MITOCW | 9. HQET Matching & Power Corrections

PROFESSOR: Question. We know how to do matching and running. We've seen an example of that. In this area, there's this label v on the fields, and we want to figure out how that affects doing matching and running. So we started out talking about the wave function for normalization calculation. So we have these fields, which we can have a bare version of and a renormalized version of in the usual way. So this is renormalized, and this is bare. Some Z factor between them that I'll call Zh for Z heavy.

And so we have to calculate at the lowest order, this diagram, send in a momentum P. This is Q plus p.? Use the Feynman rules. Use dimensional regularization. Use dimensional regularization with MS bar, so there's some extra factors. And use Feynman gauge, which is usually the simplest gauge choice.

So each of these vertices here gets a v. There's a v mu from this vertex. A v mu from that vertex. And that's where this v squared comes from.

And then there's one propagator here. It's very traditional to denote a heavy quark by two lines rather than one line, so you know which lines are heavy and which lines are light. So these two lines is a heavy quark, and that gives this propagator here, this v dot q plus P.

And then there's a relativistic propagator for the gluon, and that's the q squared. I've taken into account all the i's when I put this minus sign, and this is just the fundamental Casimir from dotting two TAs together.

OK. So when you have an integral like this-- so v squared is 1. That's one simplification. When you have an integral like that where you have a linear momentum in one propagator and a quadratic momentum in the other propagator, you don't want to use the standard Feynman trick. I call this trick the George I trick.

So it's very similar to the Feynman trick but slightly different. So you use an integral that goes from 0 to infinity, and you can convince yourself that this is true. And so then set a equal the q squared and b equal to v dot q plus p. And the reason that you'd want to use this trick rather than the usual one is that if you use the usual one, you'd get an x multiplying this and a 1 minus x multiplying that. But then you would have an x multiplying the q squared.

So when you complete this, it would be-- when you'd want to complete the square, you'd like nothing to multiply the q squared. You'd like the q squared just to be bare by itself with no Feynman parameter multiplying it. And this trick does that, because a has no Feynman parameter multiplying it, or George I parameter in this case.

OK, so we combine denominators in the usual way, and the denominator would become this. And if I kept the i epsilon it would be that if we combine these two denominators here. So this factor would be that. And we can then complete the square, right. This is some momentum squared minus whatever is left over where t is q plus lambda v, and then A is the rest of the stuff, which is this.

OK, so then this guy, now we just have our usual quadratic integral that we can use the standard rules to do. And then instead of an integral over x, we have this integral over lambda.

There's some factors that I'm dropping. So there's that e to the Euler gamma times epsilon and the 4 pi minus epsilon. So write in everything with d equals 4 minus 3 epsilon. Do that integral, which is just giving some gamma functions. And if you think about the dimensions here, so we end up with something that's d minus 3.

If you think-- if you want to look at the dimensions, d over 2 minus 2 is dimensionless, OK. The lambda had dimensions. If we go back here, that's obvious. q is dimensionful, v is dimensionless, so lambda has dimension 1.

So the dimensions of the-- actually, the dimensions of the mu to the 2 epsilon are compensating the dimensions here, and then there's one power of dimension left. And that's why if I take d equals 4, I'm getting one power of momentum upstairs, which is what we would expect for an inverse propagator for a heavy quark, is 1 factor of v dot P.

So expand this. And we get a divergence. So add a counter term for wave function renormalization. So if we did that in MS bar, then the Zh would be just this. And that would be the appropriate counter term to kill off the 1 over epsilon divergence.

I'm going to carry out the calculation today, or this discussion of matching in MS bar for everything. And I'll show you some of the slight complication that shows up in that case. But we said that we could do matching-- basically, you could have two choices here. You could either use on-shell renormalization, or you could use MS bar. What's the difference?

In MS bar, you just keep the divergence in the z. In on-shell renormalization, you keep some extra terms here, all right. And either way that we choose to do things, we should actually end up with the same matching, and I want to show you why that's true.

So in order to show you why that's true, I'll pick to use MS bar at this point, and we'll see what complication that leads to later on. OK, so that's one thing that needs to be renormalized. And if we look at this guy and we compare it to z psi, this is not the same as z psi in QCD. So this is something different, and the reason it's different is because we did a different loop integral that had this heavy quark propagator, not the light quark propagator.

So we have to renormalize also local operators. And so let's think about something that would make a heavy to light transition. So for example, if you looked at b goes to u, electron neutrino, then that would be a heavy to light transition. To b quark is heavy. The b quark is light.

So to describe that, you would use some operator that has one heavy quark and one light quark. So let me call the light quark small q and the heavy quark big Q. And so you'd have some operator looks like that. And we could write down a renormalized operator with renormalized fields and then group all the z factors into a counter term.

So let's think about doing the perturbation theory that way. So this is the renormalized operator. This is the counter term. So there's a wave function factor Zq for the light quark and Zh for the heavy quark, and then there's some Zo for the operator renormalization.

OK, so to renormalize this operator at one loop, we insert it, and we do a one-loop diagram. I'm just going to tell you the answers, but let's draw the diagram. So here's the operator inserted. Here's your heavy quark. Here's your light quark. You have a diagram like that.

And then you combine this calculation with the wave function renormalize Zh and Zq. Zq is the same as Z psi. We should've called this Zq for the light quark. Combine these things together, and then, because that graph is telling you to count this counter term, you need Zo.

And that calculation, which you can look at in your reading if you want to look at more details, just gives you something about what you'd expect. It gives you a 1 over epsilon divergence factor of g squared. So the operator here has renormalization that's just minus alpha s over pi. That's the anomalous dimension.

So what is this anomalous dimension doing? If you were to consider this kind of process in full QCD, then you'd have here gamma mu 1 minus 5, and that's a partially conserved current. So there would be actually no anomalous dimension to this operator.

The vertex graph that we just drew over there would cancel the wave function graphs exactly. There'd be no anomalous dimension. But here we have one. And that's because these guys are not equivalent anymore. You saw that the Z for the heavy quark changed, and the vertex graph also changes. And we're left with something.

And this remainder has to do with renormalization group evolution below the mass of the heavy quark. Above the mass of the heavy quark, there's no renormalize group evolution of this current. Below the mass of the heavy quark, there is.

So there's additional logs, and that's because MQ is now being treated as infinite. So things that, from the point of view of QCD, were logs of MQ have now become UV divergences, and that gives an extra anomalous dimension.

One thing you can note about this anomalous dimension is that I didn't really specify for you what the gamma was. I told you for this process it would be gamma mu 1 minus gamma 5. And the results here, actually, if you carry out this calculation with arbitrary gamma, you find that it's independent of gamma.

So you get the same universal anomalous convention for any spin structure. And that's partly related to the things that we talked about last time with the spin symmetry of HQET, which is telling you that certain couplings are not sensitive to the spin. And effectively, in this diagram, you're getting a v slash here. Well, let me not try to go through it but leave it for looking at in your reading, but we'll talk a little bit more about this in a minute. I won't try to explain where that comes from from the diagram.

So let's look at another case. The only real interesting thing that happened here was that we got-- well, the fact that the answer was non-0 was interesting, and the fact that it was independent of gamma. But the v didn't show up. And the reason that the v didn't show up in this calculation here is because there was only one v, and v squared was equal to 1. So v couldn't really show up because we had to get a scalar answer, and since v squared is equal to 1, just, it's not showing up.

So something more interesting is to look at, instead of a heavy to light transition like that, a heavy to heavy transition. So we'll spend a little bit more time on this one. So let's have two heavy fields. And I'm going to take them in a current where they have different velocities, v and v prime.

So let me imagine I went through this procedure of separating out counter term and renormalized the operator just like I did over there. So I have these two terms, two types of structures. Now I don't have a Zq, but I have two heavy quarks. So I have Zh, root Zh root Zh, which is just Zh.

So an example of this would be something like B meson changing to, say, a D star meson electron and a neutrino, so having a charm quark replace our up quark. OK, so now the charm quark and the B quark could both be thought of as heavy. They have different masses, but we take both of those masses to infinity, so we can use HQET for both of them.

So Mb and Mc are going to infinity. And there's no reason why, in this process, that the b quark and the charm quarks should have the same velocity, and so we'll give them different velocities, v and v prime. OK, so we can go through the same thing. We already calculated Zh, so we just have to calculate a graph like this or two, where you have two heavy quarks.

But the heavy quarks have different velocities. So what would that calculation look like? Again in Feynman gauge, and let me just take the external quarks to have zero momentum for simplicity, zero residual momentum. So this guy has k equals 0 and k equals 0.

So the HQET Feynman rule for this guy has a v dot q if q is the loop momenta, and this guy has a v prime dot q, OK. So the integral that have to do is that integral. And you can do this again with one of these tricks where you use lambda parameters. And then one of the handouts on the web page, I've given you the appropriate trick that's two lambda parameters for this integral.

This integral actually has both ultraviolet and infrared divergences. We're here in our discussion only interested in the ultraviolet ones, because we're worrying about the anomalous dimension right now. And again, I'm not going to drag you through the details of this calculation. It's essentially just, do the Feynman parameter, combine denominators with two Feynman parameters, complete the square, do the loop integral, do the Feynman parameter integrals, get an answer.

Combine it together with the counter terms of the wave function renormalize, and see what type of counter term you get left over. And it's more interesting than the heavy to light case. So here's what it looks like, where w is v dot v prime, and this function, r of w, is the following. It's log w plus square root w squared minus 1 over the square root of w squared minus 1. OK, so it's a non-trivial structure.

So that counter term would lead to an anomalous dimension, which depends on this r of w. So the reason that that can happen is because v squared is 1, v prime squared is 1, but v dot v prime need not be 1. So v squared was 1, v prime squared was 1, but v dot v prime is not fixed.

Well, it's not fixed simply by-- that's a poor choice of words. This is a parameter that can vary. So it will be fixed by kinematics, but it can depend on the kinematics. So let me go through this and organize it as a bunch of notes, comments. So this is what I just said. The answer depends on v dot v prime. And the way that you should think about this is that you have a current in the effective theory.

So this is in the HQET current. And just like we label it by its Lorentz index mu, we should label it also by the v and v prime, because the fields involve v and v prime. So if we thought of this as some vector current or axial vector current labeled by mu, we would also label the current by v and v prime.

So these are indices that are just indices on the current, and when you think about the Wilson coefficient or the anomalous dimension, it can depend on those indices. Now because the Wilson coefficient is a scalar, it really only depends on w. So if you think about Wilson coefficients, they depend on alpha s. They depend on scale mu. They depend on the hard scales in your problem, and the hard scales in the problem here for us are Mb times v and Mc times v prime, because those are the heavy scales of the heavy charm quark and the heavy b quark that we're removing.

But since we know that this thing is a scalar, we can just square these, dot them into each other. So it's really a function of mu of alpha s mu Mb squared, from squaring the Mb term, M charm squared, and then this w factor. Those are the scalar quantities it can depend on.

So what should I take for the w? Well, we could work that out for an example like the one I was saying that we're doing, B to D star L nu. So I want to show you how that works.

So for the B meson, for momentum, we can take it to be M capital B meson times v mu. And by momentum conservation, that's going to be the D star momentum, which we can take to be MD Star times v prime mu then some momentum transfer, which I call q. This is a different q than my loop momentum q. Sorry about that.

So then we can just take this relation, and we can square it, and we can get v dot v primes. So if you look at 2 squared, we can solve this for v dot v prime, and it's just fixed. You see that v dot v prime is fixed in terms of the meson masses and the momentum transfer, and that's the momentum transfer to the leptons here. How much momentum do they carry away?

So you could think of all those things as external. You fixed how much momentum the leptons carry away. These are fixed numbers. You look them up in the PDG, and then what value of v dot v prime to plug-in here. Now it's a function of q squared.

Q squared can vary in the process. So in that sense, it's a non-trivial function, not just a fixed number. But for any fixed kinematic configuration, any fixed q, then it would just be a number.

So if you look at this in practice, you find that this guy for this particular case goes from 1 to 1.5. So that's the kinematic range that's allowed. So it is fixed by external kinematics, kinematics that is external to the dynamics inside the loop. And then that way, the Wilson coefficient here is more non-trivial than the ones we saw earlier, which just depended on masses. Now it's depending on masses as well as this function of v dot v prime, OK?

Again, one finds that gamma T comes out to be exactly the same, independent of the choice of the spin structure. So we could do this calculation with any spin structure we like, and heavy quark symmetry in this case is all it takes to show that this gamma T is independent of the spin structure. So if you think about that from the loop graph, actually in this case, it's pretty easy to see, because remember that this vertex didn't have any spin structure. The propagator had no spin structure.

So nothing in the calculation had spin structure. So the only thing that's had spin structure is the gamma you stuck in there. So it's just a scalar times gamma, so it doesn't care about the gamma. On whatever the tree level gamma is, it just goes through. It's not touched by heavy quark, by the HQET Lagrangian.

So what is physically going on here, and what is HQET doing for you is that there's logs like this, MQ over lambda QCD and QCD. And by going over to HQET, this becomes a log of mu over lambda QCD, which is encoded in HQET operators like this current, as well as a log of mu over MQ or MQ over mu, which is in the HQET coefficient functions, HQET Wilson coefficients.

So just how we-- much exactly the same as how we talked about it for integrating out a heavy particle, the logs get split up into pieces, and the Wilson coefficient into pieces, and the matrix elements, operators. Here we're separating MQ heavy quark mass, and it's both the charm and the bottom in the case of what we're talking about. And the anomalous dimension that we calculated sums up those logs, and summing up those logs involves a non-trivial function of this w.

But we actually know exactly the non-trivial function, because we can calculate it. And it's just this guy here. OK, so the new wrinkle that can come in HQET is that the anomalous dimension can become a more non-trivial thing.

So if you look at it at leading log order, the rest is pretty straightforward. So if you go through the leading log result, you would do the same type of thing that we did before. You would match it, some scale. And at that scale, you could normalize things so that the Wilson coefficient at mu equals MQ is just 1 at tree level. And then the leading log result is the function of these various things, which in general is C of MQ times some evolution from MQ to mu, suppressing some of the dependencies.

The leading log result is 1 for this. And then this guy at the lowest order is just a ratio of alphas again. And then there's the gamma. For the purpose of solving the RGE is just, the only thing that matters is it's alpha.

So a gamma's a constant for heavy to light, just a number. So for that current, it was a constant. This solution is actually valid for both of them. And then it's a function of this w for the current, where we have two heavy quarks.

So the w dependence just goes along for the ride when you're solving the anomalous dimension equation. OK? So that's what re-summing the logs would look like. Essentially, each log is getting extra powers of this factor of gamma.

So with number four, is there any questions about that? OK, pretty straightforward. So much of this, essentially all of the story, except for this one wrinkle, is very similar to integrating out a massive particle. And the other part of the story that's similar is that the HQET matrix elements depend on mu as well.

And so in our example that we talked about last time, we had a matrix element. So let me give it to you in the context of that example. So last time we were talking about something which was a decay constant, and that's one example of this heavy to light current.

So we had our current which had one heavy quark, one light quark, and then a heavy meson. And we figured out that this was giving some a times v mu, and now I'm telling you that you should think of the a as a function of mu. The matrix element here is a function of mu. OK, so that's just a slight modification to what we talked about last time.

And again, if you want, for this matrix element, you'd want mu to be, say, a GeV or some scale that's greater than lambda QCD, and so what you would do is you would evaluate matrix elements and define that parameter at some scale like a GeV and do renormalization group evolution from the heavy quark mass down to a GeV. OK? So that's how the renormalization group evolution story would be.

You don't want to run all the way down to lambda QCD, because the anomalous dimension has to remain perturbative. So you would decide what your cutoff is for where you think perturbation theory is still valid. Often people pick something like 1 GeV or 1 1/2 GeV for these types of problems.

And again, this is just separating out all the MQs, making sure that your matrix element here has no MQs, it does have an extra cutoff mu. OK? So that's the RGE story. Let's also talk a little bit about the matching story.

So these are the perturbative corrections at the scale MQ, or alpha s at MQ. We had perturbative corrections at the w scale when we integrated out the w boson. Now we have another set of perturbative corrections at the heavy quark mass scale when we integrated out the heavy quark mass. It's different because we're now passing from something that looked like a full QCD theory with some external operators. We're now passing to this HQET theory for the heavy quark.

So if you like, we previously had Mw. We knew how to do renormalization group evolution there, and we had a Hw theory here. Now we integrate out the heavy quark mass, and we pass to an HQET theory below that scale.

So if you go back to the Hw theory, if we can call it that, and we want to match that onto HQET, then we do it with a calculation like this. And I'll just use this heavy, light example still. So here's a matrix element that you could consider for the matching. And let me write it with a bunch of schematic objects and then explain what they are.

So we use our spinners. I'm taking zero momentum here, just for simplicity. These Rs are residue factors that come in. So so far when we calculated the wave function renormalization graphs, we just took the divergent piece. And if you do that when you do the matching computation, you have to use LSZ.

And so the finite pieces come back in, and you call them residues. So these are finite residues that you have to take account of, finite in the UV sense. So UV finite residues you have to take into account if you're just using MS bar, and this here would be the vertex renormalization graph. OK, both diagrams look like this. In QCD, they're both the same type of structure.

And then in HQET, it's a similar thing. We can write down a formula for the s matrix element, the same states, now with our effective theory current. And we know how to transition from effective theory and full theory states. We talked about that last time.

And so there again would be some residue factors. And if the residue factors are not the same in the two theories, you have to account for that. And they won't be, because one of the heavy quark residue will be different than the heavy quark residue here.

So this guy here is this finite piece of this graph. This guy here will be the same as above. And this guy here is the heavy quark vertex graph, which is independent of the spin structure. It's not independent of the spin structure up there.

So we could carry out the calculations of those loop diagrams, and then we could subtract, and we could see what's left over. And whatever's left over is the Wilson coefficient. OK, so very similar to what we did before.

Calculate, subtract. What you find when you do that is that there's actually two currents. If you consider a vector current where gamma is gamma mu, then the effective field theory, which is HQET, has two effective theory currents that are vector.

So you have C1 and C2. The reason that there's two is because we have another vector to play with, which is v mu. So that can have q bar v mu replacing the gamma mu. So v mu wasn't an external thing in QCD. It was part of the dynamics. Here it's an external thing, so it's allowed to replace gamma mu as one of the structures.

And if you go through the calculation, this is the result, just to show you what the result looks like. Remember, the heavy to light case is the case where you're not getting a non-trivial function on the right hand side. So you'd get a non-zero result of order alpha s for both of those coefficients, OK?

So the reading also goes through this whole thing for the heavy to heavy case, which is more interesting. But it's not really more non-trivial than what we've talked about so far already with anomalous dimension. You get results that are functions of w. Wilson coefficient would have functions of w showing up here, OK. So I won't go through that.

Now if you wanted to carry out this calculation, it looks like it's kind of involved, this graph, this graph, this graph, all these diagrams to consider. You'd like to make your life as easy as possible, and there's actually a very nice trick here for doing that I have to mention to you, because it's kind of magical. So what is the fastest way that I could get this result?

So this is a nice trick to remember if you ever have to do a calculation like that, because it's not specific to HQET. So let's pick our infrared regulator to make the effective theory as simple as possible. We have some choice in how to pick the infrared regulator. The result for Wilson coefficients and anomalous dimensions will not depend on that choice. So let's use that freedom and make things as simple as we can.

And the choice that does that here is to use dimensional regularization for the UV, as we've been discussing, but also for the infrared. So let's use dimensional regularization for both. If you do that, you can convince yourself that all heavy quark effective theory graphs with on-shell external momentum-- so I can take the external momentum on-shell. I don't need it to regulate divergences, because I'm going to use div reg to do that.

So all the integrals are scaleless, and that means that they come out to be something that, if you think about it, is either zero or zero in a way where you have the ultraviolet divergence cancelling an infrared divergence, which is still zero. Now you have to think about the fact that there's both of these going on, because you still have to think about adding counter terms to HQET to cancel the UV divergences. But the answers are very simple, because you can throw away all the finite pieces.

If you have 1 over epsilon minus 1 over epsilon, if you multiply by epsilon, that term's not there. So 1 over epsilon gets removed by counter terms, and there's no finite pieces left over. So just use MS bar, so you just strip off that exactly, and you're left with 1 over epsilon IR.

Now if you-- so the effective theory diagrams are just simply all 1 over epsilon IR. Now the reason that this is making things simple is because you also know a fact, which is that the IR divergences in the full theory and the effective theory have to match up. So these 1 over epsilon IRs have to match up with your full theory calculation.

So if you renormalize the QCD calculation, which you can't really get around doing, you have to do that calculation. So you do that calculation in pure dim reg, same IR regulator, which is a nice regulator to use for QCD. You do the UV renormalization using the standard counter terms, and what will you get? You will get something that looks like a number over epsilon IR, and then you'll get numbers times logs mu over MQ plus other things.

This thing here just cancels with the-- if I subtract HQET, this is just canceling this. So this guy here cancels when we subtract HQET. And so the matching is then just this.

So I don't actually even have to consider calculating the heavy. If I use all these facts that I know, if I trust them, then I don't even have to calculate the HQET graphs. I just say, let me imagine that I calculate them. They're all scaleless. They look like that.

Let me imagine that I renormalized them all. The 1 over epsilon UVs are gone. I'm left with 1 over epsilon IRs. I do this calculation. I say, let me imagine that these cancel each other. And then I have the matching.

So that's exploiting all the facts that we know about effective theories and full theories to get the matching as quickly as possible by just doing the full theory calculation with a particular regulator. It's not checking anything. It's not checking that the full theory and the effective theory have the IR divergences matching up, et cetera, et cetera. But if you know that that's true, if you trust that it's true, then this is the fastest way to get the matching. Seems like magic, right?

OK, so sometimes you can exploit what you know about the effective theory to get things more quickly. So questions about that?

AUDIENCE: What is [INAUDIBLE]?

PROFESSOR: Yeah. So if you think about the loop integral, then the d here, right, and 4 minus 2 epsilon have an epsilon greater than 0, decreasing the powers of q in the numerator is making it more UV convergent. So to regulate the IR, you want to have epsilon on the other side. So this is what you need for regulating IR, and this is what you need for regulating UV.

So it may seem contradictory that you could even do both of these things at the same time, because greater than zero and less than zero. But you could always think of splitting up this integral with a hard cutoff somewhere in between and then just using this above and this below that cutoff. And the cutoff dependence will cancel when you put the pieces back together.

So it's actually valid to just do calculations. And for the most part, you can just close your eyes, and you'll get some gamma of epsilons and some gamma of minus epsilons, and those will be separately regulating the divergences. And if you ever worry about it, you can do what I said. You could put a cutoff in and check that you're not making mistakes, but for the most part, it just works automatically. Any other--

AUDIENCE: [INAUDIBLE] all of the epsilon UV [INAUDIBLE] epsilon IR.

PROFESSOR: Yeah.

AUDIENCE: [INAUDIBLE] minus epsilon UV [INAUDIBLE] zero [INAUDIBLE].

PROFESSOR: Yeah.

AUDIENCE: --negative.

PROFESSOR: Yeah.

AUDIENCE: But formally, you can set the epsilon UV plus an epsilon IR.

PROFESSOR: That's right, yeah. So formally, this is zero. And the reason that you have to worry about zero is because you have to add a counter term to cancel this. Your UV counter term, you have to still add it. And then it cancels this, and then you get something non-zero.

So the bare graph is zero. The counter term's non-zero, and there renormalized graph is non-zero. This is a subtlety that's worth remembering if you ever want to do calculations this way.

OK, so that's some of the complications and fascinating facts about HQET in the perturbative sector. Let's come back and talk about power corrections, which are-- I'll go under the title of-- well, maybe I should just call them power corrections. Better title.

So we have an effective theory. We've so far talked about it at lowest order. We stopped at lowest order. We had this HQET Lagrangian, and we talked about using that Lagrangian to carry out some perturbative calculations. What if we went to higher order in the power [INAUDIBLE] expansion, which is 1 over MQ? OK, so power corrections here means higher order in 1 over MQ.

So let me show you how you can construct those terms. So let me go back to a representation of the full QCD Lagrangian, which we had in terms of this B field and the Q field. And when we first talked about this, we just dropped all the terms of the B, but now I'm going to do something a little more sophisticated with them and really just integrate them out.

So we had this, and this was really just us writing QCD in a fancy way that was convenient for this discussion. So this is really just QCD written in a fancy way. So if we want to take this Lagrangian at tree level, we can just integrate out Bv.

This is a Lagrangian that has quadratic dependence on Bv. So you could think that the path interval in this formula here would be quadratic path interval, and those we can always just solve. And what effectively integrating out Bv amounts to is solving for the equations of motion of Bv and plugging that back in.

So the type of diagrams I was drawing before where I had this wiggly line and it was a Bv propagator, we can integrate out that by solving for the equation of motion. So we look for variation with respect to Bv bar and set it to 0. And that gives this equation, and then we solve this formally for Bv by just inverting this operator to get that equation.

And then we can plug that equation back into this equation, which is still a QCD Lagrangian, actually, but-- and then we expand. And once we expand, we match onto the HQET Lagrangian order by order, tree level. So the first term is the term we've been discussing. The next term, we drop the v dot D here, because that's small. We just have 1 over 2 Q. There's two D transverses, and they'd be higher order terms as well, but we'll stop in that order.

So this is L0, first order term, and this would be L1. They'd be higher order terms. So what is this guy? It's useful to write that guy, actually, in terms of two different things, and you'll see why momentarily.

So it's got two covariant derivatives, and both of them are dotted into gamma matrices. It involves this thing, D transverse, which, remember, is the full D minus a projection onto v. So it's something that's transverse to v. So what I want to do to simplify this guy here is I want to do the following.

I'm going to use the fact-- I'm going to write it in terms of the field strength by using the fact that the commutator of two Ds gives me a G. And the commutator of two sigmas, I'll write-- sorry. The commutator of two gammas gives me something I can call sigma. So let me write this as the symmetric piece and then the anti-symmetric piece. And I do a symmetrization in both the fields and the Dirac structures.

Doing DT slash DT slash anti-commutator. So then since it's anti-symmetric, it automatically forces that anti-symmetric. This guy is a GB nu, so for this piece, we just get DT squared. And this piece, once we track all the 2s, the i's give sigma dot G.

And so the usual way of writing L1 is as follows. You say L1, using this formula, plugging it in, has two terms. And you'll see why when I write them down that we wanted to do this. OK, so that's L1 is after plugging that in.

Now the reason to do this is that if we ask about symmetry breaking, that's something that can happen from sub-leading terms. Lowest order, we had a symmetry, heavy quark symmetry, and that is broken by these interactions. But it's actually broken in different ways by these two terms.

This term here doesn't have any spin structure, so it doesn't break the spin symmetry. It does have flavor structure because of the MQ, so it breaks the flavor symmetry. So this is a kinetic energy type correction, and it breaks flavor symmetry because of the dependence on Q in the MQ. And this guy breaks both, because it has a spin structure now, and it's a magnetic moment type term.

So it's got the sigma dot B field type interaction. This is what I mean by the magnetic moment type term. OK? So that's what the sub-leading power corrections look like, and if we wanted to use the effective theory to talk about power corrections, we could do that. We're constructing them here by knowing the full theory, just integrating out explicitly the fields, OK, which is a very nice thing if you can do it.

Now you could do it the other way, which would be to think about just writing them from the bottom up. And there is one way in which that's more general than what we've talked about, and that's because what we talked about was tree level. And if you wanted to include loop corrections, how do we know that there's not some other operator here that we missed because it just vanished at tree level, for example? We've seen examples where that happens. There's an operator that only shows up at loop level.

So we could think about it from the bottom up, even though this is a top-down effective theory, in order to make sure we're not missing anything. And if we wanted to do that, we should enumerate all the possible things, the symmetries and all that we can use to constrain the form of the operators. So let's enumerate.

So there's the power counting, of course. That's pretty simple. Here all the powers of 1 over MQ are being made explicit, and they just tell us what dimension of operator to look for, just as in our integrating out heavy particles. So we just know the dimension of the fields that we have to put in the numerator from how many MQs we're talking about.

There's gait symmetry, of course. So use covariant derivatives. Very easy to take into account. There's discreet symmetries, charge conjugation, parity at time reversal, which are symmetries of QCD if we drop the theta term. And we can impose them as well, and again, that's easy.

I wouldn't be making a list if there wasn't at least one thing that was hard and non-trivial. But discreet symmetries are easy. The thing that actually is the hardest is Lorentz symmetry. Oh, you say, just dot Lorentz indices into Lorentz indices.

But you have to ask the question whether we even have Lorentzian variance in this theory. And it turns out that part of the Lorentz group was actually broken by having this heavy quark and doing this type of expansion that we've been talking about. So if you think about the six generators of the Lorentz group, the boosts and the rotations, there's a part that I could call the transverse part, which is transverse to the velocity.

So in the rest frame, that would be M12, M23, and M13. And those are the rotations. So this i-- v is like this. So no matter what v you pick, there's always three generators that are rotations.

And then there's the boosts. And you should think of the boosts as taking v mu and then M dotted into v and then making the other guy transverse, so when we denote it like that. So the new index is transverse, and in the rest from, that's M01, M02, and M03.

And so introducing v mu actually breaks the boosts' symmetry. And if you like, you could think the reason it breaks the symmetry is because it gives a preferred frame, which is the rest frame of the heavy quark. If you have a preferred frame, then you've broken Lorentzian frames.

So that's bad. And it turns out there's actually a hidden symmetry of this effective theory that partially restores this breaking. And it restores it in exactly the amount that it needs to restore it. That is, it restores it at low energies.

And that's called reparameterization invariance, which I will write once, and then forever more, we talk about it as RPI, Reparameterization Invariance. And it's an additional symmetry that we have on v mu itself.

So let's go back and think about how we introduced v mu in the first place. So we're saying that v mu breaks part of the cemetery, but how did we decide on what v mu was? How much freedom was there when we defined v mu at the beginning?

If we're saying that it breaks, then we should know how much freedom there was, because a freedom to define different vs could restore symmetry, just realized in a different way. And that's what happens. So where did it come from?

We had P, have a heavy quark, and we split that into two pieces, MQv plus k. But this split into two pieces is arbitrary by some amount. We could move pieces back and forth between here and here, and we would still have the same theory.

We have to be careful that we're moving back pieces that don't violate the power counting. And that's what I mean by somewhat arbitrary here, not completely arbitrary. There was a point to doing this, because we wanted to separate out the big piece and the small piece. But we could always move a small piece back over here, and that wouldn't change this decomposition.

So the invariance that you have is the following. You can take v mu and send it to v mu plus some epsilon mu over MQ. And k mu comes to k mu minus epsilon mu. That moves the piece back and forth between them, and as long as I think that epsilon mu is some parameter that doesn't have a power counting in it, i.e. it's order doesn't have any MQs in it-- it's just something of order lambda QCD, say-- then that makes the power counting still true.

That was the point of this decomposition. And it allows us to move a small piece back and forth. The small piece is this epsilon. So that's a symmetry. That's called reparameterization invariance.

So we have to make sure that when we construct our effective theory that it satisfies the symmetry if we want it to be a boost invariant-- if we want to restore boost invariance to the theory. So this parameter epsilon you can think of as-- you could consider it to be a finite reparameterization symmetry, but you don't really have to worry about finite transformations. You can just do the infinitesimal. So we'll think about epsilon mu as an infinitesimal.

And it has that counting that I put over there. Now v squared was equal to 1, and that's also something we don't want to spoil. But that's easy. We just say that epsilon dot v is equal to 0. That maintains this condition.

So that means that there's three different components of epsilon, non-trivial components to epsilon. And those three components of epsilon are exactly related to the three boosts here. OK, we have a three family-- three-parameter family of transformations, which are the three components of the epsilon, which, in the rest frame, would just be the one, the two, and three.

What did we do-- what about the fields? How does the field, the Qv change under this type of transformation? Let me take the field that x equals 0 for now. So v slash on Qv, it was equal to 0. And if I do the transformation, then this v slash changes. It becomes v slash plus epsilon slash over MQ.

Let me imagine Qv changes. It goes to Qv plus delta Qv, and I have to do that on both sides. Then I can take this-- so this thing here is some order epsilon change. Then I can take this equation, and I can just solve. So the piece that's order epsilon to the 0, just satisfied. Solve for the piece that's order epsilon, and that gives me an equation for delta Qv.

So rearranging this equation, I find that 1 minus v slash delta Qv is epsilon slash over MQ times Qv. And this equation has the solution, but delta QV is epsilon slash over 2MQ. Remember that epsilon is transverse to v, so if I push the v slash-- so if I plug that solution in here and I push a v slash through the epsilon, well, then I can push the v slash through the epsilon, let it hit the QV.

That's giving a factor of 2, because it's anti-commutes. That's this 2 here. And then I would get what I wrote on the right hand side.

So if it's not obvious, check for yourself that that's a solution. So that's how you derive the change to the field under this reparameterization. And so when we talk about operators and the effective theory, we have to worry, how does the symmetry act on them? And it's a kind of a non-trivial symmetry. Was not apparent to us when we started, right. [CHUCKLES]

OK. So the full reparameterization is v mu goes to v plus epsilon over MQ. And then if I take Qv of x, then that is what I said. There's this epsilon slash over 2MQ piece. This is the transformation, so it goes to itself plus this extra piece.

And the fact that I take it at x adds one little slight wrinkle, and it just gives this extra phase factor. And that extra phase factor is exactly what encodes the change of k. So this, if you like, encodes that derivatives. Should go to derivatives minus epsilon or in momentum space, that k should go to k minus epsilon. OK, so previously we had a rule for k, but now I've encoded that in this phase.

OK? So that's the symmetry that we should look into. So what does it do? So what this does is it restores invariance under boosts, but only small boosts. The reason that I call them a small boost is because epsilon here had to be of order lambda QCD. It couldn't have been order MQ.

That's what I mean by small. And from the point of view of this theory, this is all we care about. Because we want to remain within the region where the effective theory was valid, the whole setup of the effective theory involved dividing out a large piece and a small piece. If we allow back large pieces, then the game is over, and you wouldn't be formulating correctly the effective theory, because you'd spoil the power counting.

OK, so this is this hidden symmetry we're calling reparameterization variance. And it's not special to HQET. Any time you have fields that are labeled by something, you should think about whether there's a symmetry like this.

OK, so that's the entire list of the symmetries that you should consider in order to think about doing a bottom-up approach to HQET. Simple ones, and then there's this one that's a little more complicated. So let's go back and now consider the 1 over MQ operators in general.

And it turns out that there's not any missing operators, that the two operators we have are actually the complete set that you can write down at this dimension using all the properties of the field. So we didn't miss anything from that point of view. So let me write them down again, and let me write them down in a way where I imagine that radiative corrections have come in as well. And I'll give them some Wilson coefficients, which are generically called Ck and Cf. That's the standard notation.

So this is a Wilson coefficient. That's a Wilson coefficient. This is not 4/3. it's Wilson coefficient. The name is the same as the Cf that is 4/3, but this is a little cf, not a big CF.

So if we want to-- so it's gauge invariant. It has the right parity, et cetera, et cetera. We should worry about the reparameterization invariance. So let's do that.

So at lowest order, the phase is what changes. And the leading order Lagrangian is invariant, because v dot epsilon is 0. So at order MQ to the 0, since v dot epsilon is 0, you don't get a leading order change. So our Lagrangian was variant.

This invariance, this reparameterization invariance mixes orders. It connects orders in the expansion. There was a term that was order 1, which is this piece, and there's a term that's order 1 over MQ. So the symmetry is actually making a connection between leading order and sub-leading order operators.

So we could ask about this delta L 0, and there will be a piece at order 1 over MQ. So let's just write out all these things. Transforming everything. This is our leading order Lagrangian.

After imposing the field change as well as v change, the v dot D becomes this, and the field becomes that. So there's three things here that are being changed. Expand this out. Use things like 1 plus v slash over 2, epsilon slash 1 plus v slash over 2. C equal to epsilon dot v is equal to 0.

Simplify, do some Dirac algebra, and you can boil this down to something simple, which is that the entire change is just an epsilon dot D over MQ times QV. And if you look at this, it has to cancel against something that's order 1 over MQ. And if you look at the terms that we had at order 1, which are 1 over MQ, there was a kinetic piece. We called it kinetic energy piece.

And if we do the change there, there is a contribution from the phase in this case, because we had transverse derivatives. So we can add epsilon dot D transverse. That's non-zero. And if you go through the leading order change to this guy, as well as the guy that's the magnetic guy, you find that the magnetic guy is 0 at this order. It's non-zero at higher orders, but at this order, it's zero.

And the kinetic guy does have a transformation. It has exactly the same form as this guy here. And if epsilon's dotted into the D, then it's a D transverse. But this guy has a Wilson coefficient. This guy doesn't.

So in order for these to cancel, you actually learn something non-trivial. The symmetry teaches you something non-trivial about the sub-leading Lagrangian. That Wilson coefficient has to be 1 to [INAUDIBLE] perturbation theory in order for the symmetry not to be violated.

AUDIENCE: But you are enforcing the symmetry?

PROFESSOR: Yeah. So the symmetry is boost invariance, and it seems like a reasonable symmetry to impose. Yeah. It'd be small boosts.

So as long as your scheme and your regulator don't break the symmetry, which is always something that you have to worry about in general, then this guy is 1 to all orders. OK? So if you did something like dimensional regularization and you thought you should calculate this guy, you'd just find that it would be 1. And you'd wonder, well, why is that? It's the symmetry that tells you it's 1.

OK, so you don't have to figure out that guy. The other guy, which we've called Cf, you do have to figure out, because it wasn't constrained. And at lowest order, the other coefficient is not constrained in this way. And so it does get an anomalous dimension.

And so we could calculate it. It's a good homework problem. You may see it on a future homework set.

And there is an anomalous dimension, and when you solve that anomalous dimension, you're again getting something that's the ratio of alphas to some power. In this case, it's a non Abelian power, so the adjoint Casimir. OK, so that guy does have an anomalous dimension.

I think we'll stop there today. And we'll talk more about power corrections and the phenomenology of them, how we can make non-trivial predictions of from them next time.