PROFESSOR: So this is an example in SCET one where the degrees of freedom in the p plus p minus plane looked as follows. So we had hard modes that lived out here at p squared in order some large scale q squared, then we had collinear modes for the two jets that are going to be going back to back, and I'll draw the picture in a second. Then there were some ultra soft modes as well that describe radiation between the jets. And then there can be also some lambda QCD modes, and we'll talk about them a bit today too.

So there's a lot of different things going on here, but we'll see that actually we can understand everything and organize things using SCET. So what's the picture for these modes? We have back-to-back jets. Let me draw one jet this way, one jet this way. I should use some color.

So there's one of our jets. There's another, and then there can be soft radiation. And between these jets, there can also be soft radiation physically within the jets, but there's no directional dependence to the soft radiation.

And then set that this is blue. This and this so far are orange in my picture. This is green, and then there's some hard interactions, which we've already sort of localized into a dot in this picture.

So what we're doing is we're colliding e plus e minus producing a virtual photon or z and then from that producing, if you like, a state which is full of collinear modes, state which is full of the other type of collinear modes and ultra soft particles. And this is in the center of mass frame, or that's kind of how you could think about it in SCET, or physically what you're producing is you're producing jet 1 plus jet 2 plus soft radiation or ultra soft hadrons, soft hadrons.

OK, so the first scale that you want to think about is the hard scale, and so that is just the q squared to the virtual photon or z. So if you think about the Feynman diagram, there's e plus e minus. Here's q mu. You're going to have a pair of quarks that that virtual particle can couple to, and there could be additional gluons. But the hard scale is set by the virtual photon or z.

So there's going to be some scale associated to that hard scale q. If you want to think about it as a normalization scale, there'll be some scale where we want to match from QCD onto SCET, and that's going to be the hard scale. So I'll call that scale mu h.

So then something physical has to set these other scales in the picture, and the next thing that we need to talk about is what we're going to measure. In principle, if you collide e plus e minus and you produce hadrons, you could have lots of different possibilities. In particular, you could have not dijets but trijets, so you could have a third jet and made it from a gluon.

And some kind of measurement that we do on the final state is going to restrict us to this configuration with just two jets. And the one we'll talk about to start is what's called hemisphere invariant masses, which we already mentioned earlier. So the way that you should think about this is you take all the final state particles, and you could just take the sum of all their momenta and divide it into two parts, those that are in hemisphere A and those that are in hemisphere B.

So let me augment my figure here by saying that there's two hemispheres. And then what these masses are, which I'll call m squared and then bar squared are just the four vectors squared for p, for the a, and the b guy. So written in terms of the individual particles in that hemisphere, you sum up all their four vectors for the particles in that hemisphere and square it, and then likewise for B.

And if you're measuring these two things, we talked about earlier the fact that you could specify that it was a jet by demanding that they're small. So the dijets is going to be m squared and then bar squared much less than q squared. And if you had trijets, you wouldn't-- that wouldn't be the case because if you had trijets, you'd have two particle-- two jets in this hemisphere at wide angles, and those-- that wide angle would produce a large q squared, a large and variant mass of m squared. That's of order q squared. But if you just have a single jet, then you can have a small m squared, so the small m squared limit is what forces you to have dijets.

And these guys here, it also puts us in the kinematic situation where we have our n collinear modes for one jet and n bar collinear modes for the other jet. So those are two of the degrees of freedom in our picture. So I'm reminding you of some things we talked about before, but it was a while ago so-- and the power counting parameter here is m over q.

And so if you want to say that there's a scale in this picture associated to the collinear modes, then you would say it's on a kind of jet scale, and that's of order m. OK, any questions so far? So then there's the ultra soft radiation, which is being ultra soft, it's uniform. It's not collimated in any direction. It's the thing that gives you the communication between the jets.

The jets are going to decouple from each other in the sense that interactions directly between the n collinear and n bar collinear modes is all going into the pink dot, but there can be long distance communication between the jets caused by the ultra soft modes. So, really, what I mean here is long distance communication. Short distance communication can happen too. That's what the pink dot is, but that's, in some sense, simpler. Long distance communication can only happen by this ultra soft radiation.

And we know about how ultra soft radiation interacts with energetic particles, and in particular, what we know is that it's eikonal. That came out of our Lagrangian description of ultra soft radiation. And that will have implications here. So what is this ultra soft radiation? It's radiation that has energy that's of order q lambda squared.

And so that's m squared over q energy radiation, and so the soft scale, which is the next scale down in the picture, ultra soft scale, is m squared over q. Often, people will-- because there's no soft radiation in this setup, sometimes people will just drop the ultra and just call it soft. That's very common. I'll try to always call it ultra soft here because we'll be drawing some distinctions with SCET 2 examples, and in those examples, we have soft and collinear.

In this example, we have ultra soft and collinear, so I'll keep trying to call it ultra soft. But you can see that I've already abbreviated it to u soft to make it look more and more like soft. Anyway, but in the literature, people often just drop the u completely and call it soft. Something to be aware of.

OK, so this actually-- this radiation here, this is kind of the largest scale that can show up in the soft function, and there's also lambda QCD. And that's a scale that, at least to start, we're going to leave in the soft function. So if we go back to our picture over here, that's why I put both the ultra soft mode side up on this hyperbola there, which is this m squared over q line, mu over m squared over q line. That's why I made it orange, and I also made orange this lambda QCD because they're together. And you can think of it as just that there's some modes that are capturing this entire region, which are the ultra soft modes.

Now there are two-- if m squared over q was of order lambda QCD, then those are the same thing. So there's kind of two possibilities here. You could have m squared over q of order lambda QCD, and that means that this scale is not non-perturbative. And this is what's called, for reasons that will become apparent by the end of lecture, the peak region because this is actually the region of the cross section where there's a peak. OK, that's why it's called the peak region.

And in this region, you have the following hierarchy. You have mu h is much greater than mu j. Mu j is much greater than mu s, but mu s is of order lambda QCD. OK, so if we take this to always be true, then we're in this situation.

And there is another possibility, and that is that there's really a separation between these hyperbolas that's just as large or hierarchical as the previous separations. So if this thing is much bigger than lambda QCD, that's another possibility, and there'll be a region of phase based where this is true and a region of phase based where this is true. The second one is-- here, the soft, there's-- here there is perturbative ultra soft radiation, so you can just calculate it order by order in the ultra soft scale, which is this scale, which is perturbative because it's much bigger than lambda QCD.

And this is what's called the tail region. And the analog of this statement here is that we have a hierarchy between everybody. So when we do this analysis, what we're going to do is we're going to do a power expansion. And you can say, well, the power expansion is just what you told me, m squared over-- m squared much less than q squared, but you also have to worry about this when you're doing the power expansion.

So one way of thinking about it is just exactly as I'm writing here, and that's how I want to advocate. The things you're expanding in are these assumptions. And you can see that in these two cases, there's a slightly different set up.

So usually, when you have an effective theory, you have to define this from the beginning because the way you proceed and how you set up your theory is going to depend on whether you're in this situation or this situation. But part of this story is the same in these two situations, namely the first two greater thans, much greater thans are the same. And so that, we can proceed without worrying about this. And then we can make this distinction later on, and that's what my picture was advocating for by having these guys in the same category. So hopefully that's clear.

OK, so if you're in the tail region, there's going to be power corrections, and these are actually the most important power corrections that come as powers of lambda QCD over the soft scale to some power. But you're going to have a power expansion in that. So that's the-- if you like in this situation when you're in the tail region, you can have power corrections that are non-perturbative by having ratios of these things, lambda QCD over this over that over this. But the biggest one are these guys over the soft scale, so that's the next smallest scale.

But if you neglect these, the leading order cross section in this region is perturbative. Now you know you can have any power here, and those exist. What happens in the peak region is that thinking about those as an expansion is no longer good.

And you have to take all of them and treat them as if they're order one and for any k, and that ends up meaning that there's going to be some non-perturbative function, like a Parton distribution function in the sense that it's non-perturbative, that describes part of what's going on here in this region. And that, we just see from our power counting, and our power counting already tells us that that's what's going to happen. So we can learn a lot just by thinking about the scales and the problem and thinking about the power counting.

So you could ask about other power corrections. So there's a set of power connections where you would expand and say mu s over mu j. And let me label these as kinematic because these-- the things that setting the scale for mu s and mu j, say in this picture where mu s is this and I always distinguished lambda QCD as a separate thing, these scales are just perturbative scales, and they just correspond to if you think about having some function that you're making an expansion of it. So they come about from expansion of kinematic variables basically.

So it's the m squared much less than q squared. So I mean, there's nothing non-perturbative about that. And then you could have power corrections that are lambda QCD over mu, which, if you like, are really hard power corrections. They're kind of a traditional power corrections, which are lambda QCD over the hard scale.

And the exercise that I gave in the homework, in some sense, is going towards figuring out what the sort of soft function matrix elements would be. Part of the problem-- there's other parts to it-- but would be going towards figuring out what these guys actually are and defining them, which is not known in the literature. And then finally, you could have lambda QCD over mu j, which, of course, you could write. You have to be a little bit careful because one way that this could happen is that you just have one of these guys times one of those guys.

OK, so they're not-- it's not necessarily an independent category, but you could still think about these types of power corrections as kind of something that might show up, either this way or maybe somewhat maybe directly. And anyway, that's another category. It turns out that kind of-- you can get it this way, which is something that one can deal with. And if it happens directly, then it actually is down by two powers, so these guys are kind of less important.

OK, so people deal with these guys in the literature. These guys haven't really been fully treated yet. Not very much is known about them, and this is fully treated in the literature. These ones that I said are the most important ones.

OK, so even-- we can already even outline kind of what people are doing because we just think about higher order terms of expansion, and this is an example, unlike B to C decays, where we have people going to four orders down in the expansion. Here even the first non-trivial order down hasn't really been done-- hasn't been done at all. In the second, place various expansions, so it's more complicated. And some of them have been treated, but this one hasn't.

And I should say that the way that this one is treated is really just perturbatively so it's not-- this one, in some sense, is also not treated fully in the language of the effective theory. It's just put in by hand. So in some sense, the only one that's treated properly is the top one. OK, well, we're going to work mostly to leading order, and we're going to call these guys power corrections and work with this stuff.

OK, so start with the full theory current. I'm not going to distinguish too much between pieces that come from charges, like the electrically charged and stuff like that. That's not going to be our main focus here. We can keep track of all that. Our main focus is going to be understanding the QCD effects for the jets.

So we've already done this. The current that has a psi bar psi gets matched in SCET onto something with two Wilson lines. There's the label. There's another label, and we can also make a field redefinition. And let me not put zeros on the field after the field redefinition because that makes it notationally cumbersome.

So this goes to this after making that field redefinition. OK, so this is something we talked about earlier, and the Wilson lines here are capturing diagrams, as I said earlier, where you could have, say, two n collinear is one n bar, and then this guy is off shell. So that goes into a Wilson line once you match it on to the-- so this is the [? fall ?] theory. Once you match it on to the effective theory, you get various extra gluons coming out of your operator, which is coming from these Wilson lines, and it's because you're integrating off shell particles.

So if you just look at the kinematics, then there's already actually some fairly powerful restrictions. We're in the center of mass frame. Momentum conservation says that if we think about all particles in the final stages either being n collinear or n bar collinear for ultra soft, then the initial momentum of the virtual photon or virtual C has to add up to all the final stage momenta.

And if we just look at the large part of that, in the center of mass frame, n bar dot q is just capital Q, and that's n bar dot pxn plus things that are small. And likewise, n dot q equals capital q equals n dot pxn bar, which is large order one plus small. So these guys here are lowest-- leading order in the power counting, and these guys are suppressed.

And what happens is like in the example we talked about when we were doing [? bs ?] gamma. We have these labels, omega and omega bar on our collinear fields up there. They just get fixed to the qs by momentum conservation. So I said this was generically going to happen whenever we had operators that had only one type of building block in each collinear sector, and that's exactly the situation we have. Some momentum conservation is strong enough to fix that omega is equal to q and so is omega bar.

So the first thing I want to do is show you how to factorize to the cross section in this case. So what would you do in QCD for this cross section? Well, you'd say the cross section is a sum over some restricted set of final states that satisfy the kinematic criteria that I'm interested in. You'd have a momentum conserving delta function, and you could write it as sort of leptonic or-- yeah, leptonic tensor, and then kind of a hadronic tensor, the same way we do for [? DIS. ?]

So you would start-- could start with that formula, which is really true for anything that you might want to measure, which is just imposed by kind of what states you include here. And I haven't yet specified exactly what I'm going to measure. We'll still have to put that in. So part of what SCET does is make these restrictions into something that shows up in the operators rather than showing up in the states. And ideally, we'd like to sort of get rid of the states eventually and be able to calculate-- see how to calculate things.

So in SCET, once we know that we're in a dijet configuration, we can think of the state as having been composed of sort of pieces for the different sectors. And we know that this is an OK picture because our Lagrangian for these guys factored. And when the Lagrangian factors, that means the Hamiltonian factors and the Hilbert space factors after the field redefinition.

OK, so we can put in our expansion of the current at the top of the board over there, and then put it into this formula, and see what we get. And there's some pre-factors. I'm now summing over just these states. There are still some restrictions depending on what I'm measuring, but they are different restrictions, and they're simpler.

There's still momentum conserving delta function, but I can write it out as pxn pxn bar px ultra soft. And because of the states factor, I can factor also with a little bit of work the operators. So I can write the operators. I'll just write it down. I think it's clear. And I'll tell you some of the things you have to do to get there.

So I am skipping some steps here because if we do it-- we can do it very carefully, and we come write out each step one at a time. But there's a lot of writing if one does that, so I'm kind of trying to write steps where I think it's intuitive what the results are and skip steps that some level can be filled in either by looking at the literature or you just believe me. So we have these operators that were still tied together in some sense because there is a color contraction of an index between them.

But we can get around that basically using some-- once we realized that the matrix elements factor like this, we can use some color identities and put it in this form. So there's an overall pre-factor 1 over nc, which I didn't write. There's some overall pre-factors, but the color is separate. There's a trace, if you like, over color here, and these guys here are traced into color singlets as well. Ray trace in each case. So just using some color identities, we can do that.

So once-- remember the fact that the states factor in that the operators can be written in a product means that all contractions are happening between these guys and these guys. That's why I can separate the matrix elements the way I have. And the only thing is then that there could be color indices tying these guys together, but it turns out you could deal with that as well.

When you deal with the color theories, you could get things like-- you could get a ta in here or-- well, you could get, for example, ta ta or something, but then there's some ways of getting rid of those terms related to the fact that this has to be-- you have to argue that certain things are color singlets. I'm not going to go through that.

So there's terms that we're dropping here, and those are the other high order-- those are the other power corrections. So this keeps all the power corrections of the first type, mu s, lambda QCD over mu s because those can still be encoded in our soft function, which is going to be this thing involving the soft states. But it drops the other power corrections.

AUDIENCE: How do you know that since you derived the [INAUDIBLE] by looking at the perturbative diagrams?

PROFESSOR: No, I didn't. I can derive-- I don't need to look at perturbative diagrams to draw the Wilson lines. I just derived them from the SCET Lagrangian by making the field redefinition.

AUDIENCE: So you're saying that because the Langrangian decoupled, that's good for all--

PROFESSOR: All soft particles. So far, we actually haven't needed to use perturbation theory, and the idea here is that we're not going to use perturbation theory. We're going to write down some formula that's true to all orders in perturbation theory, and we're really only trading the power expansion. That's what we're focusing on here without thinking about things as perturbative things. Although, obviously, some of these functions will be perturbative functions that we will end up wanting to compute.

OK, so let's think about what res prime means. And it's really a reminder that this formula is only really valid if we're making a measurement on the final state that really puts us in this configuration. So it's not enough to say we are in this situation. You're going to have to measure something that makes you in that dijet configuration, and that's important to remember.

And so we're going to measure the hemisphere invariant masses. So we need to cook that into our formula, and there's a very easy way of doing that. We say that the following is true. I can write one, which is equal to an integral of two delta functions. And this formula is obviously true.

And then what I'm going to do is I'm going to instead of integrating this, I'm going to move it as a d sigma d these things, and that's going to leave these delta functions inside the formula here. But since they depend on collinear and soft momenta, which are the momenta that are in these states, they're going to be sort of tightly connected to the whole thing, whatever is going on in this formula. It's not like they commute with this xn.

They only commute with the xn if I can make them into the identity by integrating over this. But once I pull this through and put it over on the right hand side-- left hand side, then you have to leave these delta functions there. They're specifying the measurement. So this is a total n collinear momentum, and this is the total ultra soft in hemisphere A. That's what the A means.

So I divided everything into hemisphere A and B. Obviously, all the n collinear particles, which are going in that direction, are in that hemisphere, but the soft ones could go in either case. And so then there's a soft momentum in hemisphere B and soft momentum in hemisphere A. Now this is unexpanded. I can also expand these delta functions, and I only need to keep the leading order piece if we're working at leading order.

And so the leading order pieces are as follows. There's an order of lambda squared piece. This is lambda squared. There's an order of lambda squared piece from the collinear momentum squared, and since this is a total collinear momentum, it can involve multiple massless particles. It doesn't have to be zero.

And then there's a cross term between the softs and the collinear. It's also order lambda squared. This order one. And this is order lambda squared, and that term's the same size. And then all the other terms are higher order. These terms are suppressed, so this is really all I need to keep.

There's also something else that effectively we've done by setting things up, and that is, in general, you could think that the collinear particles would have some perp momentum. But we aligned our axes with the jet axis, so there's actually no perp momentum there either. So that guy is really just-- we can pull out a pn minus and then we have pn plus k sa plus.

And we also know that pn minus is q. That was fixed by kinematics. OK, so this simplifies quite a bit. And we're going to drop all the powers of [INAUDIBLE] terms.

All right, so that is one thing. So then I can do what I said in words, write this guy, and it will have these two deltas that are under similar acts. They don't commute through with it-- without it. They depend on x.

Now another thing we would like to do is the factorize the measurement. So this measurement here involves the sum of a plus momentum collinear and a plus momentum in soft. And it's all in one delta function.

We'd like to write it as separate delta functions that we can associate with those different parts here, one delta-- we'd like to have a separate thing here from these guys. And that's actually very easy to do. We simply write the following. Just introduce some more delta functions.

So there's this tying together delta function. K plus is just some dummy variable. It's not the momentum of any state. It's just a dummy variable, whereas pn plus and ksa plus were momenta of particles in the state.

So this guy can then associate with the-- put them together with the n collinear matrix element, which is this guy here. I forget what color he was. And then for the ultra soft, we can associated this delta function. So that's what I mean by factoring the measurement, that we can put the piece, depended on the state, together.

And then you can see that we can move our-- well, OK, we have to also factor this delta function. I should say that too. That's the part that I'm not going to go through, but we can play similar games expanding and factoring this delta four as well. And that's besides just being a little bit tedious, it's not really any more difficult. Maybe a little more difficult.

So we factor that guy too. And then there's one other trick that we want to do, which is useful, and that is that we can write some deltas in Fourier's space. So we can always write a delta function as an integral over a phase, and that's convenient because in the phase, the momenta that are in the delta function also factor. So I can write it as a product of two separate factors like this. There's some halves. It's conventional.

OK, so this guy here, which is the guy that is associated to the state, it looks like a translation operator. if you have an e to the i x dot p, that you can use. You can put into your matrix element, and you can translate the fields, which are in this formula all at 0. They're all at space time 0. What you can do with this guy here is you can put it into the matrix element and translate the fields to point x.

And then you'd have a chi and x minus. So that's how we can deal with that guy. And so if we do a bunch of stuff like that I'm not going to go through on the board for you, after some work, we get something just kind of an intermediate step of our factorization, which I'll write out so you can get some idea of-- if I jump to the final answer, it's kind of too simple to see what's going on. So let's write this one out.

So I was focusing on the n collinear matrix element, and everything I was writing, I wrote this delta function. But there's, of course, another one for the delta n bar, and I do the same thing for him. So we got k plus and l plus by factoring the measurement there. You get a k minus and an l minus by factoring the other measurement.

One thing that comes into-- that makes this guy here a little more complicated is you have to think a little bit about residual and label momentum. But at the end of the day, the result is what I'm writing. So this is going to be completely factorizing to three independent things.

There's one set of collinear fields for the quark, and there's another for the antiquark if you like. You're specifying whether it's a quark or an antiquark with a label. So the fact that this label here is positive means that this is a quark, and maybe this should be a minus q. And then this guy is specifying the antiquark, and then there's the soft part.

OK, so this is a fairly messy one board expression, but this guy here is factored. We have the hard modes here, collinear modes, n collinear here, and bar collinear there, and the ultra soft goes here, which I've already dropped in my ultra soft. So this last guy if you think about what this matrix element is it's just some function.

We sum over all x of this. So we sum over all the intermediate states. The only thing we're fixing are l plus and l minus. So when we sum over states, we actually integrate the momentum of particles in the state if you can think of that. Those are different states.

You have to sum over them too, but it's a continuous label and therefore you integrate over the phase base. And that's what we mean when we write the sum over axis. That includes phase based intervals. And so the only-- so this momentum gets integrated over if you like, but then it gets-- there's a component that gets fixed, which is the total momentum of plus or minus momentum in each hemisphere. So this is just some function.

So that we color the same. It's just some function of l plus or l minus, and it's called a soft function. And at this stage of the game, it's actually encoding two different momentum scales. The l plus and minus as well as sort of lambda QCD, which I didn't write explicitly as an argument. And l plus and minus, well, we'll see later, but you can think of l plus and minus roughly as encoding the scale that's m squared over q. This will become more clear in a minute.

So again, we need to figure out what these guys here are. That's a little bit more work but not too much. And they're both, of course, mirrors of each other. So really, we just have to be one of them, and if you like, the other is just kind of charge congregation.

So this guy here, we're going to write in Fourier's space. That's a convenient thing to do because what we know about the Feynman rules in Fourier's space when we thought about the Feynman rules in Fourier's space, we knew that the collinear propagators would only depend on the small plus momentum. So the x-coordinates here, the x-coordinates of our fields, those corresponded to residual momenta, but when we do Feynman diagrams, there's only the plus momentum showing up. That's the multipole expansion.

So you could think about it from the [? phonographs, ?] but it's really a general property from the multipole expansion. So that means that some of these integrals here are just trivial and give me a delta function. So I'm getting delta functions in some directions because of the multipole expansion in this formula. Because I aligned the axis, there was no perpendicular momentum, so it's really just this r plus that we get, and this thing here is called the jet function.

So this is the non-trivial function that can show up. We do the same thing. We do the same thing for the case with the sum over xn bar, and it gives us another jet function. And by charge conjugation, it's the same jet function. But in this case, it would be q of sum r prime minus and some other momentum.

And so what we then do is we take this formula, plug it back into here, do all the integrals, and at the end of the day, we can actually do-- we have lots of delta functions and lots of integrals. And we can boil everything down to just two integrals left over. And that gives us our final result. So I'll write this way.

So it boils down to just involving these jet functions and the soft function, and then there's something that's called the hard function, which is just our Wilson coefficient squared. So rather than write Wilson coefficient squared all the time, we just call it another function h. So this is the factorization theorem.

So you see in this case, the hard function was just a multiplicative factor unlike dis, and what was-- where things are talking to each other. Again, things are talking to each other, but it's the soft modes-- ultra soft modes that are talking to the jets. And ultra soft momenta can change the mass, and that's what kind of the correct mass-- this is the total mass. The correct mass for the jet function is the collinear mass, and that's this thing minus this thing.

So if you wanted to guess this formula, this is how you do it. You'd say m squared, just from the kinematic relation, m squared had a collinear piece and a soft piece. The right thing to evaluate the jet function at would be just the collinear, so that's the difference.

And then the soft function could depend on these momentum, and you're basically led to a formula like this one. But we can also just, with the field theory, go through and derive it, and that's what I'm convincing you of even if I was skipping some steps. All right, so this is the dijet factorization theorem for hemisphere invariant masses.

And a lot of event shapes kind of go along the pattern that we've done here. If you think about what would be the difference if I pick some other observable both sides of these masses, pick some other thing that you could measure that would tell you that there's dijets, we just swap out the measurement part, and that would lead to-- and just see where it leads us. A lot of the steps would be exactly the same steps. So we did it for one particular observable, but steps are the same if you do others.

If you're in this situation where you have dijets, there's more than one way of measuring an event to ensure you have dijets. There's something called dijet event shapes. They go under the names of things like thrust, C parameter, heavy jet mass. All these things actually have factorization theorems that you could derive in a similar way to what we're doing here.

All right, now this is kind of like-- I didn't put any mus in, but just like we did before, we can always put the mus in. So if we put the mus in, everybody gets a mu. The h is like the mu of the Wilson coefficient, and then the Js are like-- and Ss are like mus of-- from the point of view of SCET, they're like mus of the operator. We're just switching to renormalized operators and renormalized Wilson coefficients.

Now if we draw our scale diagram again but draw it a little bigger this time and put in where these various functions are-- so somebody was blue. Not going to label everybody again. So this is the hard function. Our jet functions are kind of sitting at this scale. They had to do with the collinear modes, and then our soft function is sitting at this scale or this and this scale.

So it's clear that these things sit at different scales. And what that means from the point of view of the formula is that if you associate what scale these things want to live at, which from the point of view of perturbation theory means at what scale should I expect-- would I be able to calculate these things given that they're perturbative? Would I be able to calculate these things without encountering large logarithms? And that would be these scales that I'm writing here.

So these are the scales where we could do perturbative theory for the functions without encountering large logarithms. So we can figure out what these skills are by just looking at the perturbative theory and looking at the logarithms. But you see that it's a different scale for each of the different functions.

So in this formula, there's a common mu, which is like a factorization scale. But each of the functions wants to live at a different mu, and we're going to have to do some renormalization group evolution to put things at kind of at the scales they want to be at. And then there'll be some resummation going on, and that's going to sum logs of m squared over q squared. So we need normalization group, and that's trying to sum up logarithms, which are ratios of these scales, which expressed in terms of some physical thing is, in this case, logs of m squared over q squared.

All right, what about this situation here? Well, it turns out-- and I won't go into it in too much detail, but we can factor that guy. If these two are hierarchical, we can actually factor this in something called the soft function OPE, and even if they're not hierarchical, we can actually write it in the following way. So we can always, in some sense, whether or not we're in a situation where these are comparable or whether they're hierarchical, we can always use the following formula.

We're not guaranteed that that is possible, but in this situation, it is possible to sort of make those two situations compatible with each other with a single formula. Usually, when you design effective theories, you kind of for each different expansion, you have a different result. And making them-- whether you can put them together into a single result is not always guaranteed. But this is a situation where we can put things together in a single result where these are the perturbative part of the soft function, and this is the non-perturbative lambda QCD effects.

If you imagine that the function-- if you imagine that these two axes are hierarchical, then this perturbative guy here gives you some ls corrections, and it basically, in terms of the l plus, in terms of the momentum variable in this formula, it's giving something that's, like, a power-- what's called the power law tail. So the kind of dependence that you're getting in is some logs over an l plus, and that's called a power law since it goes like 1 over l plus, whereas these effects here that are non-perturbative live down at non-perturbative momenta, and you can actually prove that they have an exponential tail. So this is some plot of f in a one dimensional projection, so let's just think of it as one.

And this is some scale of order lambda QCD. So at some blob down lambda QCD, this describing the distribution of soft hadrons, and it's non-perturbative. It's not like I'm drawing it because I know it. So it looks something like this, whereas this part, we can calculate and it has a different dependence. This is a power law. This is an exponential.

OK, so that's kind of further factorization of the s and to get rid to make those two axis separated. Can separate the perturbative and non-perturbative corrections. In some sense, if you're in a situation where they're comparable, then whether you take a non-perturbative function and if l plus is small and you integrated against this, you're just getting back some other non-perturbative function.

So this formula is not really-- there are some things that this formula is actually doing, even in that situation, is ensuring that you're in the ms bar scheme for example. But the sort of more important in some ways is when they're hierarchical, and then you can start to expand this thing and-- because in that case, this l plus prime would be localized down here, and the l plus-- if the l plus got big, then you can start doing a Taylor series, and this formula would tell you what the-- how to put the corrections together. So there's kind of a lot of physics in that formula that I'm not going into in detail. Yeah.

AUDIENCE: So I don't really totally understand why this is valid. Are you allowed to-- does the soft function code all those below that hyperbola?

PROFESSOR: Right.

AUDIENCE: Is that--

PROFESSOR: That's right. Below the above one and--

AUDIENCE: Yeah, below the s.

PROFESSOR: Yeah, that's right. That's right. The soft function. You should always think of these things as kind of extending down to the axes in some kind of fashion. So let's drop like this.

Since they're infrared modes and the infrared is sort of down at the axis, you should always think of them as extending down to the axis. Now exactly how you want to think about how the soft modes and whether the soft modes are capturing the entire axis or the collinear-- how the collinear modes and the soft mouths kind of what the edges are is related to the power counting. And you can think of the collinear modes as kind of being held away from the axis because the soft modes are really deeper in the infrared.

AUDIENCE: But at higher power, are you going to screw these up if you say-- like suppose the soft modes have an invariant mass of lambda to the fifth, right, or they're--

PROFESSOR: Oh, yeah. No, that's-- you never have to worry about-- you could worry about putting in more hyperbolas, right, with lambda QCD over q or something.

AUDIENCE: Yeah.

PROFESSOR: And you don't have to do that.

AUDIENCE: OK, so--

PROFESSOR: You never have to do that.

AUDIENCE: [INAUDIBLE]. Why is that?

PROFESSOR: I mean, essentially, that's because those-- any lambda QCDs, what this x means is that lambda QCD is encoded in the soft-- in this kind of non-perturbative soft mode. And if you had something that at higher powers like lambda QCD, to take your example, to the sixth, the one over q to the sixth is going to come out in some coefficient. And the lambda QCD to the sixth is just going to be some matrix element of this mode that has that dimension. That's how it's going to work.

So in some sense, you could say, well, what if I put a mode down there? And I would just say, well, that mode's inside this mode. You don't have to write down something different for that. It's all encoded in the Lagrangian for that mode, and that mode has all the non-perturbative physics.

AUDIENCE: Yeah, at a higher power, I would think you're comparing--

PROFESSOR: So this is--

AUDIENCE: --two powers by--

PROFESSOR: Yeah, this is a good question.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Right, so--

AUDIENCE: [INAUDIBLE] actually significantly much smaller.

PROFESSOR: Yeah, so one way of thinking about it is like this. When you design the effective theory, you actually have to figure out what the degrees of freedom are at lowest order. And when you go to higher order, you shouldn't be having new degrees of freedom popping up at you. Yeah, and so that's kind of at the heart of what I'm saying here.

That's a kind of an even deeper way of saying it because what you're doing with the power connections is you're kind of figuring out how to put factors in the numerator if you want to think in Feynman diagrams. You're not changing the propagators. You've already figured out the propagators with your leading order theory. So you have sort of where all the poles can occur, and now you're just sort of doing perturbation theory. Perturbation theory shouldn't introduce new degrees of freedom in the-- when you're expanding in the power expansion.

All right, so to draw a picture, I want to tell you about one other observable, which is thrust. So thrust can be defined by this formula here in general, which is that you find some axis, which is actually an axis that's going to-- in our dijet configuration in the jet axis. You find the axis that maximizes this thing, and in the case of a dijet configuration, that's going to align with the jet axis. And then you do this sum, and that calculate some variable t.

T turns out to have kinematic limits between a half and one, and if you want to talk about something that's a little more convenient, you define something called tau, which is 1 minus t. And the dijet limit of tau is zero. That's why this is more. So the dijet limit of t is the limit where you just have kind of two pencil-like back-to-back things.

In that case, this goes to one. And if you wanted to find something that goes to zero, then you define it as tau. Just a little more convenient. And so it's possible to work out that when you're in a dijet configuration that tau with a bit of work you can show it's actually simply the sum of our two hemisphere invariant masses.

So obviously, in a dijet configuration, there's lots of simplifications to the kinematics. And with some work, one can show that that's true. So what this is an observable, which is simply a symmetric projection of our previous two variable observable onto one variable observable. But if we demand that, for example, tau is small, that's demanding both m squared and n bar squared are small because they're positive quantities, and we're just summing.

So demanding that this is small is demanding both of those variables is small. So tau much less than one also is a way of getting dijets, and that's a way of getting dijets just with one variable rather than two. So that's, of course, easier to draw a picture with one thing rather than two. So if we take our formula for the two variables and we project onto one, then we can do that. It looks like this.

So what happens to the jet-- the two jet functions and the soft function that had two variables is that they get projected kind of on to their symmetric projection of the two variables. So this is like-- these two are symmetric projections. So we had two variables before, and now we have one variable. And it's a symmetric projection, and that just follows from this formula.

So literally, to get this result, we would just do the same trick we did before. We'd say that this is true. Now let's move the integral d m squared and d n bar squared onto the right hand side and think about doing those integrals. And we follow through, and it would lead to a formula like this one.

AUDIENCE: J tau is j squared?

PROFESSOR: J tau is j squared. J tau is literally j j integrated over so fixing kind some of the momentum and integrating over the difference. So this guy depends on one momentum, and this guy-- these guys depend on 2, and there's one integral. And then s tau is like one integral of kind of something like this.

So let's call the integral l prime. It's kind of like that. So that's what I mean by asymmetric projection.

AUDIENCE: And so how do you know that tau can be written as a function n bar, or could another kinematic variable [INAUDIBLE]?

PROFESSOR: No, it can be written as a function of n bar. Yeah, because you're still-- if you're in subleading power, as long as you're in the dijet's configuration, we've got the right degrees of freedom. We don't need another jet degree of freedom. So if you want to describe-- you could think about the thrust distribution.

Well, let me answer your question in more detail. Let me tell you where you would have to worry about more degrees of freedom, but let me do it in a second. I'll give you-- all right, so in the thrust case and what we said about masses carries over to a formula like this, where this mu h squared being much bigger than mu j squared, much bigger than mu s squared. And then that could be much greater than or of order lambda QCD squared.

And the type of terms that our factorization theorem would be able to resum if we did the renormalization group evolution would be sums of powers of alpha and logs of tau. Tau is dimensionless variable, and so the kind of terms that would show up in the cross section are terms like that plus non-perturbative effects. So if you want to think the base level, what is inside our formula, it's all these terms. We're able to determine them from our factorization theorem, and then there's also some non-perturbative, which would be kind of-- and non-perturbative effects that in this f function.

And the factorization theorem is telling us how to compute these things and include them. So what does it look like as a picture? What does the cross-section look like? So there's a peak as I promised, and there's kind of-- and then there's a tail. And so this is tau. This is d sigma d tau.

You can always normalize. This peak is happening in the kind of non-perturbative region. So because it's a symmetric projection, there's a factor of two, but it's kind of basically happening when taus of order lambda QCD over q. And we could figure out that just by power counting. And then depending on the value of cue, that sort of for a typical configuration, tau equals 0.1 would be over somewhere over there, et cetera.

So this is the peak, aptly named because it looks like a peak. And in this peak, there's non-perturbative effects that come from the f. This is the tail. It could be called the shoulder, but it's kind of once you're out here and down here, this is what you call the tail where you have a perturbative s tau. And then you would include power corrections by expanding.

And the expansion would give you lambda QCD over the soft scale, which is q tau. So this is-- that's that expansion that we talked about, and that comes from expanding it, f, or an expansion that involves f. It's not expanding f. Basically, what these numerators are are moments of f once you do the expansion.

And then there's a region out here, and in both of these regions, you have dijets. But in this region out here, you don't have dijets anymore. That's where tau is starting to get large and you no longer have dijets. So if you wanted to think about sort of higher order corrections in the distribution out here, then you would need to perhaps think about three jets and other things. In particular, there's also a kind of shoulder here that if you really want to think about how power corrections are kind of properly dealing with that region, you think about more than two jets.

But if you just kind of restrict yourself to here and here, which is all we're going to do, then you don't-- then every mode, even if you go to subleading order, is exactly the ones that we just said. And you just have to deal with them by constructing operators in subleading order. So that's the more detailed answer to your question. Any other questions?

OK, so I'll say a few things about the perturbation theory. So so far we've kind of argued that we could get pretty far without doing perturbation theory. So we got here without doing perturbation theory. What if we actually now want to calculate these series, do the normalization group evolution? How's that going to work?

And it's actually-- in some ways, it's a combination of the two examples that we treated. We did an example where we had a Wilson coefficient that was just running multiplicatively, and we did another example where we had a function for deep and elastic scattering. It had an integral, and we're going to have both of those situations in this case.

So let me tell you how it works. So to get the hard function, you would do some matching, and the matching is actually for what people call the quark form factor. So if you wonder why people care about the quark form factor, the quark form factor is basically the h in our formula, not exactly, but closely related to the h in our formula.

So you would do some one loop matching by taking QCD graphs and subtracting the SCET graphs in exactly the kind of way we were already doing for the examples that we treated before. So those are collinear gluons. This is an ultra soft gluon.

And there's some wave function graphs that you can consider. We form the difference. And at one loop, we check that all the IR divergences between this and this cancel, and then we get the Wilson coefficient. So this is very, very analogous to the example that we did when we were doing a heavy light current.

And you see here that this thing, which depends on mu, it's mu over q. And that's what I was saying that the Wilson coefficient or the hard function should be-- the log should be minimized from mu of order q. And we see that by doing the calculation. H is the square of this thing.

So there's imaginary parts on these negative logarithms. These are minus q squared minus i 0. But when I take the mod square, this thing is real. What about the jet function? Well, if you go back to what the formula for the jet function was, the jet function is basically two fields like spit out a quark and absorb a quark. But it's a vacuum matrix element, so the quarks are just contracted.

So you're basically calculating graphs like this or, really, actually the imaginary parts of graphs like this because the way I drew it, you were summing over final states, so it would be the imaginary part. So we could just calculate the graphs without putting a cut in and take the imaginary part, and that would actually be giving us the jet function. So the jet function comes from some Feynman diagrams that look like this. And then there's one more looks like that.

So at one move, we have those three Feynman diagrams. And then we have the [INAUDIBLE] little guy. We take the imaginary part. We get the jet function.

At lowest order, it gives a delta function just to cut propagator. And then at one [INAUDIBLE], you get a delta function, and you also get plus functions. That may not worry so much about what the numbers are. I can just tell you what the result looks like.

It looks like that. There's three different types of terms that we could get. They all kind of have a power counting that makes them go like one over s. And that you know ahead of time. If you power count the operator here, it should scale one over s.

These are the different kind of structures that you can get at one loop that scale one over s. And this is kind of a symptom of there being 1 over epsilon squared divergences, and this guy here are [INAUDIBLE] the renormalized results. So we're taking care of the renormalization.

So just like in our example-- I mean, it's the same diagram really. Just this diagram was familiar because this diagram was showing up in beta s gamma, right? So we saw it had 1 over epsilon squared poles. Here, the 1 over epsilon squared poles lead to this. And, really, you can actually think of that very closely related to what we did because we were finding logs of mu squared over p squared, but here, p squared is a physical thing.

It's the s, the invariant mass that we pump into the operator. S is kind of what we put in through the-- we put in a momentum, if you like, q, where-- I shouldn't call it q-- t mu where s is t squared, right? And so before we were having logs of p squared, which were an IR regulator, but in this calculation for this jet function, it's actually a physical thing. And it's giving the momentum dependents of the jet function, but it's the right thing to stick into the factorization theorem.

And then there's the soft function where if it's perturbative, you can calculate it. And you can draw these kind of in some notation for the Wilson lines. So here's our Wilson lines in different directions. And then want this matrix element squared, and, again, you can sort of think of as kind of cut graphs like that if you like. And if we look at the soft function, it kind of has a similar structure to the jet function but now with the-- so, again, it's got a delta function and then plus functions.

Whoops. And it turns out there's no single plus function, but there is a plus function with a logarithm in this case. And then same for l minus. Same structure. It's just a product if you like.

And the reason that happens is if you only have one gluon, it's either in hemisphere A or hemisphere B. It can't be in both. So the alpha s corrections are either a function of l plus or a function of l minus, and that's why it has a kind of very simple structure at one loop.

So that gives you an idea of what these perturbative functions look like. C, if you talk about renormalization, C renormalizes multiplicatively. And so the renormalization group equation for C is just like the one we had before for beta s gamma. There's no integrals. That, again, came about from the kinematics fixing the variables.

But the jet function and the soft function have convolutions in this case. Well, they depend on this non-trivial momentum, and it's-- you can see in the factorization theorem, that is convoluting between two different sectors. And it kind of generically a hint that you're going to get out a formula like this one, which is like the PDF, but now it's a different formula. It was kind of an almost dimension for the jet function.

So we could go through that, but I was not writing down for you what the 1 over epsilons look like. But we could go through the renormalization and find these results. And actually, in this jet function case, we even know more. The general structure of this anomalous dimension is actually simpler than the Parton distribution case.

And it's the following. There's two types of terms that can show up. So the general structure of the anomalous dimension is that there's a single plus function in it or a delta function. And this single plus function is the analog in the jet function of the single logarithm that was showing up. Remember that when we decompose this guy, there could be a log mu over q term or a one term with no log of mu over q.

This is like an analog of a log, this plus function. If you integrate over s, then it's like ds over s, which is like a log. So this is a log, and integrating delta of s is like one. So the analog statement that there was two possibilities there. In this case, there's an analog of that, and there's two possibilities here. And what perturbation theory is doing is actually just computing the coefficients of these two different structures.

OK, and we're out of time, so I'll say a few more words about how you would solve, for example, an equation like this one. Next time I'll tell you how to solve it, and then we'll basically be done with our example. We'll put things back together and write down a factorization theorem that includes the resummation and then we'll go on to another example. Moving on, the next example we'll treat after this one is SCET two where we'll be dealing with energetic hadrons, some other types of examples besides jets.