MITOCW | 26. SCET for LHC

IAIN STEWART: OK, so last time, we were talking about the [? massive ?] Sudakov form factor. We're going to continue that discussion today. And we saw that in order to do this calculation, we needed to have additional regulator besides dimensional regularization.

This was an example in SCT2 where we had these rapidity divergences. And so there is this additional regulator that led to these 1 over 8 eta poles in our answer and these logs of nu as well. And we said that if you stare at these results that you could see where you would need to take those scale parameters in order to minimize these logarithms.

And it is such that in the collinear diagrams, you need to take nu of order p minus. And p minus is the hard scale. And you want you mu of order m, which is this scale of the hyperbola. And in the soft, you want nu and mu to be the same size and then both of order m.

So what we'll do in a minute is we'll-- so you can imagine what happens next. You add some counter terms, remove these 1 over epsilon poles and 1 over eta poles. You're left with something that's a finite result that's a function of the cutoffs. And you put things together.

When you put things together, the cutoffs should cancel out of the physical observable, which here was the form factor. So the form of the factorization theorem for this-- so you have some hard function. It's just the Wilson coefficient, some collinear function which I'll can cn, cn bar, and then some soft function.

And so it's the usual story in some sense, but we have this additional dependence on a parameter nu, not just mu. OK, so this same statements I was making up there kind of encoded in the arguments. You want mu to be of order m.

You want nu to be of order q, mu to be of order m nu to be of order q, mu to be of order m, nu to be of [? order mu, ?] which is order m. So I could have written nu over m here as well. OK, and then there's also some hard factor that really looks at the hard scale. So renormalization, and they have two cut off parameters.

So what we'll have, therefore, is we'll have a renormalization group for these objects in a two dimensional space-- mu and nu space. And we'll need both of those to sum the large logarithms. So just having renormalization group in mu would not be enough to solve the large logarithms in this problem.

So let's-- even before we-- so we'll go through it, and we'll write down anomalous dimensions in a minute. But even before we do that, let's just picture what we want from the running. So here's a two dimensional space where we want to do that running. So let's put mu on this axis and nu on this axis.

Break out the color. So we have some hard degrees of freedom that we're integrating out, right? But they live somewhere in this picture.

So let's say here, we have mu of order 2. The hard degrees of freedom, as I wrote over here, they don't depend on nu. We talked about that last time, and one way of seeing it is nu is just a parameter that distinguishes modes within the effective theory.

It's a regulator that was needed to distinguish soft from collinear, but if you add it up these contributions, the nu dependence and the 1 over etas all cancel. So the hard degrees of freedom only depend on mu and q. And so they don't care about nu.

So we can draw them as a line. It's hard. They don't live localized in that space.

But the other modes, we could put a dot associated with each one of them. So you have nu of order m here. [INAUDIBLE] equal m.

Mu equals q. So then we need to have a low energy mu scale, mu equals m. So the soft modes are going to live at mu equals m, mu equals m.

And then the collinear mode's going to live over here. And the way we've set up our calculation, we didn't-- we sort of had one cut off for both the cn and cn bar. So they both live at the same point.

It could have been more fancy, but that suffices for doing the calculations here. So they both live at nu equals q. We basically had a-- we didn't care about whether it was p minus or p plus. We regulated them both with nu.

OK, so to do the renormalization group, this is where the modes want to live. But we need to connect them because in this formula, they all have the same mu and nu. So the general thing that you could imagine is that there's some point mu and nu in this space, right, and that you write down this formula there. But then everybody has large logs because you're not at the right point.

And so you have to do a renormalization group evolution to that point. So the most general thing that you could think of would be that there would be a renormalization group of the hard function down to that point. And that's just the whole line moves down.

So this would be some evolution kernel uh that would run from some initial condition, which I'll call uh. I'll call this mu equals mu h, and we'll call this guy mu light. Call this guy nu light.

Call this guy nu heavy. Just have some names for them. And then these guys here would have running in two dimensions, and you have to take a path from this point to that point. And so here's a simple path.

Just go over and then up. Just go over, up. And there'd be an evolution kernel associated with each one of these things. So this mu 1 would be some kernel mu n that would run for mu light to some mu.

And then it's at fixed new. And this guy here will be some other evolution kernel which I'll call v because it's running in nu space-- so some linear evolution kernel that's running in nu from some nu heavy down to nu. And it's at fixed nu light. And then there would likewise be kernels on this side.

There would be a mu s and a vs. OK, so that would be the sort of general thing that you could possibly imagine. Now the choice at this point is arbitrary, and you know that effectively that there's consistency in the sense that you can move that point around and nothing changes.

So you can use your freedom to pick this point to make your life as simple as possible. If you pick it up here at the hard scale, you don't have to do any uh running. If you pick it over here at the collinear scale, then you don't have to do the collinear running. So we can do that to simplify things.

And that's again just the freedom-- the usual, simple freedom-- of either running the coefficients or running the operators. And it's just a little more complicated. That's what this freedom to move this point around is.

So this is like running coefficients versus operators. It's just a little more complicated because we have these two cutoffs. So let's just pick, for example, mu and nu to be mu light and nu heavy. So that's putting it over here at the orange point.

Then you would just have evolution kernels. The softs would just have to run all the way over there. So you wouldn't need mu us either. You would just need nu s, vs.

So you have vs, and vs would run from starting at nu light over to nu heavy. So this is the initial condition, and this is the final. And it's at fixed mu s. And then you'd have a running from uh down to mu light from mu heavy.

OK, so then you just have these two evolution kernels. So it would be pretty simple. So there's a consistency of both moving around in the nu space. When I move this point around, there's a cancellation between vn and vs, and there's also a cancellation between the three u's.

So there's also in this thing, there's a path independence. So I drew a particular path where I went like that. I could have drawn another path. I could have gone up first and then over.

And because the parameters nu and mu are independent parameters, the choice of the path doesn't matter. We'll see that from our calculations as well in a minute when I write down the anomalous dimensions. But this is just following from the fact that they're independent parameters.

And effectively, what that means is if you think about taking derivatives, you can switch the order of the integration of the derivatives. So mu d by d mu and nu d by d nu is the same as nu d by d nu, mu d by d mu. OK.

There are examples of effective field theories where you would set up cutoff parameters, and this wouldn't be true. You'd think maybe you have two cut offs, but actually, you only have one. There's an example of this in our QCD, where if you try to draw things in two dimensions, you find out that there's really only one path that the effective theory picks out.

But in this case, there really are two cutoff parameters. OK, so let's do that in detail by going back over here to these formulas, writing down the counter terms, deriving from those counter terms the anomalous dimensions, and then we can see how you would derive these evolution kernels in this picture. Yeah.

AUDIENCE: Can you comment on why sort of this works even though the mu and the nu and the collinear cycle aren't, like, factorizing? Like, for example, why couldn't I, just right off the bat--

IAIN STEWART: Yeah.

AUDIENCE: --regulators and never have a factorization theorem but run in different scales? Like, this seems like too easy.

IAIN STEWART: Yeah, it seems too easy. It's really because if you like the-- there's a certain-- if you think about renormalization theorems, which we know much better from mu than-- there's not really that much known for nu. There is kind of a renormalization theorem here that is related to the fact that this divergence is really related to that iconal propagator that we had.

And even when you go to higher loop orders, there's a certain universality to that propagator that's making this whole thing work. So yeah. I think this is a relatively new thing, and I think it hasn't really fully been explored.

So there's probably some interesting work to do there. All right, so let's do it. Let's see what counter terms we get and how things work out.

So we're going to do the standard thing. Think about renormalization the objects. So we'll have bare objects and renormalized objects. Each one of these guys corresponded to a single collinear field. So there's some wave function renormalization factor, and then there's some renormalization for the operator.

And together, those two things will take us from the bare to the renormalization and likewise for [INAUDIBLE] soft guy. In this case, there's no-- we just have Wilson lines. So there's just a renormalization of the object.

And this guy here, it's the usual wave function renormalization. It's just the same in SET for a collinear particle as it is in QCD. So just for the record, there's a 1 over epsilon coming from that that will modify our-- this 1 over 2 epsilon here. Let me put it together for the z factors.

[? So what ?] [? of ?] [? the z's? ?] Taking those graphs and these formulas, find the following. Think I dropped something. [INAUDIBLE].

There was this pesky factor of w squared that we talked about last time. But we'll need it for this calculation. So we simply just subtract all the poles.

Minimal subtraction. So I'm always going to only just write zn because zn bar is really the same. We set up our kinematics symmetrically. So it's really no difference between zn and zn bar.

So there's a minus sign plus sign there. Comes to 3/8 epsilon. So those are the counter terms that we get.

AUDIENCE: Did you [INAUDIBLE]?

IAIN STEWART: Yes, I did. Thank you. And we get the anomalous dimensions again just by demanding the usual thing-- that the bare objects don't depend on mu and nu.

And we can do that separately in the mu and nu. So if you like, actually, kind of along the lines of your question, [INAUDIBLE], there's a assumption built into this, the way we wrote this, right?

I've kept the full epsilon dependence in my 1 over eta pole. And that's related to saying that I could really think of doing all the eta renormalization first and then think about doing epsilon second. So the kind of-- we have built something into this that's non-trivial.

AUDIENCE: [INAUDIBLE] the order is the same [INAUDIBLE].

IAIN STEWART: You mean that the order being--

AUDIENCE: Like that you can do one [INAUDIBLE] before the other.

IAIN STEWART: Yeah, oh, that the order doesn't matter. Yeah.

AUDIENCE: That seems like--

IAIN STEWART: Yeah.

AUDIENCE: --something that you would get from a factorized expression--

IAIN STEWART: Right.

AUDIENCE: [INAUDIBLE]

IAIN STEWART: Yeah, it kind of is factorized. But it's hidden, right, because we are saying that this is kind of a multiplicative thing on that whole thing. And that's the kind of factorization, although it looks kind of like a non-factorized thing.

So let's talk about mu anomalous dimensions. And you'll see that in order for this to be true, you also don't-- it's not the case that mu anomalous dimension needs to be independent of nu, for example. So this kind of looks like it's not factoring, but magically, it is in the sense of this path independence.

So what is gamma mu for the soft function? So we just say that mu d by d mu on bare things is 0 is-- so is nu d by d nu. So that's the way that this works. And that gives us this kind of standard formulas that we're used to for the anomalous dimensions.

So zs inverse mu d by d mu with the appropriate sign is the anomalous dimension in mu. That comes from this equation for soft function. So if we look at that and we work it out, so if you look at this term, it actually doesn't contribute, because in this term, alpha s of mu times mu to the 2 epsilon is mu independent.

So that term has no contribution to the new anomalous dimension. And then this term does contribute, and there's a contribution from this term. Where you differentiate the alpha, you get a 1 over 2 epsilon. You differentiate this guy, and there's a-- if you differentiate the explicit log mu, there's also a 1 over epsilon.

Those 1 over epsilons cancel, OK? So there's no contribution from this guy. There's no contribution from differentiating the alpha in this guy or the explicit log in this guy. So the only contribution is taking the alpha, differentiating it, getting a 2 epsilon, and multiplying that 1 over epsilon. So this line switches, and there's a 2.

AUDIENCE: Set w to 1?

IAIN STEWART: And I set w to 1, yeah. w-- renormalize w. [INAUDIBLE] 1.

So likewise, for the other case, for the collinear-- and this is actually the same as [INAUDIBLE] mu and bar. And what this gives if you solve the anomalous dimension equation-- so the anomalous dimension equations are like [INAUDIBLE] sign convention like this or et cetera. So you would solve those equations.

And those equations would give you the kernels us and un. And it's a simple multiplicative rge. So if you just-- this is mu d by dmu of log s, and you just integrate the way we've done before.

And consistency says that you could run in this picture either way. And that is a relation between the anomalous dimensions. There's also an anomalous dimension for the hard function.

And if we add them all up, we get zero. We pick the sign conventions accordingly. So you could write the relation that way. And so you can calculate the gamma h by knowing these three-- gamma for the hard function.

Or you could calculate it by the counter term for the hard function. And as expected, it only depends on a log of mu over q. There's no nus in that.

So that's the mu anomalous dimension. It works as usual. And the interesting thing is that there's also a new anomalous dimension because the bear functions are independent of nu. It's the same type of formula. We just have nu d by d nu instead of mu d by d mu.

There's no nu dependence in the coupling alpha, but there is a nu dependence in this w. When you differentiate the w, you get an eta And that's where basically the contribution is coming from.

So what happens is you differentiate the w. That kills this eta. But then it looks like your result would have a 1 over epsilon pole.

But you also have to differentiate explicitly this nu here. Again, that has a 1 over epsilon pole. So those two cancel. Once you take the epsilon goes to 0 limit, they cancel, and you're just left with a finite anomalous dimension.

So here, I'm always sending epsilon to 0, and eta goes to 0. And that's what sets w to 1 and stuff like that. OK, so this guy-- so [INAUDIBLE] cancellations [INAUDIBLE] 1 over epsilon poles to get those results, and there's also cancellations of 1 over epsilon poles to get these results.

And equations like nu d by d nu of s [INAUDIBLE] gamma nu ss would give the kernel vs, et cetera. OK, so there's the explicit anomalous dimensions. Let me keep that picture up.

So if you ask what path independence means, you could say path independence could be phrased by the fact that I could take z inverse, and I can take a commutator of derivatives, right, because I'm saying that either order should get the same result. And if either order gives the same result, that must be zero. And what this formula says that the order that we're working is that you can take mu d by d mu of some, say, nu anomalous dimension, and that should be equal to nu d by d nu of the mu anomalous dimension.

That there's a connection between the two things. And if you look at the factors-- and if I wrote everything down correctly, then that's true and likewise for the collinear. So this is the statement of there not being a dependence on the path.

These are formulas that have to be true if that's going to be true. So I'm not going to write down all these kernels for you. There's a lot of them.

But just to give you a flavor for what the solutions look like, they kind of look familiar. Let me write down a couple of them. So I'll write down one. I'll write the ones for the soft.

So at leading log order, what would they look like? They're exponentials, Sudakov logarithm type formulas. They're not the same precise formulas that we had earlier, but they look kind of similar.

Let me write them down, then I'll talk about them. So these are running along straight paths. And they involve ratios of alpha, basically. [INAUDIBLE] that's what the running coupling is doing.

So you could always write things in such a way that you don't have alphas that involve nu, right, because the rga just has alphas of mu. And so if you like, when you're solving the nu anomalous dimension, this equation is very easy to integrate because there's no nus on the right hand side. You just get a log of nu. And that's why this is-- that's why we have this kind of simple log nu here.

Here I wrote alpha, 1 over alpha of nu s. But I could have write that back in terms of a log. So it really isn't.

That was just convenient to make it a simpler formula. So if you write down the evolution kernels like that, they'll satisfy these multiplication properties that this figure implies. And I won't go through that, but it's kind of neat to see how it works out.

All right, so that gives you an idea of how we would sum logs for this Sudakov form factor with these two cut offs. We just have the evolution kernels and use them as usual. But we have this more complicated picture of what type of renormalization we have to do.

And you can think about also, if you wanted to-- for example, say you wanted to do this in some calculation, and then you calculated up to some order solving these anomalous dimensions. And then you wanted to vary scales. Well, now you have a two dimensional plane to vary scales in, right?

So if you're varying the soft scales, you can kind of move around in a box around this guy where you're varying both nu l and mu l by kind of factors of 2. And so they're doing uncertainties in this kind of setup would have-- you'd have more parameters to vary than you would usually have just with mus. But really, it's just a straightforward generalization of the one dimensional picture of mu evolution to a two dimensional picture of nu evolution.

And the interesting part is the kind of overlaps between these parameters. So let me give you one other physics example of where this comes in just so that you see it's not just this one example. So I'll just go through one of them in detail, and I'll just mention one other one.

So it'll be two examples, but we'll only really cover one. So one thing you can do is do gg to Higgs, so Higgs production, where I measure a particular distribution, which is the pt distribution, pt of the Higgs. And it turns out that this process involves rapidity divergences.

So let me try to draw one picture that allows me to capture all the different degrees of freedom. So here's-- you could imagine that this is your top loop, but it's some short distance thing. And you can even integrate out the top.

Think of it as an effective operator coupling two gluons to the Higgs. And so I'm in the center of mass frame of the collision. So these guys are back to back.

So that means that one of them is n collinear, and one of them is n bar collinear. So if n and n bar collinear is coming in, annihilating and producing a Higgs-- and because of what we're measuring about the Higgs, we're only measuring the pt. If you think about what radiation you can have in the final state, well, you could have collinear radiation. So here's some collinear radiation.

And that radiation has a small pt. So that's allowed in the final state. If we have a pth distribution and we think about the limit pth much less than the mass of the Higgs, so there's some logs that you would want to sum, for example.

So you could have collinear modes in the final state that would fit within this kind of kinematic setup. But you could also have soft modes. So soft modes have the same size of pt as the collinear modes. So they would be allowed, and this propagator here would be off shell.

So it's an [? set ?] 2 problem. Because you're only constraining a pt, then it's [? set ?] 2 problem with n bar, n, and s. So pt Higgs [INAUDIBLE] order mh lambda, and modes are these ones.

So we'll SCET two. And you'd want to sum-- in this case, you'd want to sum up double logs of pth over mh. That's what you might be interested in using the effective theory to do.

OK, so you would go through the procedure of factorizing the cross section. It's an inclusive calculation in the sense that basically, in the final state, it's Higgs plus x, right, where x is collinear radiation or soft radiation. And really, the process is proton proton, for example, the Higgs plus x.

So we would want to factorize the cross section, amplitude squared. And here is kind of a sketch of how that would go. I won't go through the details. So you could think of starting with some operator where you've already integrated out the top. So you have a coupling of a Higgs to two gluons, and that happens, and this gives you variant operator with the field strengths.

And then you would do the factorization procedure. After you shift momenta to some other states, you can basically right that is the factorization for a matrix element of two currents. And you get some hard function, which is the Wilson coefficient squared of this operator.

And then you get some operators that look like this, matrix elements that look like this. So there's some proton matrix elements where one of the proteins is collinear, one of them is n bar collinear. And then there's a soft matrix element of some soft Wilson lines.

They're actually in the adjoint representation. So I wrote transpose rather than dagger. So these are adjoint representation.

They should be in the same representation as the collinear fields, and the collinear fields here were gluons, right? So when you go through the field redefinition, you would get adjoint Wilson lines for the soft. If you went through an [? set ?] 1, for example, picture, and then the g [? mu nu is ?] here would become the perpendicular polarization of the gluons.

So these things here are going to give gluon PDFs. So basically, you have a factorization theorem that involves gluon PDFs and a soft function and a hard function. Now if you look at this calculation, you have three modes.

There's an [? set ?] 2. They live on the same hyperbola. And you do have rapidity divergences in this calculation.

So it's a story like this one. And I'm not going to go in too much detail, but just let me write down the factorization theorem for you with all the mus and nus explicit. And it's really the same picture where the soft modes and collinear modes need to live at different nus.

Same mu, and then there's the hard function where you have to run from one side to the next. So it's really just-- you keep the picture on the board. It's applicable to this example too.

So if you calculate d sigma d, if you measure rapidity of the Higgs boson-- so you can call this yh. And you measure the pth squared the magnitude of the transverse momentum. There's some normalization factor, just kinematics.

There's a hard function that only depends on the Higgs mass and mu. And then your other functions can exchange [? perp ?] momenta because the [? perp ?] momenta of the soft function and the [? perp momenta ?] of these collinear functions are the same size. And what you're constraining is just the total. So the pt of the Higgs, if the initial guys coming in have zero [? perp ?] momenta, there's zero [? perp ?] momenta for the protons by design. So really, there's a balancing between the final state radiation and the Higgs, which I can write like this.

OK, there's a delta function, and then you just have objects for the different guys. So do this in the proton center of mass frame. Because of the fact that we're measuring perpendicular momentum, it allows the PDF to be a tensor.

But that kind of means it has two scalar PDFs in it. [INAUDIBLE] h. And then there's a soft function.

OK, so it's kind of got the same type of structure that we were seeing in our previous example. The external kinematics actually fixes these variables here, which are like the x. This is like the [INAUDIBLE] x.

And the kinematics of the process actually end up fixing that when you go through the factorization. This should be a plus sign here. So the only thing that can change from the dynamics is the [? perp ?] momentum.

That's what gets exchanged between these guys. So you would sum logs by having normalization group equations for these objects. And you need them in mu and nu.

The nu one is always kind of simple because of the alpha doesn't depend on nu. So it's very simple to integrate. You just get one log. But it's still an important thing because that one log is something that you can't get from the mu evolution.

OK, so these are called transverse momentum dependent PDFs. So they're not the standard PDFs that you're used to thinking about. They have this dependence on transverse momentum. And because of that dependence on transverse momentum, they also have these rapidity divergences. And these are the renormalized ones at this nu scale.

So there's a long, sordid history of these guys. They were introduced in the early days of QCD, and people have really only sort it out very recently, both in the QCD literature and the SET literature kind of simultaneously with different regulators, exactly how to make sense of these guys and define them properly and renormalize them properly. So this is kind of the last couple of years type stuff.

So another example that we could do which I won't go through in detail-- so that was example one. We could do an example that is not involving protons but involving jets in e plus e minus. So we could do e plus e minus to [? dijets. ?]

And if we did something similar to what we did here where we only measured a pt, then we would also be in this [? set ?] 2 situation. So jet broadening is a variable b for broadening where it's like an event shaped like thrust. But you only measure perpendicular momentum.

And what you measure are actually perpendicular to the thrust axis. So you still use thrust to get the axis for the jet. But then you don't measure anything like the minus or plus momentum. You just measure pts.

This would be another example, which is SCET 2, and it actually would have, again, cn, cn bar type modes. And it-- you'd [? read ?] a formula not exactly the same as this one but again involving similar types of things-- collinear jet functions that depend on pt, soft function, some constraint between them. It's a little more complicated, actually, than this pt Higgs one, which is why I chose to write down the pt Higgs one. So I write that in my notes, but just because of time, I'm going to skip it for our discussion.

So questions so far. So really, it's turn the crank once you believe some of the things I've told you. So something that might be interesting would be to work out with mu evolution, we have constraints on the counter terms at all higher orders that you can write down by consistency, anomalous dimension equations.

And I don't think there's to my knowledge in the literature an expression like that for two loops, three loops, what would the various 1 over etas and 1 over epsilons, how would they have to work out based on the consistency of the picture I've told you. I don't think anyone's done that. That would be kind of interesting. All right, so questions-- none. Good.

All right, so one final example. So presentations Monday-- they're not in this room. They're in the seminar room, the large CPT seminar room. It's on the fourth floor. And I have also made a note that you should use the blackboard, which will stop you from preparing too much information.

OK, so let's do one final example. The final example I want to talk about is Drell-Yan, which we almost kind of talked about already. But I'd like to talk about it in a little different context.

So we talked about pp to xh, Higgs boson. Classic Drell-Yan is pp to xl plus l minus. But kinematically, that doesn't really make too much difference from having a Higgs boson here.

So I will talk about this in a little more detail than I did the Higgs example. We'll go through some of the kinematics in a little more detail. So the reason that I want to talk about this process is there's kind of one function that we haven't yet seen that will show up in our discussion here, one kind spread function that's ubiquitous and shows up in all sorts of factorization theorems. And we haven't seen it yet.

So what are the kinematics? We have some momentum. Let me do it this way. Let me write it below this guy.

So pa for the proton, pb for the other proton, goes equals p for the x plus q for the l plus l minus pair. So what are the important variables? Well, there's the center of mass, energy of the collision. That's just s in Mandelstam variables, but I'll call it ECM so you remember what it is.

pa plus pb all squared-- that's the collision energy. That's ATV [INAUDIBLE] [? LHC, ?] soon to be higher. There's q squared, and that's the scale of the hard collision.

So the hard collision scale is not the center of mass energy. You take a parton out of each proton, and those collide. And they carry a fraction of that energy. You can see how much you have been looking at how hard the leptons are, and that's q squared. It's useful to talk about the dimensionless variable, which is [INAUDIBLE].

And that's the ratio of these two variables, which is always less than or equal to 1. And then you can talk about the rapidity of the leptons.

And you can write that in a kind of Lorentz invariant looking way by having a pb dot q and a pa dot q. Start using capitals. So this variable is like the theta for the leptons. So theta q.

So again, it's just like our Higgs example, but let me draw it in a little different way. You can think about doing everything in a cm frame. Protons are coming in back to back.

So then you have collinear particles because your protons are very energetic. So if this is a quark and this is an anti-quark, you produce a virtual photon. And the virtual photon produces a lepton pair.

So here's an n collinear quark and an n bar collinear anti-quark. So the anti-quark is coming in that way, quark is coming in that way. They annihilate, produce a virtual photon. That produces a lepton pair. That's Drell-Yan.

OK, so in the cn frame, we're going to have cn modes, cn bar modes, for the protons and the anti-proton. Some other variables that are useful to talk about are something that people call xa and xb, which are the Bjorken variables. And they already showed up in our previous example, but let me just define them.

So these are taking tau and taking the square root of it and then splitting up the rapidity e to the y and e to the minus y. And these are like analogs of the Bjorken variable for this problem. And it's actually these combinations that are showing up here.

So if you put in what tau is, that's explaining what these arguments were. They're just the xa and xb. And kinematics, you can work out that tau is less than or equal to xa or b and that they're less than or equal to 1. So there's some simple bounds on them.

And then finally, you can work out that x squared, if you square it, is less than or equal to ecm squared 1 minus square root of tau all squared. And one more thing that you have are sort of the parton distribution fractions. So parton time distribution variables, and they're bounded just like dis.

So xa and xb are things that are, if you like leptonic, you might call them that. But they're things that are external to the QCD, right? They're just measuring properties of the leptons, q squared, the center of mass of the equation, and y, which is a y of the leptons too.

So everything here is not a QCD variable. And then there's these QCD variables ca and cb, which are inside the parton distributions. And they're bounded. Just like we had a bound in dis, x less than or equal to c, less than or equal to 1, here, we have two analogs of that formula.

AUDIENCE: So [INAUDIBLE].

IAIN STEWART: Yeah, that's right. The rapidity of the q-- so really, this is like-- I mean, this is my kind of-- this is like log of-- the thing that's important here is the log of q plus or q minus. It's the rapidity of the q.

Yeah, you could talk about measuring individual things about the leptons, but then you would just be tacking something onto the q and taking something else out of it. All right, so what kind of-- so this is just some kinematics. What kind of limits do we want to think about?

So we already talked about one example where we would measure q perp. That's what we were doing in the Higgs case. And I'm not going to talk about that case. I'll talk about three other cases.

So if we didn't measure q perp, then there's kind of three different things we could do that I'll talk about. Let's see. Organize my board better.

I'll tell you about the kinematics of each one of these, and then I'll draw a little picture for each one. So in the inclusive process, this is the analog of what we did for deep elastic scattering. So deep elastic scattering, we set the Bjorken x variables of order 1.

Here we could say that some sense, tau is of order 1 as well as xa and xb are of order 1. And in this case, what you're saying about px squared is that it's hard. It's of order 2 squared, and if tau is of order 1, than that's of order ecm squared.

So these things are all hard variables. And in this case, the way you should think about what's happening in the process is the following picture. So you have your partons coming in or your protons coming in.

And basically, you're allowing radiation that's hard anywhere. So hard is supposed to be pink. You're not constraining the radiation. You're really allowing hard radiation [INAUDIBLE] hard.

Your x is hard and you allow jets in any direction, for example. And then somewhere, there's a lepton pair. [INAUDIBLE] purple.

But it's not constrained either. It's really fairly general. My picture is too big. I'm going to run it [INAUDIBLE].

Oh well. We'll do this. So I want to draw analogous pictures for the other cases. So in the end point, you're taking a different limit.

What you're doing is you're taking this tau goes to 1. And you can see from over here that if tau goes to 1, that forces xa and xb to go to 1. And if xa and xb go to 1, that forces ca and cb to go to 1.

So you're really talking about probing the proton in a very special kinematics where everything is going to 1. And you're basically in the hard collision, you're forcing all the energy to go into the parton that's colliding. So the proton, the full proton momentum, goes into the active parton.

Let me just say it that way-- active parton. So that changes the picture because it also-- if you look at it, when tau goes to 1, it says the outgoing energy, the full ecm squared, is going into q squared, which is the leptonic variable. So all the energy is coming in on the partons and going out on the leptons.

So you don't have hard radiation like this anymore. The only thing that you could possibly have is soft radiation. So in this case, what happens is this picture-- make my lepton a little shorter-- changes.

You still, of course, have these incoming guys, but what happens is that everything outgoing is soft. So make it green. You have soft radiation like that. And then it turns out also that in this case, the leptons end up having to be back to back because you can't have any transverse momentum.

That would be-- there's nothing for the transverse momentum to recoil against. So that's a possibility. And then there's a third thing that we'll talk about, which is what I call isolated. And it in some sense is trying to combine these two pictures here without taking a limit on tau.

So you might say, well, what's the most typical event at the LHC? Where is most of the cross-section for this process? And that would be in a situation where tau is not in the endpoint. It doesn't go to 1. It's kind of an order 1 quantity, actually.

You typically get xa's and xb's that are like 0.1 or 0.01, small x's for-- depends on what q squared you look at. But the typical ones you're interested in are small x's. So you don't want x to go to 1.

So tau can be of order 1. But if you ask what the most probable thing for the px to do, if you have these collinear particles coming in and they start radiating, well, they like to radiate collinear particles. So the most likely thing that will happen is that you'll get collinear radiation from the incoming particles.

And you can look at that by studying the following situation where you constrain px squared to be 2 ISR jets. So the picture-- we'll talk about how we do that in a minute. But-- or one way of doing it.

So the picture would then be as follows. We have these incoming guys, which now I'm going to try to draw some radiation for. So here are some colors. So here's a collinear guy in one direction.

Here's a collinear guy in the other direction. And they can radiate. They radiate prior to the hard collision here.

So you're getting some jets from these guys that look like this and then from this guy a symmetric thing like that. And then in the central part of the collision, you just allow soft radiation. So this is the isolated scenario.

And if I draw the leptons, it turns out they don't have to be back to back, but there is some constraint on their kinematics. Basically, they're back to back in the transverse plane but not longitudinally. But we won't dwell on that.

OK, so this is the sort of third kinematic configuration. So even though we're interested in one process, we've already just described to you four different ways that you could think about looking at it. And those four different ways will lead to four different factorization theorems because it's a different kinematic setup.

The first one was pt of the Higgs. That led to this rapidly divergent type factorization theorem. Inclusive we'll talk about in a minute, what it looks like. Endpoint and isolated, they all will have different formulas for the factorization theorem, and that's because they look different.

I really should say that this is ultra soft, not soft. And so this is ultra soft too here. This is ultra soft. This is cn radiation, and this is cn bar.

All right, so let's see how far we get. OK, so let's start with inclusive. So the x is hard.

And so the way you can think about it is that what you're doing is that you have-- you can use kind of an optical theorem type picture where you're cutting these forward graphs. These are the leptons here, which are the things in the final state-- so vertical photon.

And I'm squaring it. And in comes qq bar and then squaring it, so qq bar on that side as well. And then if you think about it, every kind of radiation gluon that I would put across this cut I can think of as kind of hard. And so I can integrate it out.

And so what you're going to get is if you just think about this process, when you match the cross-section, you're going to get some four quark operator in [? SCT. ?] So this will match on to an operator, which is a four quark operator where two of the quarks are n collinear-- I'll draw it like this-- and two of them are n bar because that's the external particles. We have qn, q bar, n bar.

So we're going to get that. And that we know how to write down the lowest order operator for that. That's just going to be chi bar chi, chi bar chi, chi bar chi.

So we have a four quark operator in SCET, which you can ride after doing a [INAUDIBLE] in the spin with all the n collinear fields together. You can work out constraints on the [? rack ?] structure. There's only one operator, basically.

It's not quite true. So you could have a gluon operator, an operator with four gluon fields, that's the analog of this one. Then you would have to have some higher order diagram in order to take those gluons and attach a photon. It would have to be a quark loop inside the hard function.

But you could also have a gluon operator with four b's. So let me just say there's also bn, bn, bn bar, bn bar. So you don't have a color octet structure-- no ta, ta. And again, that's like our bdd pi example where if you had that structure, it would vanish when you take matrix elements.

So there's no ta, ta, just constraints on the color contractions. Here, these guys would have to be contracted. Those guys would have to be contracted.

You can make the field redefinition. The field redefinition the y's would cancel out. So there's no ultrasofts here.

[INAUDIBLE] leading order. And if you take these matrix elements of these objects, you can kind of guess what they're going to give. This is just giving a PDF.

Each of these are giving a PDF. And they're the regular PDFs, the standard ones. Because we didn't measure a [? perp ?] momentum, we just get standard PDFs.

So some parton inside the proton, which would be a gluon from these guys-- gluon PDFs are quarks from these guys. And it depends on some x [INAUDIBLE] mu. x could be xa or x-- x could be ca or cb.

So let me just write ca. And that's really all you get. So you have a hard function and then two collinear parton distribution functions. So the cross section-- [INAUDIBLE] these limits that we talked about come from the kinematics.

[INAUDIBLE] some hard function, which is an inclusive hard function. Like in DIS, it depends on the ratio of these variables. But now there's two of them. just depends on q squared, depends on mu. And then there's times PDFs.

There's an important caveat here, which is if you want to derive this result, you have to make sure that the degrees of freedom I've told you with the degrees of freedom are the right ones. So what did I tell you? I told you we have cn, cn bar, and ultra soft, right?

And that from the kinematics, it's a SCET one. Turns out there's one other type of degree of freedom that you could worry about, and that's called the Glauber gluon? And to derive this, we must know that that doesn't matter.

What is a Glauber gluon? It's a Coulombic gluon between n and m bar particles. So it's a Coulombic type potential that goes like 1 over pt vector squared. And that's a Glauber.

It's called a Glauber exchange. So it's not an on-shell particle. It's like a potential.

But you have to know that those actually are relevant in order to get to this formula. So there's a little more work involved which I'm not going to talk about. OK, but-- and that's going to be true, actually, of all the other cases that we do as well.

So what about this threshold limit? How does this formula change? Well, you could think about how the formula changes. It's really a change of this h in the threshold limit because what's happening is that you're constraining the pink radiation and making it green. [INAUDIBLE] keep my pictures.

So if we want to go from here to here for the threshold limit, then only certain terms, if you like, in the HIJ are included. So if you wrote out that function, really what it corresponds to is that only the most singular terms in the variable 1 minus tau in these 1 minus type variables are included. So delta functions of 1 minus tau, plus functions of 1 minus tau, those are the terms that you would include from the h whereas the inclusive one has much more in it.

This keeps only particular terms. But there's also different hierarchies of scales because now 1 minus tau is small, and you want to-- for example, some logs of 1 minus tau in this situation. And that's what the factorization theorem in this threshold region does for you.

So the HIJ inclusive gets turned into the following. This is the modification of the factorization theorem. So there's a soft function for that soft radiation.

And then there's a different heart function, which is only a function of q squared. And it's like the square-- this guy here is the square of a Wilson coefficient again, and i and j are just pairs of quarks-- uu bar, dd bar. There's actually no gluons in this case.

So the terms of the gluon PDFs actually get suppressed. They're not singular in 1 minus tau, and we get this formula here. And now there's going to be some running that you have to do between the hard scale and the soft scale, and that running is going to sum up these logarithms of 1 minus tau. OK, so I'm not going to go through that in any more detail. I'll spend a little more time on this one.

So what about isolated Drell-Yan? This one is a little bit different than the other ones because we have to measure something about the process to know that it looks like this if we really want to drive a factorization theorem. Here, we were measuring tau. But now I'm telling you I don't want to measure tau.

Yet I still want to distinguish-- so I don't want to measure tau. I don't want to constrain it to be close to 1. But I still want to distinguish between these two, right?

And if I'm going to distinguish between those two situations, I need to measure something else. And we can measure something that's exactly like what we did when we did e plus e minus to [? dijets ?] because this is really like [? dijets. ?] But they're just initial stage jets rather than final stage jets.

So that's what we're going to do. We only have four jets. We have tau that's generic. So the c's are not being forced to go to 1.

And so we need something that we can observe to do that. And here's what we can do. Let's do something that's the analog of e plus e minus to [? dijets ?] but for these initial stage jets. So we say that px is a sum of momentum in two hemispheres.

The hemispheres a and b are just the dividing line of the center of mass of the collision-- so very simple, perpendicular to the beam axis. And then we just say that ba plus-- so this is just a [? four ?] momentum decomposition. But we can look at ba plus, which is like dotting the n vector for one axis into ba-- and that's a sum over particles in one of the hemispheres and then likewise for b.

So this is if I write it in terms of energies and rapidities. It'll look like this. So this is some observable ba plus. And then I can constrain that ba plus, which is exactly like constraining the plus momentum in one of the hemispheres to be small.

And what that does, if you constrain the plus momentum in this hemisphere to be small, it will allow ultra soft radiation because they have small plus momentum. And it'll allow collinear radiation because they have small plus momentum. So constraining ba plus constrains all the ba plus momentum in that hemisphere. And that puts you in an SCET one situation where we have a figure like that one.

So rather than constrain the [? perp ?] momentum, we constrain the plus. And then we get SCET one. So take ba plus to be less than or something, which you could say, well, less than or equal to some cut. Let's just call it b cut.

And that is much less than q. And then we do the same thing for b plus, which we define as nb, which is na bar. So the notation here is a little awkward, but we do the same thing here. So this is less than or equal to some cut and that that's much less than q.

And by demanding small plus momentum in both hemispheres, appropriate momentum with the small components-- these are the small components-- that puts us in a SCET one situation. So we have cn, cn bar, and ultra soft. And those are the allowed types of radiation, and that's what I drew in my figure. So hopefully that's clear.

All right, so what does the factorization look like in this case? Well, you have two types of things that are collinear if you look at the figure. There's the initial state proton, right, which came in here, and it was collinear here.

So here's a proton. And then it's cruising along, and at some point, it widens out like this and becomes a jet. So this here is collinear at a smaller scale than this here. This is a jet, right?

It's got large invariant mass relative to the proton mass. So there's actually two types of collinears, and if you really want to draw the mode picture for this, here's what it would look like. So here's our collinear hyperbola cn cn bar.

We have an ultra soft hyperbola. And then somewhere, we have a lambda qcd hyperbola. And there's protons that sit on the lambda qcd. So this is like pn bar, if you like, or let me give it a different name-- pn.

Then there's hard modes. So what we'll do is we'll not really worry about distinguishing these. We'll think of them all together.

So we'll just have [? sct ?] one, where we have ultra softs here, a cn, and a cn bar. But then we'll later have to worry about factoring. This includes the jet. This is the collinear jet.

And this is the proton, right? So both objects will be in the same function at the beginning, and then we'll have to factor them later if we want to separate those scales. It's a similar trick to what we used before in another example. It was for soft in that case.

So if we just keep these guys together, then it's just a SCET one. Just have these modes go through the factorization. And this is what we get.

So we can measure q squared, rapidity, and we can measure these two hemisphere momenta. The hard function is again the square of a Wilson coefficient just as it was in the threshold case. We get some collinear functions that are called beam functions, and they're like, they have an argument that's a plus times a minus momentum.

So this wa is like a minus momentum. We have a Bjorken x variable. And we have mu. There's one of these for each direction.

So there's one for the b direction too. And then there's a soft function. And it's a hemisphere soft function. It's really exactly analogous to the soft function that we had in e plus e minus, the [? dijets, ?] except that the Wilson lines are incoming instead of outgoing.

So it depends on two plus momenta in mu. OK, so the factorization theorem involves these b's, which are called beam functions. So this is a beam function, four parton i. And parton i could be a quark or a gluon in this case. Well, sorry, parton i here will be a quark, actually, in this case-- quark or a q bar. You have analogous functions for gluons as well that would come in, for example, if you were doing Higgs production.

So what is this beam function? It's the one object that we haven't seen an example of before. And it's really a matrix element of an operator that we've studied in many different situations.

It's just the chi bar chi operator with different arguments than we've studied previously. So it's a slightly different matrix element of the chi bar chi operator because what we're measuring is both the large momentum and the small momentum of that operator. When we measured just the large momentum, that gave us the PDF. But now we're measuring also the small momentum, which I can write like this.

So it's a protein matrix element of a chi bar chi operator where if it was a PDF, this would be 0 and 0, and we just have the minus momentum. But now we're measuring, if you like, the other momentum, which is like a plus momentum. And I wrote it in Fourier space.

So the plus momentum that you're measuring is here. Those are the two variables that are showing up on the right hand side as well as the minus momentum of the proton. So it's just a different matrix element than one we've had before.

The jet function, remember, would be vacuum matrix element of chi bar chi, and the PDF, standard PDF, would be proton matrix element of chi bar delta chi. And it's just really-- the arguments are slightly different, and that allows this thing to contain both kind of a jet, initial state jet-- in this case here, the jet function, you'd have some variable zero. In this case here, you'd have 0 and 0.

And so if you like, what this guy is is just a combination of the two things we studied before. We studied the jet function. We studied the PDF. Now this has both inside it.

And if we go through this final factorization between there and there, then that's like a SCET one to SCET two matching. And so we can integrate out the perturbative radiation, which is the jet function, right? That's perturbative.

And this is non-perturbative. We can separate those by doing another matching, and that gives a factorization theorem for the b alone that looks like this. So there's some perturbative matching coefficients which are called i.

And then you get a standard PDF. So this i is the thing that has in it the jet radiation. This is sort of an i for the jet. And then this is the PDF.

OK, and the T scale here is large. And what you're basically expanding in is you're expanding in lambda QCD over t. So if you ask what am I doing to make that separation, I'm expanding in lambda QCD over t.

There's no t here. The t is perturbative. That's the scale for the jet. And this f is a non-perturbative distribution function. And that's then the full factorization theorem putting those things together.

So these beam functions are kind of-- they are things that show up whenever you have a process where you ask for a particular number of jets. And if you asked, for example, for a particular-- if you ask for two-- if you ask for this setup plus one more jet, how would things change? Well, you'd have the same beam radiation, the same initial state radiation.

And you'd add one more jet function to this formula. So that would be exclusive one jet production. Or you could have two jets.

Then you'd have two jet functions. In each case, you have a different soft function. But these beam functions are going to be always showing up because they're describing this initial state radiation.

So any time that's an allowed thing, then it'll show up. If you start doing [? perp ?] measurements, then you might have [? perp-dependent ?] beam functions in the same way that we could get [? perp-dependent ?] PDF. In some sense, a [? perp-dependent ?] PDF is like a beam function. So that's kind of the last function that you need to know about that is kind of a generic thing that shows up with these factorizations theorems.

And we're out of time. So let's stop there. And that's it. See you on Monday.