IAIN STEWART: All right, well, it's time to start. So last time we were talking about e plus e minus to jets, and-- I should have written that-- in particular, e plus e minus to dijets. And we were talking about-- we talked about factorization and how to derive it. And I've schematically written the result here.

Last time we had a more definite formula with all the arguments made explicit. So there were hard functions, jet functions, and the soft function, and we could either think about measuring invariant masses in two hemispheres and constraining them to be small to know we have dijets, or measuring kind of the sum of the two, which we called-- which is effectively the thrust, and then we get a slightly simpler formula because we can project these guys down onto guys that'd only have-- or functions of one variable.

But either way, in either case, we have a factorization theorem that separates the degrees of freedom and kind of the picture is that we have these kind of scales in the problem. So this is new jet, new hard, and new soft, and we can use this factorization theorem to separate those scales into these functions.

And then we can use-- I started talking about using renormalization group equations in order to sum the logs between these scales. And I wanted to finish up that discussion. So for the Wilson coefficient of the operator or, for that matter, for the hard function, which is the square of the Wilson coefficient, you have a very simple renormalization group equation.

And that's because momentum conservation is enough to fix all the variables that label collinear fields. So you really just have this overall Q, which is the center of mass energy, and that's the only variable that it's fixed by kinematics. There's no convolutions. In the case of these things called jet functions, which we talked a little bit about last time, you end up with something like the Altarelli-Parisi where there's an integral.

It's a little bit different in the sense that we actually know its structure-- tolerance and perturbation theory-- it has this structure. So it only has a very particular dependence on S, has to scale like 1 over S, so one thing that scales like 1 over S is a delta function or a plus function, and basically, there's only this single type of plus function that can show up, and that's the analog actually of something that happened to this guy where we said there was only a possibility of a single logarithm showing up. OK.

So this is how far we got. Whenever you have something-- whenever you have equations where you have convolutions like this, what you should do is Fourier transform because if you Fourier transform something that's a convolution, it becomes a product. So we'll Fourier transform, and in order to be careful about convergence of our Fourier transform, we give it a small imaginary part.

And we define a position space anomalous dimension, as the anomalous dimension up there that depend on S, but now Fourier transformed and likewise, we define a position space jet function the same way, and then this formula here, which is a convolution, if you use these in the inverse transforms, you arrive at a very simple formula for the position space. Gamma J.

So the Fourier transform of a delta function is just the identity so there's this term here, becomes that term there. The Fourier transform of a plus function is actually a logarithm, and in order to get that result, you do need to have this convergence. This little i0 in there. So in general, if you have there's a general kind of relation between logs and plus functions like that, and in Fourier space, the kind of highest logarithm you get is L to the K plus 1.

OK, so these logs here, you're basically-- if you think about counting them, you should count this one over S as an extra log, and in Fourier space, that becomes explicit. You really just get a log to the K plus 1. All right, but this formula here, is something that, again, is of this kind of multiplicative form.

So what happens to the anomalous dimension in the Fourier space is that it has a multiplicative form. All right so we have mu debugging mu of J Y comma mu is simply gamma J Y comma mu J Y comma mu. So it's no more difficult than this formula up here. Just have to find the right space, which is Fourier space.

So then we can make an all order solution of that formula. Given that we know this, we can just plug-in this and integrate, and we've done this a few times now. Let me just do it formally by not doing the integrals, but writing them out in a way that you could do them to show you what the all order solution would look like.

So you can write the solution using this fact and using this fact. So we have to use that fact to convert this logarithm here into something that is just in terms of alphas.

But using those two facts, we can do the same kind of things that we've done before and write the all order solution in the following way. The integrals are contained in this W and this K.

Here's the noncusp piece, and then the cusp piece comes with two integrals because it had a logarithm, and we turned one of the logarithms into an integral as well. So hopefully I've gotten all my twos right. So we saw something similar when we were talking about the running of the Wilson coefficient of the C, because it also had this kind of form with just a logarithm and a constant term.

And it's a similar thing here. So this is an all order solution for the running of the jet function. Of course, we don't know these functions to all orders. We only know them up to some given order, and so you plug those-- whatever order you want to work-- you plug it into this formula, and then you can do this integral, and then you figure out these factors here.

And those factors are summing the logs. In the case of the jet function, what you would-- the way that you can think about this picture is that you want to-- you have to put the guys in the factorization theorem at a common scale, and so one way of thinking about it is as follows.

Let's take our hard function and do perturbation theory at mu equals mu age, and then we'll sum logs down to say the soft scale. That's one way of doing it, and we'll do perturbation theory for the jet function, at a scale mu equals mu J, and then we'll use this renormalization group here, which will just say gamma J to run this guy down to the soft scale.

In that kind of scenario, you wouldn't have to run the soft function. This is like a top down kind of picture where you're just running the objects at the higher scales down to the lowest scale. That's one way of doing it. You could also do it in an equivalent way where you run the soft function, for example, up to the jet scale, and you don't run the jet function.

And that'll give you actually the same result. But in the way-- with the information I've presented you, this is the way that you would think about doing the renormalization group, and that would sum up logarithms in the cross section. And it's very easy if you write it in position space to see what type of logarithms you're summing out.

So in position space, if we go back to a formula like the one at the top, basically, what happens is that in position space, you, again, have a product. So these convolutions here, which were also this form, if you Fourier transform, then you end up with a product. D sigma D, so schematically, D sigma Dy for, say, where y is the Fourier transform for thrust, would just be h times j in position space times the soft function in position space as well.

And there's Q somewhere, but it would be a very simple formula which just has a product. And so you can figure out what logs you're summing. The logs that you're summing, again, just as we talked about before, are simplest to describe in when you take the log because you're really summing logs in an exponent.

So if we talk about the log of D sigma Dy, then we can enumerate the types of logs that one sums as follows.

So these are the leading logs. This is next leading log, and this is next to next leading log, just like before. When we were talking about enumerating the logs, we talked about before an example, we were enumerating logs in the Wilson coefficient C. Now we're doing it for a full cross section, but because in position space, the cross section is simply a product of objects, you can think about just-- if you take the logarithm, they kind of split apart.

And it's very simple to enumerate what corresponds to the things that you would get by putting in anomalous dimensions at a given order. So if you use the leading log anomalous dimension, just the one loop cusp, then you get these terms. Get the higher terms from the running of the coupling.

If you put in the next leading log terms, then you get those terms. OK. And so you supplement that resummation which comes from this K and this omega, in general, you supplement that with sort of fixed order calculations of the H and the J and the S. And that gives you a complete cross-section at some order and resum perturbation theory.

So you could do that also in momentum space. You could write out the formula in momentum space just because after all that's what you want in the end. Position space is just really a nice way of deriving the results that then you eventually put back into momentum space in some way or another because that's where the data is.

And let me just show you what the formula would look like, just so you get a kind of picture with all the arguments. So if I were to Fourier transform the resum formula, over here I didn't write the resummation in. I didn't write in the evolution factors that correspond to this. So let me do that, but let me do it in momentum space.

So I guess I made a slightly different choice here. So I'm doing it for the thrust case. These convolutions just being integrals over the variables that are like this S prime, for example. So this is S prime integral. And this is an L prime integral.

OK so there's three integrals in this formula. I guess this is-- it's kind of a funny notation, but I could have written integral instead of writing these tensor signs. That would have been probably more clear. Do that. OK so there's the formula in momentum space.

And it's doing what I said. This factor here is running the hard function down to the soft scale, and this other factor here is running from the jet scale to the soft scale, and I didn't have to run the soft function. I should have written in here that I evaluate this guy at the soft scale.

And this guy here remember is the non-perturbative guy which we also talked about a little last time. OK so that's kind of the basic structure these use as usual are summing the logarithms, and then you put fixed order results in for this guy, for that guy, and for this guy here. And in doing so you get the [INAUDIBLE] result.

All right, so this fact that I could have done the running differently, that I could have run the soft function is a kind of consistency. So there are other ways of doing the r g, and they all lead to the same answer. And when I say the same answer, I really mean the same precisely exactly the same number.

So I really mean the same, not just sort of approximately the same but exactly the same. And the other ways of doing the RGE are the basic idea of why there are more than one way is again this fact that if I do coefficient renormalization, that that's the inverse of doing operator renormalization.

So there's two ways of thinking about doing the run. You run the operators, you run the coefficients. OK in this picture it's a little bit more complicated because we have three scales, but when I'm running the hard function, I quivalently could have run both the jet and the soft function and not run the hard function.

So let me maybe give one other way of doing this just to show you. So I could have run the hard function say, just down to the jet scale.

And then instead of running the jet function, I could have run the soft function up to the jet scale. That would be another way of doing the RGE, and that would lead to an equivalent picture. And there's more than two, because there's three scales in this. There's more than two natural things. There's three possible natural things, where you would run to either this scale, this scale, or this scale, but at the base of it, it all has to do with this equivalence that we talked about in simpler scenarios earlier.

And you can write this also as a relation between anomalous dimensions. So it implies a non-trivial things about the formula that have to be trivial in order for this to work. So for example, it implies that the coefficients of the cusp anomalous dimension in the various renormalization groups would be in a certain way that they would be related, and even in the non-cusp anomalous dimensions, which is this formula, there's a relation between the jet, hard, and soft anomalous dimensions. OK?

And that's needed-- this is an expression of this fact that you can do the renormalization group by running different objects. So you can derive this formula by just saying mu D divided by the cross section is 0. And going through it, if you were to allow, then you would find formulas like this one. OK? So questions?

I went quick, because most of the concepts here are things that we've seen before in simpler guises. The new complication is really that we have this dependence on a variable which ended up being y, but by going to Fourier space, it looked just like a simple product again, and y behaved like a simple kinematic variable. The end of the day, we want to Fourier transform back, but we have a space where things look familiar.

OK. So that's e plus minus the dijets, and that's the last SCETI example I'm going to do for a while. I'm now going to turn to doing SCETII, and we'll spend some time talking about SCETII. So in SCETII, instead of having ultrasoft interactions with collinear particles, we have soft interactions with collinear particles. So we have to call how that's going to work.

So if we take a soft particle, consider a soft particle interacting with some collinear particle, and ask what do we get out? So if we just ask about momentum conservation, and we call this q, then q is equal to q soft plus q collinear. And since we know the scaling of these two, we know the scaling of q. It's just given by whatever the larger of the scalings is.

So soft and collinear both have the same-- so let me remind you of the scaling. So this guy was q, lambda, lambda, lambda, and this guy was q lambda squared 1 lambda. So the difference between ultrasoft and collinear and soft and collinear is that soft and collinear have the same size of perp momenta. So the perp momenta are just of order lambda.

Obviously, in the minus momenta, the 1 wins, so the minus momentum of order 1. And then the plus momentum, it's the soft that wins, because it's bigger than the collinear. So this is order lambda. OK.

And that's actually much different than what we found when we added ultrasoft and collinear. Because when we added ultrasoft and collinear we got back collinear. Here, we're not getting back collinear. This is actually off-shell, from the perspective of our low-energy modes. Because q squared, the biggest part of q squared would be 1 times lambda. q squared's of order lambda which is much bigger than lambda squared.

So adding a soft and a collinear particle in SCETII immediately gives you something off-shell. That's going to make some things easier and some things more complicated. Mostly, it's going to make things a little more complicated. At least at the start, it'll looking easier.

So you could think about this from our mode picture, where we had softs that live here and collinears that live there. In order for them to interact, they have to go up to a place where we can have a common momentum, and that's a higher hyperbola, like this. It's not all the way up to the hard scale which is up here.

That was hard, but there's an intermediate scale that comes in which is the scale of this q squared. So this is q squared. This is the order of the q squared there, and sometimes this is called a hard collinear scale. This would be called a hard collinear mode, hc.

And you could even write down, if you wanted, an on-shell degree of freedom for this hard collinear mode. So an on-shell version of it, you could think of it as a mode, in theory. An on-shell version of it would have a scaling that's q 1 lambda root lambda.

So you can think of one way of approaching it which we'll take this attitude in a minute. One way of approaching this problem would be to first think about doing some matching onto a theory with this hard collinear mode and then trying to match down onto a theory that just has the collinear and the soft mode, and we'll exploit that a little later. But first, let me do a different thing and just ignore this dashed line and just think about matching directly from the QCD onto the SCETII which just has these two degrees of freedom.

So what's going to happen is that we're going to have to integrate out more things, because any type of interactions that we write down are basically giving us something off-shell. So let's do an example of this. So we'll do an example of a heavy-to-light current, but now, it won't be an SCETII current but rather an SCETI. So I just want to have one soft particle and one collinear particle. Simplest possible example, but now this is soft, and this is collinear, not ultrasoft and collinear.

All right. So we have some current. I'm just going to label the lines with c's and s's. So imagine that you attached a soft gluon here. That would give you something that's off-shelf. So this line here is off-shelf, and likewise, on this side, if this is soft coming in but you attach something collinear here, then this is off-shelf.

Before, when we had this picture, and we were doing it for SCETI, one set of attachments which still need to on-shell, in particular on this side. And on this side, we got something off-shell. We integrated it out, and we got a Wilson line. Right? Here, it's a little more complicated, because you touch alternate modes on either side. And you get something off-shell, and you just have to put out this pink line all at once, and that's going to give you another type of Wilson line.

In this calculation, we're going to get W. It's both of the collinear modes, and another type of Wilson line from the soft modes. Let's call it Sn. So how do you build up a Wilson line?

Well, first of all, think about attaching more gluons onto this line. Right? That's kind of the analog of what we did before. So just attach more soft gluons here, more collinear gluons here. This might be the first thing you would try, just adding those guys up, and that means you have a whole line of things that is off-shell, and you just calculate these diagrams.

And if you do that calculation, it will give you Wilson, lines. And what it'll give is something that looks like this, where Sn dagger is a function of the n dot As component, and W as a function of said n bar dot Ac component. So there are Wilson lines along some direction, and it looks like this. This is c. This is S.

Now, there's something wrong with this. It's not quite the right answer. The reason that the Wilson lines are the way they are is because I got the collinear ones from integrating them out from the next to the heavy quark which is the soft particle. So that's why W sits next to h. S dagger sits next to c because of the same thing on the collinear side.

But if I wanted to make them into a gauge invariant thing, I want the W to sit next to the c and the S dagger to sit next to the h. That would be the analog of what we had in SCETI, and that's not what we got. If this was QED, then that would be fine, because this and this, I could just commute them. Well, I can always do that. I can just push this guy through that guy.

But in QCD I can't, because these guys have color matrices, and they don't commute with each other. So that means that this isn't quite the whole story here. There's some diagrams that we missed. So there must be some diagrams that are non-abelian that we missed, and what are those diagrams?

These are diagrams that involve triple gluon and four gluon vertices. And what these guys do, it turns out, is they do one thing in this calculation. They really just flip the order of the W and the S.

So why do we have to consider those diagrams? So think about, instead of attaching gluons to quarks, attach gluons to gluons. So say we have soft collinear, and we attach them to each other but through a three-gluon vortex. Then, this guy's off-shell.

If that off-shell guy attaches to this guy, then it's off-shell. And if you integrate out diagrams like this, that's going to change what our result over there look like. If you go to the same order on the other side, exactly analogous thing.

OK. So these are also things that you have to integrate out. You have to integrate all these pink things out. And if you do that, then it does what I said. If you do that to all orders, what you can do with a axillary Lagrangian-type approach, obviously, you're not going to start calculating diagrams to all this, but there are some tricks to doing it. Then, you get them in the right order.

And it's easy to check, for example, that at the order of two gluons that this guy does exactly what you want. So this is then a collinear gauge invariant object, and this is then something that's soft gauge invariant, and that's nice. That's what you're hoping. That's what you're expecting.

AUDIENCE: What about diagrams like collinear soft all from the collinear line or something?

IAIN STEWART: You want to add one more gluon where?

AUDIENCE: I want to stick like a collinear right in between some softs.

IAIN STEWART: Yeah. You mean like on here, like that?

AUDIENCE: Well, yeah. Essentially, it'd be over on that diagram over there, like not even [INAUDIBLE].

IAIN STEWART: Yeah. It's the same thing, I think. So if you add this guy here, then I think it's power suppressed. Because it ends up being a power suppressed term that you don't need, if I remember correctly. Yeah. This guy's power suppressed.

AUDIENCE: You have to define like a different lambda for each one? It seems like it would be--

IAIN STEWART: No. You can think of this as a full theory diagram. Right? Where this guy is off-shell, and you haven't changed how he's off-shell. You just change the value of the momentum, but one thing you've done is you've doubled the number of hard propagators. So if I remember correctly, this guy just gives you something off-shell. I'm pretty confident, something that's power suppressed in lambda.

All right. So this is how soft collinear factorization works. So soft collinear factorization, if you think about it from the point of view of going from QCD to SCETII, it's just integrate out these pink lines, as usual, and you end up with something that's where things are splitting apart. And the reason that it's happening is because these lines are off-shell that are where you would try to interact, have interactions, and so your theory is forced apart by the fact that these things can't interact in an on-shell way. So it's different than SCETI that in that sense.

So this is kind of cumbersome, as you can see, if you wanted to think about doing it arbitrarily. Because it seems like you just have to calculate diagrams, and who wants to calculate infinite classes of diagrams for arbitrarily complicated scenarios? Right? Maybe in this case, we can do it, but actually even in this case, we have to resort to some tricks to do it to all orders. And in more complicated scenarios, it would just get even more and more cumbersome. So we'd like to have a trick that's generic, where we could get the same answer.

And the way that we can do that is by using this dashed line up in the picture. Think about formulating rather than directly this SCETII, formulate first an SCETI, and then we'll match that SCETI onto QCD. So another, in some ways, better method is to do QCD to SCETI and then SCETI to SCETII, so three steps. So we're going to first use a SCETI that doesn't have the collinear, just has the soft mode and the hard collinear mode. So it has what I call the hc in the picture and s. OK?

So that would be a mode that has P squared in this picture of order, say let's call it lambda squared. So if I say that this is for some lambda squared, then you'd have the soft mode that has P squared of order lambda squared, and this hard collinear mode that has P squared of order q lambda. And that's exactly an SCETI-type situation, where we were calling these c and us. So that's step one.

Step two is to factorize that theory, which we know how to do. In particular, the ultrasoft which is soft can be factorized with a field redefinition. So this has the advantage that at this first stage we still have a locality that protects us and helps us to understand the theory. We then factorize, and then in the third step, we match SCETI onto SCETII. And then, we're getting rid of these hard collinear modes, matching them onto some collinear modes, and then we still have our soft modes.

So we think that the hard collinear modes contain the collinear modes in the first stage. But they also contain some other stuff which is unwanted baggage that we have to get rid of, and that's why we have this second stage of matching. But you can really think of it in the picture as doing first the matching, where you integrate out the hard scale, but you contain in your effective theory this scale associated to the dashed line. And then in a second stage, you integrate out the dashed line, and that gets you to the final thing you want which is just low energy modes on this line here.

OK. So one thing this does is give a simple procedure of constructing SCETII operators, even though there's more non-locality in SCETII than there was in SCETI. Because there's non-locality, because you don't just have this one scale that could cause you, even have this smaller scale that's causing non-locality. And that's explicit in the soft Wilson line. The thing that's giving the soft Wilson line-- well, the things that are producing the Wilson lines are really modes of this scale.

OK so it's more non-local. One way of saying it's more non-local is simply that there'll be 1 over l pluses, as well as 1 over l minuses. That's another way of saying why it's more non-local.

OK so that's just an example of something that makes it look trivial. So let's say we wanted to do this calculation, this way. So then in step one, we would simply write down the current in SCETI. We would integrate out the off-shell pink lines, but there'd only be lines on one side, and we get this which is our SCETI result for heavy-to-light current.

Then, in step two, we would do the field redefinition in order to factorize this theory, and we get that result. And in step three, in order to match this result onto a current in SCETII, it's really simply a renaming here. So why is this step so easy? It looks completely trivial.

The reason that this step is so easy is that any T-product, any time-order product that you write down here, will have a correspondence in this other case. OK? So you can really-- it really is that simple. So when so if it's really that simple, then you can see the advantages of this procedure.

It's not always that simple, and let me present as a kind of theorem or just a bit of a statement when it will be this simple and when it will not be this simple. So basically, you have to know when will the T-products between these two theories match up? And the T-products will match up under the following circumstances, or they won't match up under the following circumstances. So if in the SCETI one theory you had time-order products with greater than or equal to two operators that had soft and collinear fields, then you can generate some non-trivial matching.

So I think that's best illustrated by an example again. So in this particular example, up here, we only had one operator that had both soft and collinear fields, this external current. And then we had Lagrangians, but they were totally decoupled after we made this field redefinition, the L0 Lagrangian. So they don't count as something that's mixing soft and collinear fields. We just had this operator.

But if you had two of these operators, and you wanted to go through the same procedure, then you could get something non-trivial. So let's imagine that scenario. We have two operators, so let's think of having in the SCETI, two interactions that are like this, then we could string them together as follows.

So this is a T-product between two different interactions that both had soft and collinear. I'm just taking two of these guys and putting them together. And if you look at the off-shell-ness of this line here, then P squared is of order, in our counting, it's like a k plus times a q minus which is of q times a lambda times a q. So it's really something that lives at that hard collinear scale. OK?

But you need two terms in the time-order product that, in order for there really to be a propagator like this, that you're integrating out. So this guy here will match onto something when you do this matching, where you have two soft fields and two collinear fields, like that. Because this guy here really is something that you would want to integrate out at that dashed line.

Really, it's an off-shell, hard, collinear mode. But if you didn't have two T-products, then effectively what happens is, when you change the external kinematics in order to do the matching, all what were called hard collinear lines become collinear lines, and then the matching is trivial, as it was up here. But if you're in a situation like this, then there is a line that, by changing the scaling of these external guys, it doesn't change the scaling of that internal line, and then you get some non-trivial Wilson coefficient.

AUDIENCE: [INAUDIBLE]

IAIN STEWART: So let's call these hc. Yeah, and so I start by calling them hc. Right? And then, what I want to do, when I calculate this diagram to do the matching, is I want to assign them a different scale. I want to define them to be c instead of hc. So I start out thinking of them as hc. That's what I do in step one. I can do a field redefinition. It doesn't matter.

But now, I want to match from SCETI onto II. So therefore, what I do is I take my full theory, which is SCETI, and I evaluate it with fields that don't have the right power counting for that theory. And then I do a Taylor-series expansion of all the diagrams, so that's this. I change my external fields, and I call them c instead of hc.

AUDIENCE: Oh, because you're doing the same thing with ultrasoft. So ultrasoft, it really means--

IAIN STEWART: And now I make these soft, but soft and ultrasoft, that's just really a renaming.

AUDIENCE: Ultrasoft in the SCETI [INAUDIBLE].

IAIN STEWART: Ultrasoft is equal soft. Sorry. Maybe this makes it clearer. Ultrasoft and soft just two different names for the same thing. But hc and c aren't, because hc really lived on the upper hyperbola. I put the external particles onto the lower hyperbola, but I'm frozen in here with one particle that stays in the upper hyperbola and therefore has to be integrated out.

OK. So the kind of thing that you would get by doing that is that you would get convolutions with some Wilson coefficient. In this case, they were-- well, let me just say, it depends on some P minuses. This is P1, this is P2, and then in my example over there, I guess I had some heavy quarks. OK, and there's some Dirac structures.

So it would be something like this, where you'd get some function, whatever it is, that's coming exactly from integrating all the purple stuff. So this is the Wilson coefficient, and usually, you would call these things jet functions, because they look like jet functions. So this comes from the SCETI modes that got integrated out. So the difference between this and the SCETI matching, where you're integrating out the hard scale is you're already getting sensitivity here to the plus momentum of the soft. That's one difference.

OK. So this is a general procedure for constructing SCETII. Let's see how much I want to say. So let's say a few words about power counting.

So if you take some T-product of terms in SCETI, that will scale like some power of lambda to the 2K OK? So in SCETI, you have to be a little bit careful about power counting when you do this procedure. So in SCETI, you could just assign some power. Let's call it lambda to the 2K.

And when you do a matching onto an operator in SCETII, what you're generically going to find is that there'll be a relation between the power counting here and the power counting there which is good, but there's also one thing to be aware of. So if I define eta to be the power counting parameter for SCETII-- just to give it a different name, so I can talk about a correspondence-- then basically, eta is like lambda squared. And that, well, that just follows from this formula, that if you want to identify lambda squared is lambda over q for the collinear modes, then this would be the right way of doing it. Little lambda would be square root of lambda over q.

But in SCETI for the collinear modes, you'd say there's some parameter eta which is just lambda over q. OK? So basically, it looks like you just take 2K and go over to k, because you just changed the definition of what you're calling the parameter, but there's also this plus E. That means that you can get additional suppression, and that plus E comes about from the fact that you also change the power counting for your external fields, when you do this procedure. So something that was scaling like lambda in the hard collinear theory might become lambda squared-- i.e. eta-- in the SCETII theory.

So this E which is greater than 0 comes from changing the power counting of external fields in the matching. One example is just having an external collinear quark or external collinear perp gluon, for example, where you would have c of order lambda in SCETII, and it becomes of order eta in SCETII which is lambda squared. Those are extra factors of the power counting parameter.

OK. So what do we learn, if we put all these things together? Well, one thing since soft and the collinear fields don't talk to each other, when you write down Lagrangians for them, they're totally decoupled right from the start. And so there's actually no mixed soft collinear Lagrangian at leading order. And so all the meat of this theory is coming about when you integrate out these off-shell modes, and you construct the external operators.

OK. I'll say little bit more about power counting. So there's no mixed soft collinear Lagrangian at leading order, and you can get mixed things at sub-leading order. So in that way, it's like SCETI after the field redefinition. But here, there is no analog. If you just want to go directly to SCETII, there wouldn't be an analog of the field redefinition. All the Wilson lines are coming from integrating off-shell particles.

OK. So we're going to speed up a little. I'm going to skip over one thing, which I don't think I need. So you can write down power counting formulas, like we did in chiral perturbation theory. Maybe I'll just write those formulas down. So in general, in chiral perturbation theory, we had a formula that said, if we want to figure out the power counting of any given diagram, we can have a counting for that diagram, and we know what size it is. And there's analogous formulas like that in both SCETI and SCETII.

So in SCETI it's pretty simple. You say you have something that's order lambda to the delta. Then delta, the formula for it follows. This is very analogous to what we talked about for chiral perturbation theory.

So these guys here are vertices that are purely ultrasoft, and if you have purely ultrasoft vertices, you're subtracting 8 because the measure of the ultrasoft particles is 8. I'm not deriving this for you. I could, but I think it's fairly intuitive. So if you think about the lowest order Lagrangian, say psi bar ID slash psi for ultrasofts. This would already be something that's lambda to the 8, and this is then leading order, so you subtract 8.

And this guy here is the rest, anything mixed or purely collinear. And then for that guy, you subtract 4, because as soon as you have at least one collinear particle, then you have to use the collinear measure. And so here, some operator like c bar, and dot dc was order lambda to the 4. So you subtract 4 and that's leading order.

And so what this formula allows you to do is really just you do power counting entirely in terms of vertices. You never have to power count measures for loops or power count propagators. You just count vertices.

So if you want to know how big a time-order product is, and it's a time-order product of an L3 with an L2 with this kind of setup, this will be basically you count this as lambda phi. And you could even get the absolute size right, taking into account the scaling of external particles, and that's what this constant 3 factor does. And U is equal to 1 only for pure ultrasoft. Otherwise, it's 0. OK?

So there's formulas like this, and there's an analogous one in SCETII which is a little more complicated. But it allows you just to have a power counting where you can just power count Lagrangians or operators 01 with L1, and that's lambda squared without having to worry about what propagators do I have in this time-order product? What loops do I have? You never have to ask those questions. You just have power counting at vertices, and that makes things quite easy.

AUDIENCE: When you write these expressions, one insertion of Lagrangian corresponds to a single vertex. [INAUDIBLE] that Lagrangian.

IAIN STEWART: Yeah. That's right. OK. All right. So I want to do a couple of examples in SCETII. So let's do an example.

Let's do an example which is in some ways simple, and it's an exclusive analog of our DIS example, and it's called the photon pion form factor. So it's exclusive. It's clearly exclusive. There's a pion, and really, all it's going to come in here is hard collinear factorization.

So it's simple in the way that it actually is not really exploiting the full complications of SCETII. It's really just another example, where we have hard modes factoring from cleaner modes. But they're clearly SCETII modes, because they're going to be modes for this pion.

So we're going to use again for this calculation a bright frame, which you'll see, I guess, when I write down some momenta. So what would we write down for this process in QCD, first of all? So you'd say, I have a pi 0. It has some momentum P pi. I have some current which I can take at 0, and I want to make a transition from the pi 0 to a single photon.

So you could write that as follows. I can always replace the photon by current, and then I can parameterized that matrix on it like a form factor. And if we go through things like parity, charge conjugation, stuff like that, time reversal, we find out that there's one real form factor, and there's an epsilon symbol in this guy. It has to be linear in the polarization vector. That was already true here, and there's one way of getting the indices to work out which is with this epsilon.

So what you're really after in this process would be some understanding of this form factor. That's the one piece of non-perturbative information. Everything else, Lorentz invariance is enough to tell you about. So we'd like to see if there's a factorization theorem for that form factor, if we take the limit where q squared is large.

So we have an electromagnetic current. You can think about it is up and down quarks for the pion. So there's some matrix in the 2 by 2 space which is the charge, which you can write as 2/3 minus 1/3, like that. Or if you wanted to write it in terms of polymatrices, you could write it as identity over 6 plus a tau 3 for isospin polymatrix over 2. So there's some charge matrix that's going to show up.

What happens if we expand q squared much greater than lambda QCD squared? In this formula over here, this form factor knows about lambda QCD, and it knows about q squared, and we haven't expanded. So what happens if we do expand? Can we simplify that form factor? And it does.

So I'm going to do this in the bright frame again, and that effectively means I'm taking the momentum that corresponds to the off-shell photon as the same form as we did before for our calculation in DIS, purely in the z coordinate. The momentum of the photon, it's an on-shell thing, so I can just write it as E the photons say times a light-like vector. Then, P gamma squared will be 0, as we want it to be. And if I pick this kinematics, then P pi is just P plus P gamma, so that would be a P pi. OK?

So in Feynman diagrams, what am I talking about? I'm talking about making a transition with an off-shell photon, through a diagram like this one, to a pi 0 or plus the cross graph. And with this setup with this kinematics, this intermediate line here, if we go through the scaling, it's going to be hard. So these guys with this kinematics, we make the pion collinear.

In order to see that, you can impose P pi squared equals M pi squared. Right? This is never going to be made small, but E is going to have to be tuned to be basically q over 2. So this thing is going to become, if you impose this condition, something like M pi squared over 2 q.

Then, you find out that the pion is collinear. So the pion is collinear. The photon is a photon. This line here is hard, and so we want to integrate out the hard line and match this guy in the effective theory onto some effective theory operators that would look like this. This guy is pink.

OK. So we just have to write down what type of effective theory operators that could be, and again, it's an effective theory operator of two quark fields. So it's very much like what we did before. I'm not going to be going through all the indices that I have to go through in order to keep track of charge conjugation and all those things. I'll just write down the right answers for that part.

So I could write like this. So after doing hard collinear factorization, there's again operators which is two quark's. Some hard Wilson coefficient which is this pink thing, integrating out the pink line that sits in the middle. This obey current conservation. Dimensional analysis fixes this 1 over q.

Charge conjugation, actually, also provides constraints on the Wilson coefficients, just like it did in DIS. So charge conjugation tells us that there's a relation between flipping signs. But we just impose on this operator all the symmetries and things that we can think of and see what it says about the operator. You can think about just writing the operator down based on this picture without ever calculating any diagrams, just knowing what you're after. And then imposing on it all the symmetries that should be conserved.

So one is dimensional analysis. That gave the q. Current conservation, that's partly responsible for the epsilon which is also like a parody.

Go through flavor and spin, and you find that this has this structure, and there's some constant that I threw in here. And you actually know from the flavor that it's got two photons. Right? And I'm not ever adding any more photons, so that means there's two factors of q hat. And so there's a q hat squared, which I can stick inside this gamma.

And it has to be a color singlet, because it's going to have to have a non-trivial matrix element with pi 0 which is a color singlet. And since it has to be-- so there's no TA inside the gamma that's what I mean. There'd be nothing for the index A to contract with.

And since it's a color singlet, that means if you were to think about soft interactions here, they would just cancel. Right? If I were to think about putting in soft interactions and integrating them out, I'd get an S dagger S that would cancel out. So that's the sense in which this is a simple example.

OK. So then, we would have a formula where we could equate the effective theory result with the full theory result. So we can equate matrix elements, and this is one way in which this example is different than dependent elastic scattering. We're really doing a matching at the amplitude level. Remember, the form factor was a parameterization of the amplitude, and if we do that--

If you like, if you think about the two currents that we had, and we've integrated them out. Now, think about just this operator here. The matrix element pi 0 to vacuum that we had to finding this thing which had two currents just becomes a matrix element of that operator.

Again, like our DIS example, we form the sum and difference at the momenta. We can think of forming P bar plus which is P dagger plus or minus P. And one of these guys just gives the total momentum, just the minus with our sign conventions. So that gets fixed which, in this case, it just gets fixed to the pion momentum which is q.

So if you like, you've inserted the operator here. You have some collinear lines, and then you have a matrix element of the pi 0, and then you have gluons. But you know that whatever momentum comes in has to be the momentum of the pion, and that's the one momentum constraint. So that means that the answer just involves one convolution again which is the one that's unconstrained.

So there's a Wilson coefficient that depends on the other unconstrained guy which was P plus, and then there's a matrix element, where I can write in a delta function with that P plus, the usual kind of way that we've done. But here, it's a little different matrix element than the one we saw on DIS, because it's not forward. It's vacuum to pi 0. But it's actually the same operator that we were talking about in DIS. It's just a bilinear operator with two collinear quark fields, two collinear quarks in dress with Wilson lines.

So one can go through a similar type of matrix element analysis for this operator, and it gives something that's called the light cone function. So let me define that for you. This matrix element can be written in terms of an object that has a dimensionless variable. It's an analog of the part-time distribution function but for this matrix element that we're dealing with here which is vacuum to pion.

So this has some similarity to the formula we had in DIS. There's a delta function there. There's a delta function here. There's a dimensionless variable z, dimensionless variable z there, and this is a non-perturbative function.

So this is what's called a light cone distribution function for the pion. So generically, when you have an exclusive process, and you're producing some hadron that's very energetic, like a pion, this is the type of thing that's going to show up, one of these light cone distribution functions. This example of photon to pion is in some sense a very, very simple, exclusive process, the simplest one in some sense.

OK. So we could take this formula, plug it back in there, and then we'd have a factorization theorem. And I think that you can imagine what that would look like. I could write it, instead of an integral over W, as an integral over z, and that would be the factorization theorem involving this 5 pi.

AUDIENCE: [INAUDIBLE]

IAIN STEWART: Yeah.

AUDIENCE: [INAUDIBLE]

IAIN STEWART: That's right. So you should think of the z as like-- so the way to think about the z is as follows. So think about like when you initially produced these guys here, think about all the momentum going this way. When you initially produced them, after you integrated out the hard interactions, you had z and 1 minus z is the possible split of the quarks fields in the operator.

So one of these guys carries z. Effectively, what this 2z minus 1 is doing is one of these guys is carrying z, and one of them is carrying 1 minus z. OK? The sum of these is 1, and that's the analog of the statement that the whole total momentum should be the pi 0 momentum. But you don't know how to split how much goes into each one of those.

And what the wave function is, it's all the linear interactions that subsequently rearrange this thing before you annihilate it with the state. So all these things are dressing up the pion state. They're producing the pion pole. So this is like 5 pi.

And so what you have is an operator that depends on z, a wave function that depends on z. So this is a Wilson coefficient that depends on z, and your final factorization theorem is exactly of that type, that you sort of-- of an integral of z of c of z Which also can depend on q and mu and then 5 pi of z and mu.

AUDIENCE: Is there a sense of the 5 pi z as universal?

IAIN STEWART: Yeah. It's universal.

AUDIENCE: You can use it for other [INAUDIBLE]?

IAIN STEWART: Absolutely. Yeah. So as long as you can factor it so that it's these fields ans that pion, then you have this matrix element, you get this guy. Some people try to measure things about this guy, it's moments and stuff.

All right. One thing that happens here has to do with this integral over z which is still in some sense an unsolved problem in SCET. So I have to mention it. So when you do this integral, you could ask, what does the c of z look like? And it turns out that c of z will have in it terms that go like 1 over z. And so you get integrals that look like dz over z of 5 pi of z.

So at lowest order, this integral would show up. And it turns out that, for this matrix element here, you don't get anything worse than that. You never get 1 over z squared. And this integral here, because of properties of this 5 pi, is finite. There's no problems.

But there are examples known in the literature, where that's not the case, where you actually get 1 over z squared, and people have some understanding of the physics that's happening in those cases. But there's not a complete understanding of how factorization works in those cases. OK? So that doesn't happen in this example, but there are other examples of exclusive processes that would lead to 1 over z squares, and then this integral is not well-defined.

And people understand that there's a cut-off that's coming in, and they understand that that cut-off actually has to do with some rapidity. But how to explicitly write down an analog factorization theorem that involves those cut-offs and has renormalization group is an unsolved problem, unsolved SCETII problem. OK? But for the example we did, everything's kosher and beautiful.

All right. So I think what I'll do next time, since we're out of time I'm not going to start my second example now. So next time, we'll do an example that does involve soft fields, both soft and collinear fields in SCETII. That's where we're going, but we'll leave that to next time.