now to separate momentum scales, we define our momentum p^μ to be a sum of a large momentum components p_ℓ^μ called the *label* momentum and a small momentum p_r^μ called the *residual* momentum.

$$\begin{aligned} p^\mu &= p_\ell^\mu + p_r^\mu \\ p_\ell^\mu &\sim Q(0, 1, \lambda) \\ p_r^\mu &\sim Q(\lambda^2, \lambda^2, \lambda^2) \end{aligned} \tag{4.15}$$

This decomposition is similar to the one found in HQET where the quark momentum is $p^\mu = m\, v^\mu + k^\mu$. Although at the end of the day all momenta will be continuous, it turns out that it is quite convenient for understanding the multipole expansion to interpret the p_ℓ as defining a grid of points, and the p_r as defining locations in the surrounding boxes. This expansion is only necessary for the p^- and $p^\perp$ momenta since there are no label p^+ momenta, so we have a grid as shown in Fig. 5 (for convenience we show only one of the $p_\perp^\mu$ components). Note that any momentum p^μ has a unique decomposition in terms of label and residual components. Since $p_\ell \gg p_r$ the spacing between grid points is always much larger than the spacing between points in a box. This setup has the advantage of allowing us to cleaning separate momentum scales in integrands, arranging things so every loop integrand is homogeneous in λ.

In practice the grid picture is a bit misleading, since actually the boxes are infinite and with momentum components (p_ℓ, p_r) we are really dealing with a product of continuous spaces $\mathbb{R}^3 \times \mathbb{R}^4/\mathcal{I}$ where $\mathcal{I}$ are a group of relations that remove redundancy order by order in λ. ($\mathcal{I}$ includes the set of RPI transformations that we will discuss later on.) Nevertheless it is very convenient to derive the rules for integrals on the

label-residual space by working with a more familiar discrete label and continuous residual momentum picture, and then taking the continuum limit.

Thus if we are integrating the collinear momentum p over a certain region, we will write

$$\int d^4p \to \sum_{p_\ell \neq 0} \int d^4p_r \tag{4.16}$$

where we do not include $p_\ell = 0$ in the sum over all p_ℓ values, because $p_\ell = 0$ does not define a collinear momentum. Indeed the $p_\ell = 0$ bin corresponds to the ultrasoft modes. For an ultrasoft momentum p we simply have

$$\int d^4p \to \int d^4p_r \,. \tag{4.17}$$

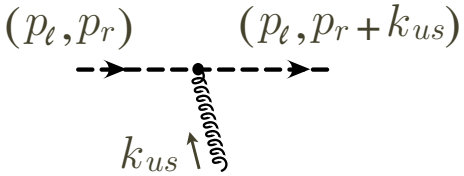
With this momentum separation we can also label our fields by both components

$$\tilde{\xi}_n(p) \to \tilde{\xi}_{n,\,p_\ell}(p_r) \,. \tag{4.18}$$

We also have separate conservation laws for label and residual momenta

$$\int d^4x \; e^{i(p_\ell - q_\ell)\cdot x} \, e^{i(p_r - q_r)\cdot x} = \delta_{p_\ell, q_\ell} \; \delta^4(p_r - q_r)(2\pi)^4. \tag{4.19}$$

Every collinear field carries both label and residual momenta, they are both conserved at all vertices, but Feynman rules may depend on only one or the other of these components. For example, what was previously a nonconservation of momenta for an interaction between collinear and ultrasoft particles now becomes two separate conservations of momenta.



An example is shown in the figure above.

Finally, since all fields carry residual momenta the conservation law just corresponds to locality of the field theory with respect to the Fourier transformed variable $p_r \to x$. Therefore we transform the residual momenta back to position space to obtain our final collinear quark field

$$\xi_{n,p_\ell}(x) = \int \frac{d^4p_r}{(2\pi)^4} \, e^{-ip_r x} \, \tilde{\xi}_{n,\,p_\ell}(p_r) \,. \tag{4.20}$$

We will build operators using these fields. Altogether, the above steps allow us to rewrite our hatted collinear field $\hat{\xi}_n(x)$ as

$$\begin{aligned} \hat{\xi}_n(x) &= \int \frac{d^4p}{(2\pi)^4} \, e^{-ip\cdot x} \, \tilde{\xi}_n(p) = \sum_{p_\ell \neq 0} \int d^4p_r \; e^{-ip_\ell \cdot x} e^{-ip_r \cdot x} \, \tilde{\xi}_{n,\,p_\ell}(p_r) \\ &= \sum_{p_l \neq 0} e^{-ip_\ell \cdot x} \, \xi_{n,\,p_\ell}(x) \,. \end{aligned} \tag{4.21}$$

We can identify several facts about label conservation for the field $\xi_{n,p_\ell}(x)$

- Interactions with ultrasoft gluons or quarks leave the label momenta of collinear fields conserved.
- Interactions with collinear gluons or quarks will change label momenta.
- The label n for the collinear direction is preserved by both ultrasoft and collinear interactions. Only a hard (external) interaction can couple fields with different collinear directions.

Now that we have separated momentum scales in our fields we would like to do the same with derivatives that act on these fields. Since $\xi_{n,\,p_\ell}(x)$ contains only residual momenta, we know that

$$i\partial_\mu \xi_{n,\,p_\ell}(x) \sim \lambda^2 \xi_{n,\,p_\ell}(x). \tag{4.22}$$

We also define a label momentum operator such that

$$\mathcal{P}^\mu \xi_{n,\,p_\ell}(x) \equiv p_\ell^\mu \xi_{n,\,p_\ell}(x). \tag{4.23}$$

Recall that $\mathcal{P}^\mu$ and p_ℓ^μ only contains the components $\overline{\mathcal{P}} \equiv \bar{n}\cdot\mathcal{P} \sim p_\ell^- \sim \lambda^0$ and $\mathcal{P}_\perp^\mu \sim p_\ell^{\perp\mu} \sim \lambda$. Therefore we have $n\cdot\mathcal{P} = 0$. Also

$$i\bar{n}\cdot\partial \ll \overline{\mathcal{P}}\,, \qquad\qquad i\partial_\perp^\mu \ll \mathcal{P}_\perp^\mu\,. \tag{4.24}$$

The main advantage of the label operator is that it provides a definite power counting for derivatives. It is also notationally friendly in that it removes the necessity of a label sum. We can see this by rewriting our field $\hat{\xi}_n(x)$ in terms of label momenta

$$\begin{aligned}\hat{\xi}_n(x) &= \sum_{p_\ell\neq 0} e^{-ip_\ell\cdot x}\,\xi_{n,p_\ell}(x) \\ &= e^{-i\mathcal{P}\cdot x}\sum_{p_\ell\neq 0}\xi_{n,p_\ell}(x) \\ &\equiv e^{-i\mathcal{P}x}\xi_n(x)\,.\end{aligned} \tag{4.25}$$

In the last line we defined $\xi_n(x) = \sum_{p_l\neq 0}\xi_{n,\,p_l}$. Since the label operator allows us to encode the phase factor involving label momenta as an operator, we can suppress the momentum labels on our collinear fields if there is no reason to make them explicit. For field products we have

$$\hat{\xi}_n(x)\hat{\xi}_n(x) = e^{-i\mathcal{P}\cdot x}\xi_n(x)\xi_n(x) \tag{4.26}$$

where the label operator acts on both fields. Consequently, conservation of label momenta is simply encoded by this phase factor and is manifest at the level of operators.

Lastly, we must deal with anti-particles and gluons. For the anti-particles, we expand our Dirac field into two parts

$$\begin{aligned}\psi(x) &= \int d^4p\,\delta(p^2)\theta(p^0)\big[u(p)a(p)e^{-ip\cdot x} + v(p)b^\dagger(p)e^{ip\cdot x}\big] \\ &= \psi^+(x) + \psi^-(x)\end{aligned} \tag{4.27}$$

we then associate each part with a collinear field and expand as a sum over label momenta.

$$\begin{aligned}\psi^+ \longrightarrow \hat{\xi}_n^+(x) &= \sum_{p_l\neq 0} e^{-ip_\ell\cdot x}\xi^+_{n,\,p_\ell}(x)\,, \\ \psi^- \longrightarrow \hat{\xi}_n^-(x) &= \sum_{p_\ell\neq 0} e^{ip_\ell\cdot x}\xi^-_{n,\,p_\ell}(x)\,,\end{aligned} \tag{4.28}$$

where both have a $\theta(p_\ell^0) = \theta(\bar{n}\cdot p_\ell)$. Because of charge conjugation symmetry it is convenient to combine the particle and anti-particle fields back into a single field. In order to do this we have to deal with the opposite signs for their phase. To do this we define

$$\xi_{n,\,p_\ell}(x) \equiv \xi^+_{n,\,p_\ell}(x) + \xi^-_{n,\,-p_\ell}(x) \tag{4.29}$$

where p_ℓ has either sign, but one picks out particles and one picks out antiparticles. Thus the action of the fields ξ_{n,p_ℓ} and $\bar{\xi}_{n,p_\ell}$ is that for

$$\begin{aligned} \bar{n}\cdot p_\ell > 0 \ : &\qquad \text{a particle is destroyed or created} \\ \bar{n}\cdot p_\ell < 0 \ : &\qquad \text{an antiparticle is created or destroyed} \end{aligned}$$

The sign convention for the label momentum is easy to remember since it is in the same direction as the fermion number flow. With this definition, we may write

$$\hat{\xi}_n(x) = e^{-i\mathcal{P}\cdot x}\xi_{n,\,p_\ell}(x)\,, \tag{4.30}$$

and all the manipulations we were making with particle fields carry through for the fields that have both particles and antiparticles. For collinear gluons, we proceed analogously to find

$$\hat{A}^\mu_n = \sum_{q_\ell\neq 0} e^{-iq_\ell\cdot x} A^\mu_{n,q_\ell} = e^{-i\mathcal{P}\cdot x} A^\mu_n(x) \tag{4.31}$$

where

$$A^\mu_n(x) = \sum_{q_\ell\neq 0} A^\mu_{n,\,q_\ell}. \tag{4.32}$$

Since the gluon field $A^\mu_n = A^{\mu A}_n T^A$ where $A^{\mu A}_n(x)$ is real we also have

$$[A^{\mu A}_{n,q_\ell}(x)]^* = A^{\mu A}_{n,-q_\ell}(x)\,. \tag{4.33}$$

Once again for $q^-_\ell > 0$ the field A_{n,q_ℓ} destroys a gluon, while for $q^-_\ell < 0$ it creates a gluon.

With our conventions the action of the label operator on a bunch of labelled fields is

$$\mathcal{P}^\mu(\phi^\dagger_{q_1}\phi^\dagger_{q_2}\cdots\phi_{p_1}\phi_{p_2}\cdots) = (p_1^\mu + p_2^\mu + \cdots - q_1^\mu - q_2^\mu - \cdots)(\phi^\dagger_{q_1}\phi^\dagger_{q_2}\cdots\phi_{p_1}\phi_{p_2}\cdots). \tag{4.34}$$

Thus it gives a minus sign when acting on daggered fields. It is also useful to note that if we differentiate an arbitrary collinear field $\hat{\phi}_n(x)$ that it yields

$$\begin{aligned} i\partial^\mu\hat{\phi}_n(x) &= i\partial_\mu \sum_{p\neq 0} e^{-ip\cdot x}\phi_{n,\,p}(x) \\ &= \sum_{p\neq 0} e^{-ip\cdot x}(\mathcal{P}^\mu + i\partial^\mu)\phi_{n,\,p}(x) \\ &= e^{-i\mathcal{P}\cdot x}(\mathcal{P}^\mu + i\partial^\mu)\phi_n(x). \end{aligned} \tag{4.35}$$

In the last line, we can suppress the exponent if we assume that label momenta are always conserved. Effectively, by introducing the label operator we have replaced the ordinary derivative operation by

$$i\partial^\mu\hat{\phi}_n(x) \to (\mathcal{P}^\mu + i\partial^\mu)\phi_n(x). \tag{4.36}$$

4.1.4 Final Result: Expand and put pieces together

At last, we may construct our final leading order Lagrangian. We begin with the previously derived result:

$$\mathcal{L} = \bar{\hat{\xi}}_n \left(in\cdot D + i\not{D}_\perp \frac{1}{i\bar{n}\cdot D} i\not{D}_\perp \right) \frac{\not{\bar{n}}}{2} \hat{\xi}_n. \tag{4.37}$$

Changing $i\partial_\mu \to (\mathcal{P}_\mu + i\partial_\mu)$ and $\hat{\xi_n} \to \xi_n$ and expanding our derivative operators, we have

$$\begin{aligned} in\cdot D &= in\cdot\partial + gn\cdot A_n + gn\cdot A_n \\ iD_\perp &= \underbrace{(\mathcal{P}^\mu_\perp + gA^\mu_{n\perp})}_{\sim\lambda} + \underbrace{(i\partial^\mu_\perp + gA^\mu_{\perp,\,us})}_{\sim\lambda^2} + \cdots \\ i\bar{n}\cdot D &= \underbrace{(\overline{\mathcal{P}} + g\bar{n}\cdot A_n)}_{\sim\lambda^0} + \underbrace{(i\bar{n}\cdot\partial + g\bar{n}\cdot A_{us})}_{\sim\lambda^2} + \cdots \end{aligned} \tag{4.38}$$

where the ellipses again denote additional $\sim\lambda^2$ terms that can be dropped in our leading order analysis (but later on we will see are required by gauge symmetry when considering power suppressed operators). Keeping only the lowest order terms, we have the following lagrangian

$$\boxed{\mathcal{L}^{(0)}_{n\xi} = e^{-ix\cdot\mathcal{P}}\ \bar{\xi}_n \Big(in\cdot D + i\not{D}_{n\perp} \frac{1}{i\bar{n}\cdot D_n} i\not{D}_{n\perp} \Big) \frac{\not{\bar{n}}}{2}\, \xi_n}, \tag{4.39}$$

where the collinear covariant derivatives are

$$\begin{aligned} iD^\mu_{n\perp} &= \mathcal{P}^\mu_\perp + gA^\mu_{n\perp}\,, \\ i\bar{n}\cdot D_n &= \overline{\mathcal{P}} + g\bar{n}\cdot A_n. \end{aligned} \tag{4.40}$$

Remarks:

- Both terms with covariant derivatives in the $(\cdots)$ in $\mathcal{L}^{(0)}_{n\xi}$ are of order λ^2 so the leading order Lagrangian is order λ^4 (recalling that the fields scale as $\xi_n \sim \lambda$). Since for a Lagrangian with collinear fields $\int d^4x \sim \lambda^{-4}$ this gives us an action that is $\sim \lambda^0$ as desired. The superscript (0) on the Lagrangian denotes this power counting for the action.

- All fields are defined at x, and derivatives for this coordinate scale as $i\partial^\mu \sim \lambda^2$ so the action is explicitly local at the scale $Q\lambda^2$.

- The action is also local at the scale of $\mathcal{P}^\mu_\perp \sim Q\lambda$ since these derivatives occur in the numerator. It only has non-locality at the hard scale through the inverse $\overline{\mathcal{P}} \sim \lambda^0$. The fact that there is locality except at the hard scale is a key feature of SCET$_{\rm I}$. Some attempts to tweak the formalism described here, in order to simplify SCET, lead to actions that are non-local at the small scale $\sim\lambda^2$ because they integrate out some onshell particles, while leaving other onshell particles to be described by an action. We will avoid doing this, taking the attitude that low energy locality is a desired property for the effective field theory.

- If we are considering a situation without ultrasoft particles, and without hard interactions that do not couple to a particular component, then the interaction of collinear fermions alone could equally

well be described by the QCD Lagrangian. Indeed, even in the presence of ultrasoft fields we can write a Dirac type Lagrangian that is equivalent to Eq. (4.39) by

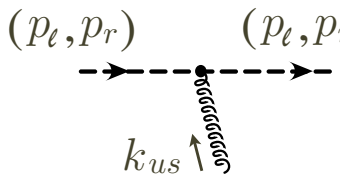
$$\mathcal{L}^{(0)}_{n\xi} = e^{-ix\cdot\mathcal{P}}\,\bar{\Xi}_n i\not{\mathcal{D}}\,\Xi_n\,,\quad \Xi_n \equiv \begin{pmatrix} \xi_n \\ \varphi_{\bar n}\end{pmatrix},\quad i\not{\mathcal{D}} = \frac{\bar{\not n}}{2} in\cdot D + \frac{\not n}{2} i\bar n\cdot D_n + i\not{D}_{n\perp} = i\not{D}_n + \frac{\bar{\not n}}{2} gn\cdot A_{us}\,. \tag{4.41}$$

Integrating out $\varphi_{\bar n}$ exactly reproduces Eq.(4.39). This Lagrangian is not equivalent to QCD due to the coupling to the ultrasoft gluon field, and the zero-bin subtractions related to $p_\ell \neq 0$ that will be discussed later on. But this form does make it more clear why the collinear particles share many of the properties of the full QCD Lagrangian (for example, we have the same renormalization properties for the gauge coupling).

The computation of the propagator from $\mathcal{L}^{(0)}_{n\xi}$ is also greatly simplified without the need for any additional power counting. Specifically, Eq. (4.39) gives the collinear quark propagator

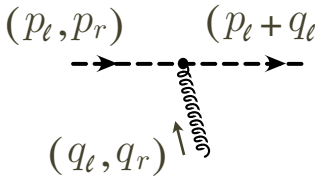
$$\frac{i\not n}{2}\frac{\bar n\cdot p_\ell}{(\bar n\cdot p_\ell)(n\cdot p_r) + (p_{\ell\perp})^2 + i0}\,. \tag{4.42}$$

The leading order Lagragian is smart enough that it gives the correct propagator in different situations without having to make further expansions. This is important to ensure that the leading order Lagrangian strictly give $\mathcal{O}(\lambda^0)$ terms, while subleading Lagrangians (and operators) will be responsible for power corrections. For example, if we have an interaction with an ultrasoft gluon then



$$= \frac{i\not n}{2}\frac{\bar n\cdot p_\ell}{(\bar n\cdot p_\ell)(n\cdot p_r + n\cdot k_{us}) + (p_{\ell\perp})^2 + i0}\,, \tag{4.43}$$

while if we have an interaction with a collinear gluon then



$$= \frac{i\not n}{2}\frac{(\bar n\cdot p_\ell + \bar n\cdot q_\ell)}{(\bar n\cdot p_\ell + \bar n\cdot q_\ell)(n\cdot p_r + n\cdot q_r) + (p_{\ell\perp} + q_{\ell\perp})^2 + i0}\,. \tag{4.44}$$

4.2 Wilson Line Identities

With the label operator formalism there are several neat identities that we can derive for Wilson lines. In particular we can show that all occurences of the field $\bar n\cdot A_n$ can always be entirely replaced by the Wilson line W_n. As an example we will show how this is done for the Lagrangian $\mathcal{L}^{(0)}_{n\xi}$. In position space the defining equations for a Wilson line are $W(x,x) = 1$ and its equation of motion, which we can transform to momentum space

$$\begin{aligned} i\bar n\cdot D_x W(x,-\infty) &= 0 \qquad \text{(position space)} \\ &\Downarrow \text{Fourier Transform} \\ i\bar n\cdot D_n W_n &= (\overline{P} + g\bar n\cdot A_n) W_n = 0\,. \end{aligned} \tag{4.45}$$

With this definition, the action of $i\bar{n}\cdot D_n$ on a product of W_n and some arbitrary operator is

$$\begin{aligned} i\bar{n}\cdot D_n(W_n\mathcal{O}) &= (\overline{P}+g\bar{n}\cdot A_n)W_n\mathcal{O} \\ &= \big[(\overline{P}+g\bar{n}\cdot A_n)W_n\big]\mathcal{O} + W_n\overline{\mathcal{P}}\mathcal{O} \\ &= W_n\mathcal{P}\mathcal{O} \end{aligned} \tag{4.46}$$

So we have the operator equation

$$i\bar{n}\cdot D_n W_n = W_n\overline{\mathcal{P}} \tag{4.47}$$

and with $W_n^\dagger W_n = 1$ we have

$$i\bar{n}\cdot D_n = W_n\overline{\mathcal{P}}W_n^\dagger\,, \qquad\qquad \overline{\mathcal{P}} = W_n^\dagger i\bar{n}\cdot D_n W_n\,, \tag{4.48}$$

as operator identities. Since by collinear gauge invariance we can always group $\bar{n}\cdot A_n$ with $\overline{\mathcal{P}}$ to give $i\bar{n}\cdot D_n$, the first identity implies that we can always swap $\bar{n}\cdot A_n$ for the Wilson line W_n. Inverting these results also gives useful operator identities

$$\frac{1}{i\bar{n}\cdot D_n} = W_n^\dagger\frac{1}{\overline{\mathcal{P}}}W_n\,, \qquad\qquad \frac{1}{\overline{\mathcal{P}}} = W_n\frac{1}{i\bar{n}\cdot D_n}W_n^\dagger\,. \tag{4.49}$$

The first relation allows us to rewrite $\mathcal{L}_{n\xi}^{(0)}$ as

$$\mathcal{L}_{n\xi}^{(0)} = e^{-ix\cdot\overline{\mathcal{P}}}\;\bar{\xi}_n\Big(in\cdot D + i\not{D}_{n\perp}W_n^\dagger\frac{1}{\overline{\mathcal{P}}}W_n i\not{D}_{n\perp}\Big)\frac{\not{\bar{n}}}{2}\,\xi_n\,. \tag{4.50}$$

It is also useful to note that we can use the label operator to write a tidy expression for the Wilson line which is built from fields that carry both label and residual momenta:

$$W_n(x) = \Big[\sum_{\rm perms}\exp\Big(\frac{-g}{\overline{\mathcal{P}}}\bar{n}\cdot A_n(x)\Big)\Big]\,. \tag{4.51}$$

4.3 Collinear Gluon and Ultrasoft Lagrangians

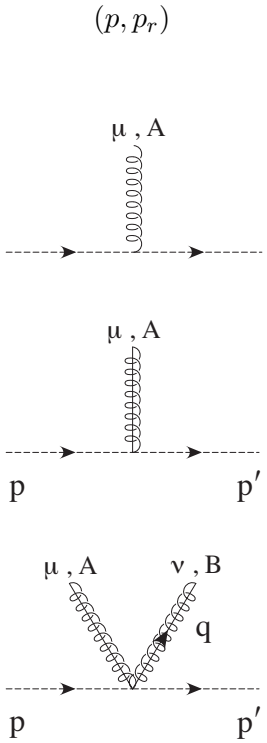
To derive the collinear gluon Lagrangian, we treat usoft and collinear degrees of freedom separately by letting A_{us}^μ represent a background field with respect to A_n^μ. We begin with the gluon Lagrangian from QCD:

$$\mathcal{L} = -\underbrace{\frac{1}{2}\mathrm{Tr}\big\{G^{\mu\nu}G_{\mu\nu}\big\}}_{\text{Gauge Kinetic Term}} + \underbrace{\tau\mathrm{Tr}\{(i\partial_\mu A^\mu)^2\}}_{\text{Gauge Fixing Term}} + 2\underbrace{\mathrm{Tr}\big\{\bar{c}\,i\partial_\mu iD^\mu c\big\}}_{\text{Ghost Term}} \tag{4.52}$$

where $G^{\mu\nu} = \frac{i}{g}[D^\mu, D^\nu]$. Expanding the covariant derivative as we did in the quark sector we keep only the leading order terms. For a covariant derivative acting on collinear fields the leading order terms are

$$iD^\mu \to i\mathcal{D}^\mu = \frac{n^\mu}{2}(\overline{\mathcal{P}}+g\bar{n}\cdot A_n) + (\mathcal{P}_\perp^\mu + gA_{\perp,n}^\mu) + \frac{\bar{n}}{2}(in\cdot\partial + gn\cdot A_n + gn\cdot A_{us}). \tag{4.53}$$

Recall that the field A_{us}^μ varys much more slowly than A_n^μ, so we can think of A_{us}^μ as a background field from the perspective of the collinear fields (even though it is a quantum field in its own right). The gauge fixing and ghost terms for the collinear Lagrangian should break the collinear gauge symmetry, but we do not want them to gauge fix the ultrasoft gluons, and hence they should be covariant with respect to the A_{us}^μ connection. Since by power counting only the $n\cdot A_{us}$ gluon can appear along with the collinear gluons



$$= \; i\,\frac{\bar{n}\!\!\!/}{2}\,\frac{\bar{n}\cdot p}{n\cdot p_r\,\bar{n}\cdot p + p_\perp^2 + i0}$$

$$= ig\,T^A\,n\!\!\!/_\mu\,\frac{\bar{n}\!\!\!/}{2}$$

$$= ig\,T^A\left[n\!\!\!/_\mu + \frac{\gamma_\mu^\perp p\!\!\!/_\perp}{\bar{n}\cdot p} + \frac{p\!\!\!/'_\perp \gamma_\mu^\perp}{\bar{n}\cdot p'} - \frac{p\!\!\!/'_\perp p\!\!\!/_\perp}{\bar{n}\cdot p\,\bar{n}\cdot p'}\bar{n}_\mu\right]\frac{\bar{n}\!\!\!/}{2}$$

$$= \frac{ig^2\,T^A\,T^B}{\bar{n}\cdot(p-q)}\left[\gamma_\mu^\perp\gamma_\nu^\perp - \frac{\gamma_\mu^\perp p\!\!\!/_\perp}{\bar{n}\cdot p}\bar{n}_\nu - \frac{p\!\!\!/'_\perp\gamma_\nu^\perp}{\bar{n}\cdot p'}\bar{n}_\mu + \frac{p\!\!\!/'_\perp p\!\!\!/_\perp}{\bar{n}\cdot p\,\bar{n}\cdot p'}\bar{n}_\mu\bar{n}_\nu\right]\frac{\bar{n}\!\!\!/}{2}$$

$$+ \frac{ig^2\,T^B\,T^A}{\bar{n}\cdot(q+p')}\left[\gamma_\nu^\perp\gamma_\mu^\perp - \frac{\gamma_\nu^\perp p\!\!\!/_\perp}{\bar{n}\cdot p}\bar{n}_\mu - \frac{p\!\!\!/'_\perp\gamma_\mu^\perp}{\bar{n}\cdot p'}\bar{n}_\nu + \frac{p\!\!\!/'_\perp p\!\!\!/_\perp}{\bar{n}\cdot p\,\bar{n}\cdot p'}\bar{n}_\mu\bar{n}_\nu\right]\frac{\bar{n}\!\!\!/}{2}$$

Figure 6: Order λ^0 Feynman rules: collinear quark propagator with label p and residual momentum p_r, and collinear quark interactions with one soft gluon, one collinear gluon, and two collinear gluons respectively.

A_n^μ, only this component is needed. Therefore we replace $i\partial^\mu \to i\mathcal{D}_{us}^\mu$ for all the ordinary derivatives in Eq. (4.52) where

$$i\mathcal{D}_{us}^\mu \equiv \frac{n^\mu}{2}\overline{\mathcal{P}} + \mathcal{P}_\perp^\mu + \frac{\bar{n}^\mu}{2} in\cdot\partial + \frac{\bar{n}}{2} gn\cdot A_{us}. \tag{4.54}$$

The resulting leading order collinear gluon Lagrangian is then

$$\boxed{\mathcal{L}_{ng}^{(0)} = \frac{1}{2g^2}\mathrm{Tr}\big\{([i\mathcal{D}^\mu, i\mathcal{D}_\mu])^2\big\} + \tau\mathrm{Tr}\big\{([i\mathcal{D}_{us}^\mu, A_{n\,\mu}])^2\big\} + 2\mathrm{Tr}\big\{\bar{c}_n[i\mathcal{D}_\mu^{us}, [i\mathcal{D}^\mu, c_n]]\big\}\,.} \tag{4.55}$$

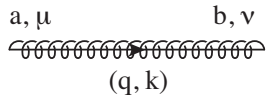
For the Langrangian with only ultrasoft quarks and ultrasoft gluons, at lowest order we simply have the QCD actions. Using a general covariant gauge for the ultrasoft gluon field we therefore can write

$$\mathcal{L}_{us}^{(0)} = \overline{\psi}_{us} i\not{D}_{us}\psi_{us} - \frac{1}{2}\mathrm{Tr}\big\{G_{us}^{\mu\nu}G_{\mu\nu}^{us}\big\} + \tau_{us}\mathrm{Tr}\big\{(i\partial_\mu A_{us}^\mu)^2\big\} + 2\mathrm{Tr}\big\{\bar{c}_{us}\, i\partial_\mu i D_{us}^\mu c_{us}\big\}\,, \tag{4.56}$$

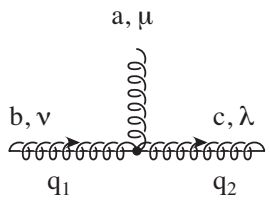
where $iD_{us}^\mu = i\partial_\mu + A_{us}^\mu$. All the terms in $\mathcal{L}^{(0)}$ have a power counting of $\mathcal{O}(\lambda^8)$, but we subtract 8 for the ultrasoft measure d^4x which is why we label the Lagrangian as (0). Note that the choice of gauge fixing parameters τ and τ_{us} for the collinear and ultrasoft gluons are independent, which is related to the fact that there are independent gauge symmetries that define these connections.

All together this allows us to write down the full leading order SCET$_{\rm I}$ Lagrangian with a single set of quark and gluon collinear modes in the n direction, and quark and gluon ultrasoft modes,

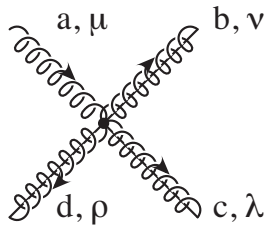
$$\mathcal{L}^{(0)} = \mathcal{L}_{n\xi}^{(0)} + \mathcal{L}_{ng}^{(0)} + \mathcal{L}_{us}^{(0)}. \tag{4.57}$$



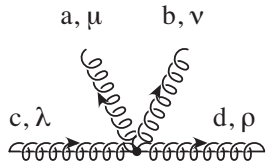
$$= \frac{-i}{\bar{n}\cdot q\; n\cdot k + q_\perp^2 + i0}\left(g_{\mu\nu} - (1-\tau)\frac{q_\mu q_\nu}{\bar{n}\cdot q\; n\cdot k + q_\perp^2}\right)\delta_{a,b}$$



$$= g f^{abc} n_\mu \left\{\bar{n}\cdot q_1\, g_{\nu\lambda} - \tfrac{1}{2}(1-\tfrac{1}{\tau})[\bar{n}_\lambda q_{1\nu} + \bar{n}_\nu q_{2\lambda}]\right\}$$



$$= -\tfrac{1}{2} i g^2 n_\mu \Big\{ f^{abe} f^{cde}(\bar{n}_\lambda g_{\nu\rho} - \bar{n}_\rho g_{\nu\lambda}) + f^{ade} f^{bce}(\bar{n}_\nu g_{\lambda\rho} - \bar{n}_\lambda g_{\nu\rho}) + f^{ace} f^{bde}(\bar{n}_\nu g_{\lambda\rho} - \bar{n}_\rho g_{\nu\lambda}) \Big\}$$



$$= \tfrac{1}{4} i g^2 n_\mu n_\nu \bar{n}_\rho \bar{n}_\lambda (1-\tfrac{1}{\alpha})\left\{ f^{ace} f^{bde} + f^{ade} f^{bce}\right\}$$

Figure 7: Collinear gluon propagator with label momentum q and residual momentum k, and the order λ^0 interactions of collinear gluons with the usoft gluon field. Here usoft gluons are springs, collinear gluons are springs with a line, and τ is the covariant gauge fixing parameter in Eq. (4.55).

4.4 Feynman Rules for Collinear Quarks and Gluons

For convenience we summarize some of the Feynman rules that follow from the collinear quark and gluon Lagrangians. We do not show the purely ultrasoft interactions which are identical to those of QCD, nor do we show the purely collinear gluon interactions which are also identical to those of QCD.

The Feynman rules that follow from the leading order collinear quark Lagrangian are shown in Fig. 6 where each collinear line carries momenta (p, p_r) with label momenta $p^\mu = \bar{n}\cdot p\, n^\mu/2 + p_\perp^\mu$ and residual momentum p_r^μ. Only one momentum p or p' is indicated for lines where the Feynman rule depends only on the label momentum. For the collinear quark propagator we have contributions from both quarks and antiquarks which give:

$$\frac{i\not{n}}{2}\,\frac{\theta(\bar{n}\cdot p)}{n\cdot p_r + \frac{p_\perp^2}{\bar{n}\cdot p} + i0} + \frac{i\not{n}}{2}\,\frac{\theta(-\bar{n}\cdot p)}{n\cdot p_r + \frac{p_\perp^2}{\bar{n}\cdot p} - i0} = \frac{i\not{n}}{2}\,\frac{\bar{n}\cdot p}{\bar{n}\cdot p\, n\cdot p_r + p_\perp^2 + i0} \tag{4.58}$$

The Feynman rules between collinear gluons and ultrasoft gluons are shown in Fig. 7 with a collinear gluon in background field gauge that is ultrasoft covariant and specified by the parameter τ.

4.5 Rules for Combining Label and Residual Momenta in Amplitudes

In practical calculations the grid picture in Fig. 5 is not to be taken literally. Doing so would correspond to using a Wilsonian EFT with finite cutoff's (edges for the grid boxes) that distinguish the size of momenta.

Instead of this, we need to use a Continuum EFT picture where the EFT modes have propagators that extend over all momenta, but integrands which obtain their key contribution from the momentum region these modes are built to describe. The terms needed to correct the (otherwise incorrect) ultraviolet contributions of the resulting Continuum EFT are included as perturbative Wilson coefficients for low energy operators. The Wilsonian and Continuum versions of EFT are really two different pictures of the same thing, in much the same way that two different renormalization schemes may represent the physics in different ways, but in the end still do encode the same physics. Nevertheless there are many practical advantages to the Continuum EFT framework, and it makes setting up SCET much easier. In particular it allows us to use regulators like dimensional regularization which naturally preserve spacetime and gauge symmetries. To setup up SCET in this continuum framework we need to understand how the redundancy $\mathcal{I}$ in the label-residual momentum space $\mathbb{R}^{d-1} \times \mathbb{R}^d/\mathcal{I}$ (for the case with d-dimensions) is resolved, given a pair of momenta components $(p_\ell, p_r) \in \mathbb{R}^{d-1} \times \mathbb{R}^d$. The upshot is that in the simplest cases the residual momentum can simply be dropped or absorbed into a label momentum in the same direction (making it continuous), while in the most complicated cases the formalism leads to so-called 0-bin subtractions for collinear integrands. These subtractions ensure that the collinear modes do not double count an IR region that is already properly included from an ultrasoft integrand. For future convenience we list the rules in this section, but caution the reader that some parts of this section are best understood when read together with one of the one-loop examples from section 7, and also after having read the discussion of the reparameterization invariance symmetry in section 5.3 that describes the redundancy $(p_\ell^\mu) + (p_r^\mu) = (p_\ell^\mu + \beta^\mu) + (p_r^\mu - \beta^\mu)$ which specifies $\mathcal{I}$.

For an arbitrary tree level diagram in SCET we will have some set of external lines that are either ultrasoft or collinear (and either in the initial or final state), and also a set of collinear and ultrasoft propagators. For the external lines that are ultrasoft we have only residual momenta k_{us}^μ and the onshell condition $k_{us}^2 = 0$. For the external lines that are collinear it suffices to take label momenta $p_\ell^- = \bar{n} \cdot p_\ell$ and $p_{\ell\perp}^\mu$, and a single residual momentum p_r^+. This amounts to picking β^μ above to contain the full p_r^- and $p_{r\perp}^\mu$ components. The onshell condition for the collinear particles is then simply $p_\ell^- p_r^+ - \vec{p}_{\ell\perp}^{\,2} = 0$. All propagators for intermediate collinear and ultrasoft lines are then simply determined by momentum conservation as usual. At leading order in λ this perscription for tree diagrams simply amounts to the same thing as dropping any ultrasoft momentum components k_{us}^- and $k_{us}^\perp$ from collinear propagators, though of course these momenta can still appear within ultrasoft propagators. At higher orders in λ these ultrasfot momentum components can also appear from collinear propagators through Lagrangian insertions, which yield terms like the second one in Eq. (4.11).

For loop diagrams and loop integrations we need several rules for operations on the label-residual momentum space. Internal collinear lines should be considered as carrying loop momenta with two parts $q = (q_\ell, q_r)$, while ultrasoft propagators only carry loop momenta k_r. There is a seperate momentum conservation for the label and residual momenta. After using momentum conservation we have label momenta from either external collinear particles or collinear loops, and residual momenta for external ultrasoft particle, external collinear particles from p_r^+, and from collinear and ultrasoft loops.

First we note that if we integrate over all label momenta q_ℓ and residual momenta q_r that this will be equal to an integration over all of the q^μ momentum space, since it does not depend on how we divide the momentum into the two components. For notational convenience we denote the label space integration as a sum rather than an integral. In d-dimensions we have

$$\sum_{q_\ell} \int d^d q_r = \int d^d q \,, \tag{4.59}$$

where we have recombined the label and residual momenta for the minus components, and the $(d-2)$ $\perp$-components. This is relevant for combining the two collinear loop integrations back into a single d-

dimensional integration. In particular at leading order in λ after having used momentum conservation there will always be one q_r^μ for each collinear loop integration, where q_r^- and $q_r^\perp$ do not appear in any collinear or ultrasoft propagator, and hence not in the integrand $F(q_\ell^-, q_\ell^\perp, q_r^+)$. We can therefore use this residual momentum integration in Eq. (4.59) to obtain a full integration

$$1)^{\rm naive}: \quad \sum_{q_\ell} \int d^d q_r \, F(q_\ell^-, q_\ell^\perp, q_r^+) = \sum_{q_\ell} \int d^d q_r \, F(q_\ell^- + q_r^-, q_\ell^\perp + q_r^\perp, q_r^+) = \int d^d q \, F(q^-, q^\perp, q^+) \,. \quad (4.60)$$

In the first step we use the fact that F is constant throughout each box in the grid picture of Fig. 5 so its the same with the first two arguments shifted by residual momenta. (In the continuum EFT picture its the same property, F does not depend on residual momenta in these components.) In the final equality we then combined the momenta back into a standard dimensional regularization integration as in Eq. (4.59). Essentially at leading order in λ Eq. (4.60) amounts to the same thing that would be achieved by never considering the split into label and residual momenta in the first place, and simply writing down the integrand without ultrasoft momenta appearing in the $-$ or $\perp$ components in collinear propagators, which corresponds to the lowest order term in the ultrasoft multipole expansion (and is an easy way to think about the outcome of the above formal procedure). We have called this rule $1)^{\rm naive}$ because there is one final complication that we will have to deal with, namely that the integration on q_ℓ must avoid producing additional divergences when this collinear momentum enters the ultrasoft regime. We denote this fact by $q_\ell \neq 0$ if q is the momentum of a collinear propagator. These are referred to as 0-bin restrictions.[4] We will discuss the change needed which handles this complication below. Often the results for collinear loop integrals are called "naive" if one uses Eq. (4.60). The result from this naive result will be correct if the added terms which properly handle this complication turn out to be zero, which happens in some cases.

At higher orders in λ there will be dependence on the residual momentum components from higher order terms in the multipole expansion of the collinear propagators. If these terms correspond to the momentum components q_r^- and $q_r^\perp$ that do not appear inside any ultrasoft propagators then the resulting integration is zero

$$2): \qquad \sum_{q_\ell} \int d^d q_r \, (q_r)^j F(q_\ell^-, q_\ell^\perp, q_r^+) = 0 \,, \quad (4.61)$$

where $(q_r)^j$ denotes positive powers of the q_r^- and $q_r^\perp$ momenta, $j > 0$. Here Eq. (4.61) is like the dimensional regularization rule, $\int d^d q (q^2)^j = 0$ for $j > 0$, which is a consequence of retaining Lorentz invariance with this regulator. Eq. (4.61) is the analogous statement in the residual momentum space and ensures that we do not obtain nontrivial contributions from higher order terms in the multipole expansion, unless the residual loop momentum corresponds to a physical momentum for an ultrasoft loop integration. Both ultrasoft loop integrations and ultrasoft external particles introduce residual momenta into propagators that can not be absorbed by a rule like that in Eq. (4.59). If we consider a case with an ultrasoft loop integration, then there will be dependence on the residual momentum also in an ultrasoft propagator, so the integration will give

$$\sum_{q_\ell} \int d^d q_r \int d^d k_r \; F(q_\ell^-, q_\ell^\perp, q_r^+, k_r^\mu) = \int d^d q \int d^d k \; F(q^-, q^\perp, q^+, k^\mu) \,, \quad (4.62)$$

which in general is nonzero. This integrand corresponds to a mixed two-loop diagram with one loop momentum with collinear scaling and one with ultrasoft scaling.

[4] After imposing momentum conservation we get a set of such restrictions, one for each collinear propagator. For example $q_\ell \neq -p_\ell$ if there is a collinear propagator carrying momentum $q + p$.

Finally let us consider the implications of the zero-bin when combining label and residual momenta. Rather than Eq. (4.59) we can have

$$\sum_{q_\ell \neq 0} \int d^d q_r \,, \tag{4.63}$$

where $q_\ell \neq 0$ is simply a label to denote the fact that the label momentum q_ℓ must be large in order to correspond to a collinear particle carrying total momentum q. If $q_\ell = 0$ then the particle would instead be ultrasoft, and we will often have included another diagram in SCET to account for the different integrand that accounts for the proper expansion in this special case. Thus these zero-bin restrictions avoid double counting between the SCET fields, which effectively means double counting from the resulting loop integrations. It is easy to determine what the set of restrictions are for any diagram, since we have one such condition for every collinear propagator. At leading order in λ only the zero-bin subtractions corresponding to collinear gluon propagators can give non-zero contributions since operators containing an ultrasoft quark together with collinear fields are power suppressed. In a continuum EFT these zerobin restrictions are implemented by subtraction terms which can be determined as follows

$$\begin{aligned}
1): \qquad \sum_{q_\ell \neq 0} \int d^d q_r \, F(q_\ell^-, q_\ell^\perp, q_r^+) &= \sum_{q_\ell \neq 0} \int d^d q_r \, F(q_\ell^- + q_r^-, q_\ell^\perp + q_r^\perp, q_r^+) \\
&= \sum_{q_\ell} \int d^d q_r \, F(q_\ell^- + q_r^-, q_\ell^\perp + q_r^\perp, q_r^+) - \int d^d q_r \, F^0(q_r^-, q_r^\perp, q_r^+) \\
&= \int d^d q \, F(q^-, q^\perp, q^+) - \int d^d q_r \, F^0(q_r^-, q_r^\perp, q_r^+) \\
&= \int d^d q \, \left[F(q^-, q^\perp, q^+) - F^0(q^-, q^\perp, q^+) \right] .
\end{aligned} \tag{4.64}$$

Here the integrand F^0 is derived from expanding the integrand for F by taking the label momenta that appear in its first two arguments to instead scale as ultrasoft momenta $\sim \lambda^2$, expanding, and keeping the dominant and any sub-dominant scaling terms up to those that are the same order in λ as the original loop integration. If the original integrand $F \sim \lambda^{-4}$, then this corresponds to keeping just the terms up to $F^0 \sim \lambda^{-8}$, which is often the leading term. (Together with the standard scaling for the collinear measure, $d^d q \sim \lambda^4$ and for the residual measure $d^d q_r \sim \lambda^8$ these two integrands give contributions that are both the same order in λ.) In the last line we combine the subtraction term back together with the original integrand, since the integration variables are after all just dummy variables. This set of steps makes it clear that zero-bin contributions are encoded by subtractions.[5] The scaling for the subtraction is shown pictorally in Fig. 8. The F^0 term subtracts singularities from F that come from the region where the collinear momentum behaves like an ultrasoft momentum. In general when the subtraction integration is non-trivial there will always exist a corresponding ultrasoft diagram where the integration is ultrasoft from the start, which precisely corresponds with the contribution that the subtractions is allowing us to avoid double counting.

In general, when one has a continuum EFT with modes that live in a two dimensional space, such as those in Fig. 8, one has subtractions induced by the presence of modes at smaller (or equal) p^2. Therefore

[5] In fact, an alternate formulation of zero-bin subtractions that avoids the use of notation like $q_\ell \neq 0$ is to note that in a theory with both collinear and ultrasoft modes, each collinear propagator is actually a distribution, like a generalized +-function, that induces these subtraction terms. The fact that we drop higher order terms in the λ expansion when determining F^0 implies that we are making the minimal subtraction that avoids double counting IR singularities. Indeed there in principle could still be a double counting by a "constant" contribution, but such constants will be properly taken care of by the matching procedure. The minimal subtraction also ensures that the matching result remains independent of the IR regulator as desired.

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Spring 2013