MITOCW | 20. SCET Wilson Coefficients

IAIN STEWART: All right. So let's get started. So last time, we had talked about factorization in the effective theory. And there is one type of factorization, which is this hard-collinear factorization, which is a factorization between the low energy physics described by the operator and the Wilson coefficients. And we decided we wanted to think about that in the following way, as having convolutions between variables that appear in the Wilson coefficients and variables that appear in the operator.

And the way that we can think about that is just what the most general possible thing is that we can write down for the Wilson coefficients and of course, it can depend on large momenta. And so large momenta here includes the label operator. So large momenta always show up in Wilson coefficients and in this case, that includes momenta that are the large component, the order lambda 0 component of collinear fields.

So this is hard-collinear factorization and it just comes out in a natural, standard way in this effective theory because we've set up the effective theory to have the right low energy degrees of freedom. So even though it's a little more complicated than just a simple product, we see what the variables are that can connect the two things just by power counting, essentially. The large momenta are the ones that are order lambda 0.

We can do this generically if we recall the definition of our Wilson line. We can see how this would carry over to a generic case in the following way. So that was one way of defining the Wilson line.

And as an operator, we had relations as well. So we had this relation and in particular, we could take this to any power and that just takes the p bar to any power like that. So you can really think of this as given any function of this operator i m bar dot d you can always write that as a Wilson line, a function of p bar, and a Wilson line. And what you want to do is you want to stick these Wilson lines in the operator and you want to put this function in the Wilson coefficient.

So you could think, lets me start with operator. I can throw in i m bar dot d's because these guys are order lambda 0, these are collinear derivatives. I could just allow it to stick as many of those as I like in my operator. But given that I put in any function of them, I could always do this and the right way of thinking about it is that this function here is determined by matching. It's your Wilson coefficient.

And that's what we were effectively doing up here. But in some sense of our general discussion you can see that even if you had multiple places in the operator where you could insert these derivatives, then you would just do this. And what would happen is at the end of the day, you get a function of all the possible large momenta that you can form from the operator.

So maybe one more line. So think of it, you can always separate out. Make a split like we did over there with the integral over some variable and then leave something that you can stick in your operator, make it like this. So this could go in the operator and then that's the coefficient.

AUDIENCE: If I stick this in [INAUDIBLE]?

IAIN STEWART: Well, here I didn't write it. So if I wanted to put in i m bar dot d's, I'd stick them between. Right? If I wanted to use the formula. If I wanted to use this kind of logic I wouldn't have written this and I would have said let me stick an arbitrary function of i m bar dot d in here.

AUDIENCE: But Between before you even write--

IAIN STEWART: It has to be between them because of the gauge invariance.

AUDIENCE: Right. So before you write down the Wilson line. So say you start and say, OK--

IAIN STEWART: Oh, yeah, so there's still-- right. Right, right, right. This Wilson line here came from this h. So that's still going to be true. And what I'm saying, if you think about this operator, I could dress it up by putting any function of i m bar dot d's right in here because that would still be same order and gauge invariant.

AUDIENCE: OK, but this an alternative--

IAIN STEWART: But then if I use this formula, it would sort of push this w. That would cancel there and then this w comes back and then you get the if. OK. So in general, the right way of thinking about this is as follows. We can encode this in some notation by just setting up a convenient set of building blocks which are gauge invariant objects under the collinear gauge transformations. And so we need to have a fermion field, but we know that the fermion field, generically we can make it gauge invariant by multiplying by Wilson line like that. And because of this, what we were just discussing here, generically, we can be in our Wilson coefficient, sensitive to the momentum of this object. And so we can denote that by defining the following thing, which is just a chi field that carries some momentum which is the overall large momentum of this product of fields. And sometimes this guy goes by the name of the quark jet field because if you were to produce a quark in the hard scattering process, the quark would be represented in your operator by this w dagger c. The Wilson coefficients would talk to that quark through the large momentum and that quark, in the low energy theory, would evolve into a jet. So it goes by that name. You could also think of it as a parton field because as it turns out and as we'll talk about later on, if you were to think about the parton distribution function and what pulls a quark out of the proton, it's exactly this operator. And then that quark's momentum, what momentum that quark carries is exactly picked out by this delta function as well. So these are the objects you'll usually want to work in terms of.

We can do something similar for the gluon, which is a curly b and I'm going to define it. If we wanted to get a gluon, the natural thing is to use a field strength because then you get a gluon without any derivatives. But I want my object to be dimension one, so I'm going to define it in the following way. So here's a field strength commutator of covariant derivatives, these are collinear derivatives. And I throw Wilson lines around it to make it gauge invariant, and then to make it dimension one I throw in a 1 over p bar. And if you start expanding this out, then the first term will be the a n perp gluon and so you have this b just has a n perp gluon in the same way that the first kind of order term in this chi is just to the fermion if I set the Wilson line to 1. OK? So it's starting out as a perpendicular gluon but it's dressed up in a way that makes it gauge invariant in dimension one. And then just like we did here, we can put a subscript on it with a omega to say that we fixed the large momentum of it and sometimes there's a convention that that's done, you have to decide if you want the Wilson line-- I mean, the delta function to be a w minus p bar or w minus p bar dagger. Anyway, it doesn't really matter that much because it's just a sign of what you mean by a w. If it's an outgoing gluon, this is a more convenient convention. OK, and so this delta function here and this p bar here, when I put these square brackets, what I mean by that is that they don't know how to act outside. They just act on the operators inside. So these are objects that exist by themselves and they don't care about other things that are multiplying them later. So that's the gluon analog of the quark.

So it turns out that we can show the following. That if you want to build operators that are subleading order, a complete set of things to do that is as follows. It's this chi n, the bn perp, and then you could also need p perps. And then you could also have ultrasoft fields, where it's really kind of similar to how you're used to building operators in effective field theories. But if you just want to talk about the collinear sector, the only things you need are this chi n this bn field, and then p perps and no other things.

AUDIENCE: That's to go just one order?

IAIN STEWART: All orders.

AUDIENCE: So why aren't there any--

IAIN STEWART: I'll talk about it. Yeah. I'll show you.

AUDIENCE: OK.

IAIN STEWART: Yeah. I mean, intuitively you would think these are the physical degrees of freedom, right? There's two of them so that's intuitive. But I'll show you how you get rid of all the other ones. It kind of matches up with what you want. You could think that these are the physical gluons that you're producing and your operators can be set up so that there's not any spare use components. OK. So we'll get there.

So let me introduce a bit of notation. I have to do. So let's consider this thing. A covariant derivative sandwiched with Wilson lines on either side and this i dn here is like p bar plus. So it just involves the collinear gluon field. So then if I have that operator and I just take n bar i dn, then that's p bar. if I take i dn perp mu, then you can show that this guy is p perp u plus g bn perp mu.

So this one is just a relation we talked about before. This one is not as obvious, but if you take this guy, you can let this derivative p perp act on the Wilson line or act through. If it acts through, that's this term. If it acts on, then I can manipulate the operator so that it is exactly this form. And part of that comes from the fact that-- I can explain it to you here. I have it in my notes and you can look at it later, but let me just explain it in words.

If you looked at the term in this commutator that was other order, where the i m bar dot d was sitting here. It would hit the Wilson line. You'd get 0. So that term was just put in to make it look like a field strength. OK? So really you have i m bar dot d dn perp, without the comma, without the brackets.

But then this combination here, you can use the identity that we had with it and basically if you push this guy through here, you're canceling the p bar so it's just giving you the Wilson line. So that's basically how you go from here to here. OK? So basically what this is saying is I don't need to consider covariant derivatives because I can instead consider p perps and b perps and that's equally good.

OK. Now if you do a similar type of thing in the n component, then you can derive a similar type of object. So this is the same object as we have over here, but instead of having an i dn perp, I'd have an n dot d. So d dot n perp replaced n dot d. And that looks like it could be something that you'd build operators out of and and also, furthermore, why not have operators that depend on this n dot partial? OK? So for p bar, we don't have to worry about that.

So OK, I said we'll include this, we'll include that. I didn't say we have include this, this, and this. So there's three things I have to argue away here. The derivatives that are p bars, those just go into the Wilson coefficient. So if we had p bar chi n comma omega, for example, then that's omega chi n comma omega and this is put into the Wilson coefficient. So that's why we don't have to worry about having p bars.

In some sense, we do have p bars. They're all in the Wilson coefficients. So it's really these other two that we have to worry more about and those actually can be simplified using the equation of motion. So if you have i n dot p partial on chi n, then equation of motion when you write it out in terms of these objects, it has some form and basically, you can just get rid of those terms.

So the equation of motion allows us to get rid of i n dot partials that are on chi n's. So that's why we don't have to worry about those. And this is just like saying that in our leading order action, there wasn't i n dot partial. But that's just like saying there was time derivatives in our leading order action in some standard effective field theory.

But then in the higher dimension operators, you can always use the equations of motion to get rid of those time derivatives. And here we're using the equation of motion to get rid of i n dot partial. So it appears in the leading order action, but then we don't have to have it in any other subleading operator and that's why it's not one of the ones that's included in the list. And if you had i n dot partial on curly b, that's also part of the equations of motion of the gluon field.

Now when you do the gluon equations of motion, there's components because it's a vector and one of the other components allows you to get rid of the n dot b. So there's another term that you can rearrange, and I won't write it out because the equation is rather messy but give you some idea. There's another component that looks like this where I can get rid of all the n dot curly b's using the gluon equation.

So the gluon equations of motions allows me to get rid of both of these things and basically after I've done that, I just have the objects that I've told you we can use. So after using equations of motion, we can get down to those objects. And any other thing that you might dream up can be reduced to these objects. So I'm not saying that I went through a complete list here.

For example, what if you had a commutator of 2 d n perps? Right? You might say, oh, that's some new thing. It's one of these. You can also reduce that too. So let me list that one just to give you some idea that there's others you might think of dreaming up.

So you can reduce all of this to be that. OK, so for the collinear sector this is enough to build higher dimension operators, just these three. And then for the ultrasoft sector, so I guess this is two. We do need ultrasoft derivatives and ultrasoft field strengths and ultrasoft quarks.

So this part is really just similar to the story that you'd have for a standard low energy effective field theory. Like integrating out a massive particle, you can use the equation of motion. One thing that is worth commenting about is the connection between one and two. So one is collinear and two is ultrasoft and prior you might think, well, they're totally independent. But we saw last time that reparamaterization invariance connects them.

So if I've decided that this is the type of basis I want to use for my operators, then what is the reparameterization connection? You can rewrite what we said last time in terms of these curly d's because it basically just means moving the Wilson lines that we had in this formula last time. They were around the ultrasoft operator last time and if I just move them over to the collinear operator, then I get this curly d. Right?

So the RPI connection is connecting the curly d collinear to the d ultrasoft and then you can write this guy as the p perp plus the b perp. And likewise if you do the same for the n bar sector and you move the Wilson lines from this term to that term, then it looks like these two combinations. So that's just rewriting what we had before but now in terms of this type of notation. OK? So that's enough to build operators at higher order and then we write down Wilson coefficients for those operators that are functions of a large momentum and then we start doing physics with them. So any questions about that?

AUDIENCE: For the equations of motion, you always just use the leading order equation of motion--

IAIN STEWART: Yeah.

AUDIENCE: --into a higher order term?

IAIN STEWART: Yeah, these are the L0 equations of the motion. That's what I'm doing here or writing here. Yeah, there's one more actually. There's three but it's only needed at some very high order. OK. So the next thing I want to talk about before we start doing explicit examples and going through processes is how loops work in this effective theory.

And we're going to have to come back and talk about our grid that we have for the split up of momenta and how should we actually think about it in practice and then we'll also deal with how matching and running work. So I'm going to do this in the context of an example and again, I'm just going to pick the simplest example that has only one jet just to make our lives a little bit simpler.

So we'll consider our heavy, light current and I think actually once you see how it works in this example, you'll understand what all the general features are of doing loops in SCET. So we had operators that we constructed, lowest order operator. And let me write it in this way which was prior to making our field redefinition. I could just write it this way if I want.

And gamma for beta s gamma is a tensor operator. And there's a photon field, which is a field strength, f mu nu. So let's just think of that as all part of gamma. OK, and that's also appearing here and I could use the spin structure properties of these things to reduce the sigma and mu nu but that's not really going to be part of our story so let's not bother with that.

So what I do? I just compute the QCD loops and the SCET loops and I compare them. And if SCET is the right effective theory for this limit where I have an energetic photon and am back to back with an energetic strange quark, then I should match all the infrared divergences in that QCD computation, I should be able to extract a matching coefficient, I should be able to determine what the C is from that calculation, and I should be able to run the operator. I should be able to make this into m s bar, do some renormalization of this operator, and then do some renormalization group evolution of that Wilson coefficient.

Well, let's just think about computing the graphs. So we have to decide, when we compute the graphs, how to regulate them? Right? And we need to use the same infrared regulator in the full theory and the effective theory. So here's how I'm going to regulate them. We could do this different ways and it's useful to understand that various answers that we get are independent of the regulator.

So I'm going to take p squared not equal to 0 for the strange quark. So there's infrared divergences associated with the strange quirk and I'm going to take p squared not equal to 0 for them. I'm going to use dim reg for the heavy quark.

So I could take the b quark to also be offshell. That would make the formulas even more complicated. So instead of doing that, I'm just going to allow epsilon to regulate the b quark. These guys are pretty easy to track so that won't be a problem. And I'm going to use Feynman gauge. I don't have to do that, but that's a nice gauge for doing calculations.

So what are the diagrams where I have my photon and I have a vertex diagram like this and then I have wave function renormalization? And so let me, in the usual kind of cavalier way, denote the wave function renormalization, which is just the multiplication by the appropriate z factors. My diagram's like that. And this is a standard QCD calculation and we can carry it out.

And what does the answer look like? So this diagram here has double logs and single logs that involve p squared, which is our IR regulator, and then it has some terms that are finite, which I'll just note by-- so p here is the momentum of our strange quark going out and pb is the momentum of our b quark coming in.

And we're not going to talk much about these finite terms but what I mean by finite here is terms with no IR divergences. The IR divergences are these single logs and double logs and then there's some remainder that I can write that way. I'm expanding it for small p squared. So p squared is not equal to 0 but I take the limit p squared goes to 0 and then that limit, these are the IR singularities.

And then there's some function of b dot pb, which is just an order one thing. And then b squared, of course, is pb squared. So that's sort of the remaining kinematic variables this could possibly depend on.

AUDIENCE: [INAUDIBLE]?

IAIN STEWART: No.

AUDIENCE: Is it finite?

IAIN STEWART: It's really finite. So yeah. Yeah, so let's see. Yeah, I guess I carried out the so there is a z ten for the tensor and when I write this-- yeah, let's see. So, OK. Let me do this. I think it's better. So I added up all these graphs and I want to add one more. So there's a counterterm for the tensor field.

So let me change what I was going to say and do it this way. So the sum of these four graphs is this. OK? So there's no UV divergences. And I'll just tell you what the z factors for these three are, then you could figure out what this graph is by subtracting the two.

So the tensor current in QCD has a z factor, looks like that. There is a z for the heavy quark. And if I include the finite residue as well as the divergent pieces, this looks like this. I think I'm going to have to make one adjustment to my formula here.

Let me just fix something here. Yeah. So if I want it to be the way I say, then what do I have to do? So this guy should be three halves and there will be one more divergence. I think that's right. So this 2 over epsilon ir here is that 1 over epsilon ir and the UV renormalization is taken care of once I have the z tensor.

There's divergences in this diagram in these ones. but there's one left over, and that's taken care of by z tensor. So there's no 1 over epsilon UVs after I take care of this guy. So that's just by the definition of what z tensor should be and then everything else is as I wrote. So there's either divergences that are associated to this strange quirk offshellness, that's these two terms.

There's an IR divergence associated, a heavy quark going onshell, that's that term. And that's the sum of diagrams. OK, so what about SCET? So there's going to be, in SCET, collinear diagrams and ultrasoft diagrams.

So I'm going to use Feynman gauge for everything again. This is not something I have to do, but this is what I'm going to do. So let's start with the ultrasoft loops. So there's a vertex graph. So using the notation that we've adopted where the collinear quarks are dashed and the heavy quarks are double lines, we have a diagram that looks like that. There's some free factor.

Let's focus on what the loop looks like. So this loop here, the k that's going through this loop is just a residual k. There's no label k for this loop because it's an ultrasoft gluon. So when we write down all the terms in this loop, it's just standard field theory. There's nothing special about it.

Since I'm taking the strange quirk offshell, the propagator that I get is this shifted iconal propagator where basically the fact that it's offshell it gives me this extra term there. That's regulating some minor divergences. And we want that because we want to regulate the IR divergences in the same way in the full theory of the effective theory. This is proportional to that and if I put in all the factors.

So it does have double logs of the p squared, doesn't have single logs of the p squared, and actually if we look at the double logs, the coefficients also don't match with what we had over there. And so there's going to be some other diagrams that are going to involve double logs of p squared as well. One thing that we can note here is that if you think about the scales in the problem, remember that our loop integral was totally homogeneous in the power counting, the lambdas were totally homogeneous.

And if you think about p squared scaling like lambda squared, which is natural size for an external collinear momentum, all right. If p squared scales like lambda squared then so does p squared over m bar dot p. And this is a dimension one thing and this is the ultrasoft scale, right? For some ultra soft momentum and so if you want to look at these logarithms and you ask, what scale is this effective field theory diagram sensitive to? It's sensitive to the ultrasoft scale.

So the logs are not large logs, they're order 1 logs as long as mu squared mu is of order lambda squared, which is the scale for the ultrasoft momentum. And that's what we expect from this ultrasoft diagram, that it's telling us about physics at the ultra soft scale and that's what we see setting things up, doing the calculation. So you could also think about a wave function diagram with an ultra soft gluon, but this guy 0's out since in Feynman gauge you just get n mu n mu, which is 0.

So there's no so z for the collinear quark field from an ultrasoft loop is 0. That wouldn't necessarily be true in some other gauge, but in some other gauge this diagram would also change. All the diagrams would change. And so if there was a non-zero contribution in this diagram, it would just be taking care of gauge invariance. And then finally, there's ultrasoft loop on the heavy quark.

And that's an HQET diagram, has nothing to do with the collinear quark. So from that diagram we would just get the z factor, which is the appropriate z factor in HQET with r regulator. If I specify ultraviolet and infrared divergences, that's this, and this infrared divergence here is actually exactly the same as the one that we had over here. OK, so the ultrasoft sector, there's nothing really tricky about it. It's just write down the diagrams, do the loops.

AUDIENCE: So for the [INAUDIBLE], isn't that IR divergences of those diagrams cancel with every automation [INAUDIBLE]?

IAIN STEWART: So what are we looking at, right? So it depends on what we're looking at here. And so, yes, in general that would be true, right? If you were calculating some cross-section, which was IR finite cross-section and that, of course, would depend on defining what you mean by measuring this quark. Right?

So the IR divergence would become some physical scale like the mass of a jet. The IR divergences would turn into something physical if you put this into a physical cross-section. And that's exactly basically what would happen. These p squareds would become the mx squareds that we talked about when we talked about beta s gamma. But here what we're interested in doing is a matching calculation.

So we fix the external state, still have a particular number of partons, and we want to compare the full theory calculation with the effective theory. The effective theory should reproduce the infrared divergences. So really all we care about is not that this is infrared finite, but rather that the effective theory has the same infrared divergences.

And we'll see that in the end when you can think about is rather than canceling in for divergences as you're thinking in the full theory, we've matched the full theory onto the effective theory and that matching gives us the Wilson coefficient. And then we take effective theory and all the cancelations that you're thinking about between real and virtual graphs will occur in the effective theory too. So you can just think about the effective theory virtual-real graphs, then the cancelation will take place there.

But then you're thinking about that cancelation later, at a lower scale, which is what you actually want to do. Because what I just was telling you about IR divergences becoming different things in the final state becomes very trivial once you're in the effective theory and we'll see sort of exactly how that works later on. But first we have to talk about linear graphs.

So in the collinear graphs we had graphs like this one where we can take a gluon out of the vertex here. That corresponds to taking it out of the Wilson line. So let's label the graph in the following way. This would be k plus p this will be p This will be k. And if we follow our rules for what this is, we would write a sum over labels and then an integral over residuals. And let me put the residual integral in dimensional organization.

Let me try to squeeze in everything, which won't be possible. So I'm going to write out the components to make clear whose residual on whose label. So in the denominator there's one more term. So the plus guys are always residual, the perp and the minuses are always labeled. The short way of saying what I'm trying to squeeze in here.

So the denominator has these three terms, this guy here, this guy here, and this guy here. And when I label the diagram like this, you should think that each of these has a label and residual part. So k you can think of as a pair, k label, k residual for now.

And remember the importance of doing this. The importance of doing this was related to always being able to identify what the lowest order term was on the right here. OK? And that actually becomes more important once you start to think about diagrams where you would add like an extra ultrasoft gluon somewhere in this picture. Then, of

Course when that ultrasoft gluon feeds its way through this loop, you gotta make sure that it's only the lowest order piece that's showing up. In this case here, we just have a collinear loop and we don't have any sort of real ultrasoft momenta from ultrasoft particles besides the heavy quark, which is just an external particle to the loop. So what we want to do with this is we want to turn it back into an integral.

Just let me say it this way. And if it wasn't for these restrictions here, then that's very easy, actually. and then I'll talk about why it's true. I claim that if we ignored the restrictions and those restrictions are ensuring that we don't double count between our collinear and our ultrasoft degrees of freedom. So they are important, but let's ignore them for a minute.

If we do ignore them, we would just get the following. Where I just basically stop thinking about residuals and labels, write everything as a full momentum, and write down exactly the same thing I just wrote. So this is a full k squared and this is a full k plus p squared. OK? So what I'm going to do is I'm going to first ignore these restrictions and then I'm going to tell you how it would work, how this actually does turn into this. What are the rules for doing that. And then I'll come back and I'll tell you what these extra conditions do.

So really this looks like maybe it's trivial, but we should think about it. And it's almost trivial but not quite. So really what we're doing here is we're combining back together labels and residual momentum, right? And the place that we have to worry about that is in the perp in the minus space. And recall we had the grid and the grid is sort of our way of guiding, our guidance to see how to put things back together.

So we had vectors that lived in this space and this is the label and then there's the residual, if we want to point to some place in that space. This picture was like a Wilsonian effective field theory because the picture makes you think of sharp edges. But the real effective theory that we're doing is a continuum one and so you have to expand your brain a little bit and think that each of the boxes in this picture is actually an infinite space as well, because the residual space doesn't have restrictions like that that would spoil Lorentz symmetry.

So each grid point really is specifying an infinite space of residual momenta. And it's R4 or Minkowski space, so the momenta components are real numbers and there's some rules. So what are the rules? So I'll tell you what the rules are without restrictions for now and then we'll come back and I'll tell you what the rules are with restrictions.

So rule number one is the simplest, and it just says that say I had the following, I'll use a one dimensional notation. Say I had a sum over kl's and an integral over all kr's. Well that's just the same as not having split it up and doing an integral over everything. Because we split this thing into boxes and if we really integrate over all boxes and sum over all labels, then we should just get back the full integration over all the momenta.

So that's true for each, for minus and perp momenta. But really what we have to do is we have to do this with some integrand, right? So the type of integrand that we have is following. If you think about the components that we're talking about here, which are the minus and perp ones, then in our integrand up there, there's no minus or perp residuals.

So it's just a function of the labels. Right? And so what that says effectively is that this is a constant function in this box, in each of the boxes. And effectively, the way that you use this formula is you do the following. You say, well, if it's a constant function in the box, I could evaluate it at a different point in the box and I'd still get the same value.

So in particular, I could evaluate it k label plus k residual and then I can use that formula there to say that this is just an integral with a continuous k of the function evaluated at that continuous k. So that's the way that number one works. We are integrating functions that are constant in the boxes and so then it's kind of trivial, how to put the boxes back together. Yeah?

AUDIENCE: So this d kl division doesn't have kl plus--?

IAIN STEWART: Yeah, so what I mean by it is-- yeah. So I'm using here for each minus or perp momenta, one dimensional notation. Yeah, and so I have three of these. I have to do this. And for each one of them it's true that what I'm saying. But of course, if I tried to write that on the board then it would confuse the point, I think, which is, in some sense, a simple point. OK? So this is, in some sense, saying that this whole split up that we were doing was not really needed.

We could've just written a continuous integral and that's kind of what this is saying, right? We could have just written a continuous interval and not worried so much about all the split up that we were doing. The place that we have to be careful about is these restrictions, and we'll come back to that. And the other place that we have to be careful about is when you have the multiple expansion. But as long as you take care of that, then basically this is always happening.

So the reason why this works is the following. For every label loop momentum that there is in any diagram, there's always going to be some corresponding residual that's not specified by delta functions in terms of external momenta. And effectively, therefore that we can absorb in order to do what we just said.

So we can always go back from this discrete type notation back to a continuous notation. The discrete notation was just helping us to set up the expansion and be careful about it, but we can always go back to the continuous one because there's always a kr that has this property. OK. So now in general though, you might have some more complicated thing. And if I'm going to give you some rules I should give you a complete set.

So we have to append our list of rules by the following one. So if I thought about doing what I just said over here but I went to some higher order, then what could happen? Well then these kr minus and kr our perps could show up. They'd never show up in the denominator of our propagators if they were just collinear propagators, but they could show up at some point in the numerator.

And we need actually a rule like this one, which would be clear if we were regulating d kr in dimensional regularization, that the power of divergences are getting set to 0. And that is basically to maintain the run symmetry in the residual space we need a rule like this for j greater than 0. So that doesn't come into our calculation, but included for completeness.

So when would these integral kr actually do something non-trivial? It would do something non-trivial if we had an ultrasoft loops and collinear loops at the same time. So in the case where we just have collinear loops, it's basically up to this issue about the restrictions that we'll talk about. It's basically that we just could have done everything as continuous and ignored this split.

But if we have both ultrasoft particles that are participating through the loops and/or then in general these will give non-trivial loop momenta in the residual momenta. And hence there will be some that we can't just absorb in the fashion that we just said. So there will be, in this situation, residual momenta. Some residual momenta will be absorbed in the same way to turn the integrals into continuous ones, but other ones won't be absorbed.

And that's because the ultrasoft propagators rate would involve the lr plus, the lr minus, and the lr perp in the denominator, so we don't have this rule to apply. We can't do what we said here with the constant boxes because now the functions are depending on that variable. So we just have to do the integral. So you could have something that looks like just in a schematic formula. Let's have there be 2 kr and an lr.

We're in kind of an obvious notation. I'm saying that the function could depend on lr residual, it doesn't depend on kr residual. So we absorb kr back into the sum to make it continuous and then this integral we just have to do. OK, but we're still, in the end, just doing an integral. So this guy come from ultrasoft propagators, for example. OK?

So those are the different cases that you can get. But nevertheless, even if you have ultrasoft particles and propagators floating around, there will always be a residual momentum associated to of what we were doing over here that you can absorb in the same way I stated, so let's proceed along these lines and see where it takes us and then we'll come back and put in this additional restrictions.

AUDIENCE: Do we ever have to do a discrete sum? Like is there a sum over ll?

IAIN STEWART: Yeah, so you never have to do a discrete sum.

AUDIENCE: That's good.

IAIN STEWART: Yep. The discrete sum is really just a way of thinking and I'll show you in a minute that you even can avoid thinking about it once you know what to do. So it's guiding you towards the right answer but it's not really something that you have to think about this grid picture. It's really just if you get confused, you can always think about it but if you're not confused, you don't have to think about it.

OK, so this guy is what we said, proportional to what we said. It's just in this case, the reason why it was so simple is because k residual is one component of this guy, and then there's these other three, right? And the other three, we just absorb the sum and the integral back together and there's nothing further to talk about. So that's why this is, in some sense, very simple.

So we just do that, and what do we get? Get some 1 over epsilons, we get some logs of p squared, and I'm putting in even the constant term since it's pretty simple. 4 minus pi squared over 6.

So if you think about where this came from, there was a place that this Wilson line came from was attaching this guy over here and then integrating out that offshell propagator. So this is a vertex diagram. We had a vertex diagram, which is an ultrasoft gluon, and now we've added an identical type up topology. Except we drew it different because this line was offshell so we drew it as a Wilson line and that's the right way of thinking about it. But if you think about where it came from, it was the same topology and we've added another vertex diagram.

So the effective theory has two type of vertex diagrams, this collinear one and the ultrasoft one and we're going to add them together. And in this one, you see the logs are minimized at a different scale, mu squared of order p squared, which is the right scale for a collinear loop. Because remember, the collinear particles lived at a larger p squared than the ultrasoft ones.

So there's a larger scale that would minimize these loops, it's a collinear scale. So the effective theory is capturing the physics at that scale. Squeeze in another diagram here. If we do the collinear wave function renormalization, then this is non-zero. This was 0 for the ultrasoft gluon and but for the collinear gluon, it's non-zero and it's exactly, actually, the same as the full theory. It's the same as massless QCD.

So you can do the diagram using the effective theory Feynman rules, then that's what you find. But you could also understand why that's true. And reason why it's true is nothing in this diagram really specifies a frame. We called all the particles collinear, but it's not attached to anything.

So we could just take the whole diagram and boost back to the frame where everything's kind of soft and then we would usually be thinking about it in terms of a full theory field. So that's effectively why this diagram like this that doesn't have any reference, unlike this one which has a reference because it's attached to the heavy quark. That's why it's the same as QCD. Yeah?

AUDIENCE: How do you know what the epsilon position for the Wilson line is set?

IAIN STEWART: Oh, for this one?

AUDIENCE: m bar dot k.

IAIN STEWART: Yeah, for this one. Yeah, for this guy it actually doesn't depend on the i epsilon prescription up to a-- yeah, if I remember correctly. Yeah. That's true, I believe, when I'm being a little bit cavalier with it but once I put it the zero bin restrictions, I'm not sure if that's true anymore.

AUDIENCE: OK?

IAIN STEWART: Yeah. I mean, really what solves that is that in a minute I'm going to be talking about the fact that there is a subtraction term here and once you put the subtraction term in, that i0 is not relevant. But I think even if you do this diagram with arbitrary i0-- because what's happening is m bar dot k going to 0 is related to some of these 1 over epsilons and we'll talk about that more in a minute. Yeah, I'm not 100% sure. It could be that there's some sign that might flip. I'm not 100% sure.

AUDIENCE: Even if they get the wrong side, the difference would be subtraction.

IAIN STEWART: I guess I know that-- yeah. What I know is that-- yeah, let me answer your question later. It'll be easier because I'm trying to say a bunch of things that depend on something else that I haven't explained yet. So what are the other possible topologies we could write down?

So we could write down this one, where we take two attachments in the Wilson line and just loop them back up but that's proportional m bar squared and so that's 0 in our Feynman gauge. And likewise, there's a looping back up in the vertex in the wave function renormalization, but this guys scale is power law divergent and so we can just set it to 0 and dim reg. You don't have to worry about it. OK, so that's all the diagrams.

Let's think about doing matching, i.e let's think about comparing QCD and SCET by adding up the diagrams. In QCD we carried out the randomization, we added the z for the tensor current. Let me just write again the answer looked like. In SCET we didn't carry out renormalization yet, so let me call this the bare SCET result for now.

Once we add the ultrasoft and collinear diagrams together, the logs of p squared match up exactly with the full theory. So this is the first sign really that it makes sense to be thinking about adding these loops. Even though they were the same topology, we are correctly reproducing those logs of p squared in the full theory.

And then there's some other pieces. I'll write out all the other pieces so you see what they look like. Well, maybe I won't I won't write the constant. So there's all the effective field theory terms. So these terms here we can match up with these terms here. So that's good.

These terms here, which remember in the full theory were finite, and these terms here, which in the effective theory, are finite, the difference of those is going to give the Wilson coefficient. Now we said that the Wilson coefficient could be a function of p bar, so what's going on with that? Well, if you look at momentum conservation in this process of beta s gamma-- so I probably should've said this earlier.

So when you look at beta s gamma, if you look at momentum conservation then the p minus of the strange quark has to be equal to the p minus of the b quark, but that's just mb. OK, so actually p minus is equal to mb by kinematics. So mb's in this result here you shouldn't think of as p minuses, and that's this p bar that was in our Wilson coefficient is just getting set to mb because of some delta functions that are specifying kinematics. So that leaves the 1 over epsilon terms.

And so what we'd like is that those terms are associated to renormalization. Of the effective theory, right? I wrote that the effective theory was bare. But if I want to do that, then I have to ensure that all these epsilons that are appearing here are really ultraviolet divergences. If they're infrared divergences, then doing that doesn't make sense. And that's actually the remaining issue that we have to deal with.

I just wrote epsilon, that means I'm ignorant to what they are. And if I knew this one was epsilon IR because it came from the wave function renormalization of the heavy quark and that was the same on both sides. It was the same diagram, it was the wave function renormalization diagram in the full and the effective theory. So I could match up that one.

These ones just came out, but it turns out that so far with what we've done, some of these epsilon here are IR. And so the IR divergences aren't matching up and the reason is because we didn't put in those restrictions on our sum over labels.

OK. So we have these restrictions, k label not equal to zero and k label not equal to minus pl. Those are the restrictions that I'm talking about. The place that those restrictions came from was k was the momentum of the gluon. kl not equal to 0 is saying that this is the restriction that the gluon is collinear. Because kl equals 0 is the ultrasoft gluon and this is the restriction that the fermion is collinear in the loop and that's why there was two of them.

So these are called zero bins. Zero because it's where the ultrasoft momentum lives and from the point of view of collinear, that's zero. Imposing these restrictions is removing the zero bin, if you like and what these restrictions do is they avoid double counting and the way I've said it that's, I think, clear.

So far in our calculation we haven't avoided double counting and that's the problem. OK. So we have to modify our rule or we extend our rule to include the case where we have these restrictions. In an extended version of rule two that house restrictions.

So really what we want to do is that and we want to think about that as an integral. So here's how we can manipulate these to think about it as an integral. That sum over all kl's, but then we'll subtract the limit of this f where we take the f and we let the kl go to the place.

So we integrate over everywhere, including the place we don't want to go, and then we subtract it back. So what is this fl of k? This f of kl goes to 0. it's defined by taking the scaling limit of the collinear momenta towards the ultrasoft.

So you take your collinear momenta, which are the minus in the perp here, and you scale them towards an ultrasoft momentum in whatever components, i.e you start counting the kn's as order lambda squared and you keep the leading order piece or you keep the piece that's the same size as this term. And then that defines what this f is. Once you take that limit and you expand, then that's what the f is.

So what you're doing here by doing this procedure is you're basically setting things up so that this guy is integrated over in a way that we can combine back into an integral. But then we have to subtract an overlap of when that integral would go into the ultrasoft region. But the overlap we're subtracting is also an integral, so we have a difference of integrals. And you should think of the second integral as integrating over the square where the zero bin was.

So if you think about our picture where the collinears were up here, ultrasofts are down here and you thought about there being some box, you're taking this scaling limit when this guy goes down into that box. You add up all the boxes, that's this, and then you subtract out that box again. And that avoids having a double counting in that region. So then this guy here, you can do the same kind of trick as before.

So continuing with the equation. kl is going to 0, well, we can think of it just as a function of kr's. And so effectively what we get in the end is an integral over all k of f of k, the full f, evaluated with a continuous momentum minus some f that's expanded and avoids there being overlap in this box.

So rather than having these discrete sum with the restriction, we have a difference of integrals and this subtraction term avoids the overlap in that region. So all the discrete sums are good for is a means of figuring out what limits you need to take to generate these subtractions. Once you've done that, everything's a continuous integral. And this is called the zero bin subtraction.

OK, so if you like, one way of phrasing what's going on is that the collinear propagators are really distributions. They're distributions that know that they should have a subtraction in order to not overlap the other region where we have another degree of freedom. So you can think about it that way and having this sum it's just a way of encoding that. But in the end, it looks like some kind of plus function where you have a subtraction.

AUDIENCE: So it seems like in the second term you would have to relearn the expression because initially in the [INAUDIBLE] k you ignore kr minus and kr perps.

IAIN STEWART: I'll show you how it works in an example up there. So let's go to our example. Yeah, so what you just said is not quite the way you should think about it. You should think about it that you've given the effective theory f and now I'm saying that that f, as given, with its momentum as given still has an overlap with the ultrasoft region that I want to subtract. So I'm taking a limit of that f, I'm not add anything back to it. So this guy, integral dk.

So this is what we wrote before. And now if I take the ultrasoft limit of it, of the k, then I would write this. OK? So when I take the scaling limit, if you think about k squared, k plus, k minus, minus k perp squared, when I take the scaling limit of all momentum being ultrasoft, the components of the k are still homogeneous. So there's nothing there to expand. Some expansion happened in this k plus p term.

One of our n bar dot k changes its power counting, but it's still n bar dot k. There's nothing to expand there. So this is what the subtraction looks like and then the numerator, the n bar dot k can be dropped relative to the n bar dot p, which is large and external. OK, so this is taking the ultrasoft limit of this.

AUDIENCE: Do you have to subtract off the the ultrasoft limit of the quark as well?

IAIN STEWART: Yeah, so in general I would have to subtract off the ultrasoft limit of the quark as well. And when I do that, what I find is a term that's power suppressed and so I drop it. But in general, I would have to do that as well. That's right. And so it looks like an ultrasoft diagram except it's got this n bar dot k inside of the v dot k that we had in the ultrasoft diagram. And if you do a power counting with the loop momentum scaling as ultrasoft, then you have this piece is of the order the same size as this piece and that's why you keep only that term.

AUDIENCE: Do you have to worry about a higher power?

IAIN STEWART: No. So the prescription we have is that we drop the higher powers

AUDIENCE: But if I were to do say--

IAIN STEWART: Oh, if you did the higher power--

AUDIENCE: --lower power higher power zero bin?

IAIN STEWART: No. So if we did the higher power, then this would start out at higher power. And then when we took the limit of it, it would end up just starting at that power or higher.

AUDIENCE: Right, but there was a--

IAIN STEWART: Yeah, you don't have to. No.

AUDIENCE: No?

IAIN STEWART: No. Yeah.

AUDIENCE: Is there a reason?

IAIN STEWART: Yeah, so really what you care about subtracting here are the log divergences and that's what this minimal subtraction is doing. By keeping the piece that's scaling the same way, you're removing the log divergent pieces. And it's the log divergent pieces which are giving one of our epsilons. The pieces that you would get from the higher expansion, they would all be kind of like power law divergent terms from the point of view of the power counting and we just don't have to worry about those.

And another way of saying it is, it's not that I'm removing absolutely this whole integrand in that region, right? There could still be a constant, for example, that comes from that region. But if there's a constant that comes from that region, I don't care. What I care about removing is any spare use IR singularities. And for those I can make a minimal subtraction, which is just the first term.

All right, so I want to finish this discussion. So if we do this, we get an answer, which I will try to write on the board for you. So now I'm going to distinguish all the epsilons and then we'll see what this subtraction does.

So if I was careful and I distinguished all the epsilons in our original calculation, it'd actually look like this. And then the subtraction piece gives an extra contribution and it's actually scaleless in the n bar dot k here. So there's a scaleless loop in this guy.

So it actually vanishes if the epsilon IR and the epsilon UV are said to be equal. But what it does is it converts the epsilon IRs that are in the first expression into epsilon UVs, which is what we want. So once you add up these two things, the epsilon IRs are canceling and the epsilon IRs that were coming in the original formula, those were coming about because of this bad behavior as n bar dot k goes into the limit of n bar dot k going small. You can think about that roughly as where the ultrasoft is.

This is subtracting off that behavior and the remainder then is coming from only having divergences for a n bar dot k goes to infinity, which is a proper collinear ultraviolet divergence not from n bar dot k going to 0. So once you put the two together, the epsilon IRs cancel and then we get exactly, actually, the same expression we had before but where all those 1 over epsilons are 1 over epsilon UVs.

AUDIENCE: So the epsilon has come from second term?

IAIN STEWART: From both terms. So they both have epsilon IRs but they cancel between them.

AUDIENCE: OK.

IAIN STEWART: Yeah. And the remainder is just epsilon UVs, so all the epsilons that I wrote my earlier formula would be now epsilon UVs once I take into account the subtraction. So I could have just ignored the subtraction and that's often what people do. If they know that the zero bins are giving a scaleless integral, they say, well, let's ignore the subtraction, we'll just say that all the epsilons are UV and the zero bin makes them UV. But if we really want to look and see that things are working properly, we should take the subtraction and calculate it and make sure that that's true.

But we could have just taken the answer that I wrote down earlier and said those epsilons are ultraviolet and let's throw them in a counterterm and calculate an anomalous dimension. So we'll proceed that way next time, but we now know that actually they are ultraviolet divergences. So next time we'll take the ultraviolet divergences and we'll define from them a counterterm And we'll see how we get an anomalous dimension and what kind of logs we sum by using that anomalous dimension.

So the zero bin that's scaleless in this particular example is not always scaleless. So sometimes it could give a nonce. Depends on the problem you're dealing with. So sometimes you can set things up so that it's scaleless and then you just basically can ignore it. But that's not always true, so you do have to think about whether it's really going to be true for what you're doing. If it is true, then you can effectively ignore it because it's sort of just making the physics come out right, making sure there's no overlap.

But if your regulators set up so that it's scaleless, you can just get around it. But in general, that might not be true. If you had more scales in the problem, if you're doing some calculation that had some jets of finite size then that won't be true typically.