PROFESSOR: All right. So where were we last time? So we had derived the Fermion Lagrangian which is at the top of the board, and really the only difference between this and what we talked about earlier is that we had to perform the multipole expansion. So we distinguish between collinear momenta that are scaling like a collinear momentum and momentum that are scaling like an ultrasoft, and only the collinear pieces were showing up in here, and same with the collinear gauge field. It was only showing up in here, whereas both were showing up in here.

There's only a single type of derivative here, because ultrasoft and collinear momenta are the same size in the end component. But they're different in these other components, so we had these labels in this label operator that picked out the collinear momenta. So we could define collinear covariate derivatives, if you like, with this subscript n, whereas this D is like a full derivative that involves both types of gauge field. And then at the end of lecture, it was kind of rushed, but I was talking about the collinear gluon Lagrangian, and I said there is a set of replacement rules that we could make to effectively do the same thing that we did here. And now I've just written out for you with that Lagrangian is with those replacement rules.

So curly D is basically just taking all the derivatives and taking only the leading order pieces. So we take the collinear pieces in this component and this component. We take the ultrasoft pieces in this component. And then if you wrote the original Lagrangian as a commutator of two covariant derivatives, you'd just replace it by the leading order pieces, and that's what the leading order action will be.

Then, we had to think about gauge fixing. Since this is the collinear gluon Lagrangian, we should think about collinear gauge fixing. This here as a general covariant gauge fixing with parameter tau, and then there's a corresponding ghost term.

And in this Lagrangian, the usual way it would look, this would be i partial, and then I said that, because we don't want the collinear gauge fixing term to break ultrasoft gauge invariance, we're going to turn that i partial into a covariant derivative under-- we're going to include this piece here at the n dot a ultrasoft to make it ultrasoft gauge invariant at lowest order. OK? So that's really the only-- other than just doing this most naive thing by just replacing derivatives by covariant derivatives, and you might think, well, I'll just keep this as a partial derivative. Since it's to doing gauge fixing, I don't need to make it covariant, but we do want to make it covariant under the ultrasofts, and that's why I write this curly D ultrasoft.

OK. So that gives you the leading order Lagrangian. Once you put together this with what you want for the ultrasoft, then you would have the full leading order Lagrangian, and the ultrasoft part is actually very simple. So for the ultrasoft part, we just take a full QCD action for the ultrasoft field. So this is just q ultrasoft bar i D slash ultrasoft q ultrasoft. And likewise for the gluon piece, it's just QCD, and we would do the gauge fixing without thinking about any complications, like these ones, and these are just involving ultrasoft fields, so just QCD, just ultrasoft fields. OK?

So the only place-- and you take everything here together-- the only place that the ultrasoft and collinear fields talk to each other in the Lagrangian is in this single component n dot partial, n dot a ultrasoft. And that comes about basically because of the power counting, that this is really the only place that these two things can interact. And we'll talk about the implications of that later on, but that's the leading order Lagrangian.

So once you have this piece, and this piece is pretty straight forward with this additional complication of worrying about what gauge symmetry means, which we'll talk more about later. But we had to be careful not to break it, when we introduced this term, because we wanted to in some sense have a gauge fixing both for the collinear gluon and a separate gauge fixing for the ultrasoft gluon, and that's why we wanted to do this. And then this piece here is simple. It's just QCD, because it's, if you'd like, it's the lowest energy mode, and it doesn't know about any of the complications that we had for the collinear modes. OK? So any questions about this so far?

OK. So everything that we did in deriving these actions is tree level. All the steps that we did were tree level. So you can ask, if I start to do loops, will there be some Wilson coefficient that shows up somewhere here? Will I generate some new operators that I don't see here? Those are reasonable questions, and that's what we're going to address next.

So to go further, we'll use symmetries, and we're actually going to consider three different symmetries, gauge symmetry, which I've been promising you for a while, reparameterization invariance, and spin symmetry. Where I'll put a question mark by this last one, because we need to answer the question whether there even is a spin symmetry. These two here, this number one and number two, will turn out to be quite important. Number three is not so important.

So reparameterization invariance here will be like reparameterization invariance in HQET, except now it's different. We've introduced the parameters and then n bar, and we'll have to see what kind of symmetries we have with respect to that choice of basis factors that we made. But it otherwise will be analogous to our discussion of HQET.

So let's actually first dispense with the one that in some sense is the least important, this number three. So first, let's revisit our spinners a little bit. So if I put together the information that we talked about when we derived the equation at the top of the board there for the Lagrangian, we worked out at a tree level we have this formula that relates the fields.

So from that formula, if we just project onto the spinner pieces, we can write down a formula that relates the spinners. So throwing away the gauge fields, we have in momentum space the u of p, the spinner, would be related to whatever spinner we have for this cn field which I call Un by that formula. And then if you take this formula, you also see that if I hit it with a projector, n slash n bar slash over 4, if I hit the u with a projector, it's going to kill this piece. Because the n slash n bar slash can be pushed through the p perp slash, then n bar squared is 0, so that kills that second piece. So we also have this formula.

And then once you have this formula, you have the formulas that we wanted for that. OK? But this actually, this spinner here, this Un is not exactly the same as the spinner that we talked about earlier. So let me come back. Let me come to that in a minute.

So first thing you might consider is whether, when I take this cn field, and I take the Lagrangian up at the top of the board, do I just get the collinear propagator that we talked about? And indeed, you do. It's kind of obvious for the momentum-dependent parts, and really you might only worry about the spin. And so if you consider this formula, and you consider the sum over spins of u u bar which is what's going to appear in the numerator of the propagator for the Fermion when you're driving the Fermion propagator, you get a sum over the physical spins.

Then, from this formula, this is a projector on something you know how to do the spin sum for which is the full theory spinner. And that spin sum is p slash, so this is like p slash sandwiched between projectors. And you could work out that that's exactly the numerator that we had before after a little bit of Dirac algebra.

So that part works as expected. If we take this Lagrangian, and we work out what the propagator is, we get exactly the propagator we got from expansion. So quantizing lc0 gives us that propagator.

But the situation is not quite the same for the spinner. And in some sense, this is not-- this point is not absolutely crucial, but actually, there's a little simplification I want to do in order to discuss the spin symmetry, and in order to do that at this point, it's important to understand this point. So that I'm going through this in some sense, because then it'll be very easy to discuss what spin symmetry we have in the theory.

So this guy is actually not equal to our expanded spinner which putting in some normalization I could write like this. So this is what we got by expanding, something like this which is very simple where this u is equal to 1, 0 or 0, 1. in the Dirac representation. But that's actually not what we get if we just take the formula up here and use this. So let's see what we do get.

So if you use the formula up there, then you have the following. So here's the full theory spinner with a conventional normalization. So this thing and this thing here is the projector. OK? So you could work out what that product is, and it turns out that you can write it in the following way which is very closely related but not precisely the same as what we had before.

So you can write it in what looks like the same form as we have over here, except this curly U, curly capital U, is a more complicated object. And it's just whatever I get by multiplying these two things out which turns out to be something I can write in that form. So it's some two-component spinner, but it's got momentum dependence, unlike our simple 1, 0 and 0,1.

But everything we said about Un really depended only on the fact that it could be written in this form in terms of some two-component spinner-- the fact that n slash on it was 0, the fact that it had projection relation. So these formulas here, if you have a formula like these, these formulas here are true. OK? So actually, it would be true whatever spinner we have there.

So why should I want this U twiddle spinner and rather than the U spinner? That actually has to do with this reparameterization invariant, so it'll become clear when we talk about that. But these extra terms in U relative to those for the other guy, the simple U, actually ensure the proper structure under reparameterizations.

And basically, it'll become clear in a moment, but basically, if we wanted to get this U spinner, we should have a slightly different projector, which I'll call Pn prime. So we could have used a different projector which is this one, and then we could have come up with another projector which was the Pn bar projector which would satisfy that the sum is 1, if we wanted. And this projector here, when acting on the full theory field, would have given something that would have been proportional to this combination over here. OK?

So you just have to believe me. I don't want to go through the algebra, or you can check it yourself. But this projector here is not invariant under that symmetry of reparameterization variance. When we talk about RPI, it'll be clear why we want a projector which is this projector and not the slightly different projector which has this extra n slash over 2.

OK. But nevertheless, the important point that I wanted to emphasize is really that we have this way of decomposing the spinner, the true spinner, for our field cn in terms of a two-component object, U twiddle. OK? So we are able to do that. And if you want to talk about spin symmetry of the theory, it's easier if you use a two-component notation. So the reason that I wanted to go through this is really to have on the board this equation, which I can then use to motivate writing down to two-component version of cn. And once I have a two-component version of SCET, then it's very easy to see what the spin symmetry is.

So let's write down a two-component version of our collinear quark Lagrangian. You can do the same thing, of course, in HQET right down the two-component version rather than a four-component version. If you have a four-component version, it has this projection relation. If you have the two-component version, the projection relation is built in. The reason to consider the four-component version is if you want a couple this object to four-component fields, like the ultrasoft field, then it's, of course, a nice thing to have a four-component version, but some things are easier in two components.

So take cn, and write it as follows, where this phi n has two components. And I've set things up, so the dimensions of cn are equal to the dimensions of phi n. OK? So I can take that formula, plug it into our SCET Lagrangian, and then I can, using the Dirac representation for the gamma matrices, write out a Lagrangian for this phi n. So that requires doing a bit of algebra, which I will take you through. And then we get an equivalent Lagrangian but in terms of this field, and it looks as follows.

OK. So it's almost independent of spin but not quite. There's this sigma 3 sitting there. If sigma 3 wasn't there, it'd be like HQET, where you have an SU2 symmetry. The fact that sigma 3 is there means you don't have an SU2 symmetry. And really, all you have is a U1 symmetry, and that U1 really corresponds just to helicity.

So in four-component notation, that U1 would be the following. It's projecting a spin operator onto perpendicular indices anti-symmetric in both of them, and in the two components, that just becomes a sigma 3. So obviously, if we do an exponential rotation with respect to sigma 3, sigma 3 commutes with sigma 3, so does the identity. And so we could have a rotation of this guy by that, and that's the only spin symmetry that you have is this helicity.

And so because of the coordinates we're using, this corresponds to the spin along the direction of motion which is the three direction, if you like, which sometimes we denote by just saying it's along the direction n which is then more independent of how we pick our axes. So this symmetry here is actually related to what you would call the chirality in QCD. So it's not really a new symmetry. So this is just related to the chiral rotation.

So if you look at gamma 5 times c and gamma 5 in our representation would be 0, 1, 1 0. And then if I write it out in terms of this two-component thing, then that is just giving me 1 over root 2, and it's swapping up and down. And so that just means that phi n has gone to sigma 3 phi n. So multiplying by gamma 5 is actually the same as multiplying by sigma 3 in the two-component notation. OK?

So this is not really new symmetry, and actually all the usual things that you would say about chiral rotations in QCD would apply here too. So chiral rotations, of course, are not exact chiral rotations are broken by masses. Chiral rotations are broken by non-perturbative effects. You can worry about anomalies. They are useful in perturbation theory for quantifying operators, and that remains true here, but if the collinear fields were non-perturbative, then you should worry about those other things as well.

OK. So spin symmetry, there's not really anything new there to talk about, but along the way, we saw we could write SCET in two-component form which is kind of nice. So let's talk about something that's more important which is gauge symmetry. Is there any questions about the spin before we talk about gauge symmetry?

AUDIENCE: So the extra term you got in determining the theta n bar, that is why you had the minus n slash [INAUDIBLE] and why you don't have the full SU2?

PROFESSOR: Yeah. I looked at it once, but it's hard for me to remember the answer. I think not, but I think if you do the other version, then you just get something more complicated here but still breaks SU2. Yeah.

AUDIENCE: Is it-- I'm just looking for [INAUDIBLE].

PROFESSOR: So they're two disconnected fact. The thing that I wanted to motivate that I could use this formula, and I wanted to be honest with you about where that came from, and that's why I told you this other fact. But I could have jumped right here and said, remember, and glossed over it, and that would have been fine for this discussion. So they're in some sense two disconnected things, at least in my mind.

AUDIENCE: Do you have some intuition for what theta n bar is?

PROFESSOR: For theta n bar?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Oh. Yeah. It's really just saying-- my intuition for it is really just that, when you're producing energetic particles, it's what we did at the beginning. When you're producing energetic particles, these are the spin components that you have at lowest order, and that leads to, if you like-- the fact that you have this form, and you have this projection relation is kind of non-trivial, even though you don't have a spin symmetry. There is something to it. It's not like a SU2 symmetry of the spin, but it does, for example, when you go and look at form factors, it does lead to non-trivial relations for those form factors.

So I don't call it a symmetry. I don't think of it as a symmetry, because it's not a formal group theory statement of Lagrangian, but I think it's basically-- well, OK. Let me not speculate any further. It caused some confusion in the literature actually as well.

All right. So what about gauge symmetry? So we're doing a non-abelian gauge theory. A gauge transformation is something like this. And in order to just talk about gauge symmetry, what we need to do is we need to talk about the gauge symmetries that are related to the two type of gluons that we have.

We have a collinear gluon. We have an ultrasoft gluon. You know that the field acting like the connection for that gauge symmetry. So there should be, in some sense, a gauge symmetry associated to the ultrasoft gluon and a separate one associated with collinear gluon, if the meaning of these things as gauge fields is going to have any sense to it.

So rather than have just this set of gauge transformations that we can make in QCD, we should have some more complicated structure of gauge symmetry in the effective theory, because we have a more complicated structure of gauge fields. So in order to talk about U's, we're going to want to divide them into camps. And the useful thing to think about is that you want to have U's that, first of all, leave you inside the effective theory.

And really what this means is that you want talking about gauge symmetry to commute with your discussion of talking about power counting. So by way of example of this, say you had a gauge transformation that was large, that made a formula like that one, that i partial on it was scaling like q. OK? So this would be a very large momentum associated to that spacetime dependence. Then, if you talked about the gauge transformation which would be U times c, even if this guy has collinear momentum scaling, the new guy would not because of this fact. OK?

So if we want to talk about gauge symmetries, we want to talk about symmetries that take us to the fields that we've already decided to have a certain power counting. And we don't want the gauge symmetry to mess that up, so we're going to demand that that's true. So given that, we can think about dividing up the gauge symmetry into pieces, into pieces that have different scaling in terms of how big the transformation are, how the coordinates are behaving.

And it's convenient for this discussion to divide them into three parts. A global transformation, where this guy is not spacetime dependent, a collinear transformation, which I'll call Uc, where i partial on Uc scales like a collinear momentum, and then an ultrasoft U Us, where the i partial scales like an ultrasoft momentum. And basically, what we want to do is we want to connect this gauge transformation to the gauge field for the collinears and this transformation to the gauge field for the softs. OK?

So one complication that we have to deal with is this momentum labels that we were talking about. Because the way that we distinguish between these types of momenta and these types of momenta was dividing things into these labels and residuals. Gauge transformations are something that's very nice in position space but not so nice in momentum space, and that's because gauge transformations are local transformations. Local means local in x not in P. So what is simple multiplication in position space becomes convolutions in momentum space.

So if you go over to this hybrid notation that we developed, then you would have the following. At this field, if I do a transformation, it goes into something like that. Where the way that you should think of what this sum over q is in the following way. But if you have a transformation in position space that looks like that, then in a momentum space, that simple product is convolution. OK? So this is one dimension, and you'd have that correspondence. And so this complication that we have here just correspond to the complication that you'd have for any gauge symmetry that you want to write in momentum space.

But we can be slick about this and introduce a notation and then dispense with it, because it's really not a technical complication. It's just a notational complication. So let's do that in the following way.

Let's let Uc which is a matrix with two indices be defined by this Uc of bl minus ql. So I'll say define a matrix. Then we can use a matrix notation for the convolutions, where we sum over repeated indices, and that takes care of this notational complication. So what is meant by this is that the Pl, ql entry of this matrix is a number, and that number is given by this thing that we call Uc Pl minus ql. OK. So then the formula up here could just be written in terms of that matrix as a kind of matrix multiplication.

OK. So what are the transformations once we adopt that? Let me write them down and then talk about them. So cn of x goes to Uc of x cn of x, where you should understand this thing is a matrix that has two indices, like this, also dependent on spacetime. And I just dot over the repeated indices in this thing which is like a vector. So this is like a matrix, and this like a vector. That makes it look like it's the usual notation which is the point of introducing this notation.

So for the collinear gauge field, it's going to be basically that it, as we said over here, that it becomes the gauge field of that transformation, and so that's what this would be. That would be the standard gauge transformation, and then there's one little complication here, which is I wrote curly D ultrasoft, the ultrasoft derivative, the thing that has the n dot a ultrasoft field in it.

So the reason that I wanted to do that is because I'm thinking here of the ultrasoft field. When I'm thinking about collinear fields, I'm thinking of the ultrasoft field as a background field. And if I was to do a background field gauge transformation, then I would have a covariant derivative here under the background. So this n dot a ultrasoft is like a background field to the collinear gauge field. So this, if you want to make an analogy of what this is like, it's like the quantum gauge transformation of a field that has a background.

If you look up how background field gauge works and background field transformations, you'd have the same thing. You'd have something covariant under the background, and that's what we're going to do here as well. Yep?

AUDIENCE: Kind of a silly question. So UC of x is a matrix. So what exactly UC of p [INAUDIBLE]?

PROFESSOR: Well, no. This is just a number. So just think of this as like this is some vector, and this is some other vector. The difference of these are some label on this thing. Right? And for each value of this, I get a different number, and then for each value it's based on, I get a the different number, but this thing is just a number. And I'm saying think of it like a matrix in the sense of, because we are thinking of these as discrete--

AUDIENCE: But isn't UC [INAUDIBLE]?

PROFESSOR: Oh, yeah. Yeah. That's another matrix. It's an orthogonal matrix. It's also a matrix in that space. Yeah. Yeah. Yeah. So here, I'm defining a matrix in a new space which is this momentum space, and that's just really my way of making transformations in momentum space look like as simple as they do in position space by just having a matrix notation for the convolutions.

AUDIENCE: So that number is essentially a matrix.

PROFESSOR: It's a matrix, yeah, in that space. Good point. What about the ultrasoft fields? Well, ultrasoft fields shouldn't transform under collinear gauge transformations because of exactly the same logic that I gave you a minute ago about why I shouldn't be able to inject a hard momentum like this one into the effective theory.

If I want my ultrasoft fields to remain ultrasoft, then I can't let them transform, because that would spoil their power counting. So q ultrasoft under this Uc transformation just goes to q ultrasoft, and A ultrasoft has to go to A ultrasoft. So these things should not be touched by that transformation.

AUDIENCE: So why cannot make the ultrasoft and the collinear talk to the--

PROFESSOR: Well, we don't want that. You could think of trying to develop effective theory, where they would mix. Right? But whenever you have a Lagrangian that mixes two fields, the first thing you try to do is to force them not to mix. So it's better, say I set the theory up so that they mixed. The first thing I would try to do to that Lagrangian is make field redefinitions to go to some form of the Lagrangian that wouldn't mix. OK?

So I prefer, if you like, to develop the one that doesn't mix right from the start, and you could view it that way. So you could always think of I'm going to tell you like two sets of transformations. You could always think of formulating linear combinations of those transformations. Say these are the fundamental ones, not the ones you wrote down but these other ones. Right? And then by some manipulations, one could go to the ones I'm talking about.

All right. So now, the other set of transformations that we want are these ultrasoft ones, and for that type of transformation, we don't have the same type of problem with the collinear. We can't say the collinear fields don't transform, because that type of transformation doesn't spoil the collinear field being collinear. We can always add ultrasoft momentum to it.

So these fields cn and An will transform like background fields-- sorry, like quantum fields in a background. So from the point of view of the cn fields which are the shorter distance fields, you can really think of like what we're doing here is exactly like what you would do if you had quantum and background gauge transformations. And the physics of thinking of the background as a longer wavelength mode is exactly the right way of thinking about how these fields which are shorter wavelength think about the ultrasoft fields. So what that means is that they transform like matter fields in the appropriate representation. OK?

So with that in hand, we can write down how things transform under the ultrasofts. A Fermion just transforms like that on the left, and this, unlike our notation over there, this is one number for all entries in the vector. This guy doesn't have any indices. OK? So this is like transforming all the terms in the vector by the same overall color matrix, overall number in the momentum matrix base.

So this guy should transform like an adjoint, and you can read the adjoint as a U on each side. So there's two ways of writing the transformation here. You could write it as like this. If you wrote down an adjoint transformation, then this guy would have two indices A and B, and you'd have a transformation like that, where this is in the adjoint. But if you want to write it in terms of the U ultrasoft that's e to the ita, which is in the fundamental, then you write it like this, and those are equivalent.

So why are they equivalent? So they're equivalent because-- well, OK. Maybe I'll tell you later. Yeah?

AUDIENCE: I'm always confused about why n dot is a background field to A ultrasoft, because it actually has the same size of momentum, so its wavelength is the same size.

PROFESSOR: The wavelength is not the same size. The component's power counting is the same size, but the n dot An field carries momentum which have P squared which is much larger.

AUDIENCE: Right. P squared [INAUDIBLE] wavelength, right?

PROFESSOR: So I'm talking about the wavelength which is related to that P squared. So you think about the n dot An field carries a large P minus momentum still. Right? Even though it has a small-- it carries in order a 1 P minus momentum.

So if it's confusing, think about just the P minus momentum. The n dot An field, right, has P minus, which is the what are lambda to the 0. So just let's think about wavelength with respect to P minus.

AUDIENCE: That's definitely true, yeah.

PROFESSOR: Right. Whereas the n dot A ultrasoft field carries P minus, which is what are lambda squared, so just thinking about those two.

AUDIENCE: You were saying conflating [INAUDIBLE]

PROFESSOR: Yeah. Yeah. All right. We have room here to squeeze the final transformation. So the final transformations are just what you'd expect, that the ultrasoft fields transform as regular gauge fields. So let me just squeeze that in.

All right. So there's the regular transformations that you'd have for a gauge field. Yeah?

AUDIENCE: So I just want to clarify. These two analogies for background field transformations, they're not disconnected. Right?

PROFESSOR: They're not disconnected in the sense-- that's right. Yeah. So if you take the background field notation, and you just really think about that, then this is how you would transform the quantum field into the background gauge transformation. And this is how you would transform the quantum field under the quantum gauge transformation. Yeah. That's right. Yeah.

AUDIENCE: So even if you made these to separate the gauge transformation for just [INAUDIBLE] they still have the same quantum numbers?

PROFESSOR: Yeah. That's right.

AUDIENCE: Are they going to meet at some level?

PROFESSOR: We're going to make sure they don't.

AUDIENCE: OK.

PROFESSOR: Yeah. In some sense, we really want-- the picture is this one that we had earlier, where we really want the collinear fields to live up here, so we had these hyperbolas. Right? We want this guy to live up here and this guy to live down there in momentum space. So this was P minus. This is P plus.

So we want them to live in different places. They have the same quantum numbers, except we want them in momentum space to live in different places and to describe different degrees of freedom. Right? So everything that we did here is exactly related to this picture, that we want to talk about this guy as having larger momentum, talk about gauge transformations that shuffle things around with components that have large momentum versus small momentum.

All right. So we're going to hypothesize that these gauge transformations are the fundamental symmetry of the effective theory. And I tried to motivate them physically by this analogy with background field gauge and thinking about a longer wavelength mode with respect to a shorter wavelength mode which is in this picture exactly the fact that this hyperbola is higher than this one. Maybe the hyperbola should stay on the board. OK? So they multipole, where the multipole expansion that we did between these fields and the statement that this is a longer wavelength mode is exactly the fact that there's a separation between these guys. So we'll take these to be the fundamental transformations that do not get corrected by power corrections.

And I'm not claiming that that's necessarily a unique way of getting to the effective theory. You could think of there might be an example in the literature of where people were thinking of other gauge transformations. These are the ones that people talk about now, but in the early days, I think, there is an example in the literature, where people are thinking of other transformations, and you could prove that they could be connected to these ones. But we'll just take the ones that are the most useful from the start and not worry too much about what we could have done.

All right. So there's no mixing in the power counting, and that's something that we like. Set up that way. If there was a mixing, if we connected things that were different sizes in lambda, then you could say, well, that the power counting, that the gauge transformation induces power suppressed terms. And you could end up with some situation where you're trying to connect things at different orders, and we don't have that here.

And that's because, if we want to think about this thing as a symmetry, it should really be a symmetry of the leading order action that shouldn't require some other sub-leading terms necessarily to compensate for it, at least for a gauge symmetry like this. We'd like to think that what the meaning of gauge symmetry is and the redundancy encoded in gauge symmetry is all something that you're talking about at the order of the leading order action. And you're not talking about redundancies in fields in sub-leading order, et cetera.

OK. So let's do an example, now that we have this, of how it comes in. And I want to come back to our example of our heavy-to-light current and talk about gauge symmetry there. If you just took the tree level result that we had in that case, then we had something that just involved these two guys, and this hv field was ultrasoft.

This guy was collinear. So if you did a Uc transformation, then under our rules, what you'd find is that the cn transforms, and this guy doesn't, the h doesn't. So that would not be gauge invariant.

But actually, we saw that this also wasn't the lowest order current. There was this additional thing, the Wilson line showed up, and that Wilson line is going to fix this issue. It's going to make it gauge invariant.

So we have to ask, how do Wilson lines transform under gauge symmetries? And that Wilson line in position space, I called it something like this. I can't remember if I used a twiddle or a bar. We'll use a bar here for the position space Wilson line.

So in general, in just a regular gauge theory, the Wilson line between two points transforms at the endpoints. So you get U of x. Then, you get a U dagger of y. So if you like, what we would say for this Wilson line is that we have a U of x on one side and a U dagger of minus infinity on the other side.

Minus infinity is really long wavelength physics, and so we have to worry a little bit about the overlap between our different types of gauge transformations. I divided them into three camps-- collinear, ultrasoft, and the global, so let me do the following. To avoid some double counting with what I called U global, let's just simply take, since U global acts the same everywhere, let's just simply take Uc dagger at minus infinity to be 1.

So this is at some particular spacetime point. We take it to be 1, and this U global will transform at that point. We then know there's no overlap between Uc and U global. OK? Then, that's enough to ensure that.

So with that, our guy under a collinear gauge transformation just transforms on one side. If you like, what's happening is that the coordinate here corresponds to this very large momentum, and we're stretching ourselves out to long distance physics which is corresponding to the smaller momentum. So when I think about transformations that should be collinear, I want them to be associated to the x. And I want to associate other types of transformations to what's going on at infinity which is really a spacetime of its own, which is this ultrasoft spacetime that's sitting at the collinear infinity. So that's why I want, actually, it to look like this.

AUDIENCE: [INAUDIBLE] just with the collinear fields?

PROFESSOR: Yeah.

AUDIENCE: OK.

PROFESSOR: Yeah. So this guy here is a path-ordered exponential, and it just involved the n bar guy. Yeah. So that's what it was in position space, and in momentum space, they called it W, and it was a sum over a bunch of things. I'm not going to-- we had this formula. Let me not write it all out again.

As a short form for that formula, we can write the following formula which is given our notation that we've developed a useful way of thinking about it. So if we formally expand out this exponential, and we just keep the order of that expansion, where the P bar acts on all the fields to the right, that will correctly reproduce this denominator, the sum over permutations is something that we still have to do. It was 1 over n factorial, k factorial in here. M factorial with this notation. OK? But this is like a little shorthand for that messy formula.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah?

AUDIENCE: I think it's related to the gauge transformation equation before.

PROFESSOR: Yeah.

AUDIENCE: How do you get the i epsilon prescription? [INAUDIBLE]

PROFESSOR: So that actually is related to the fact that, when you look at the-- for this Wilson line with the i epsilon, it's not related to the gauge transformation at infinity, but it's related to the fact that the path goes that way. So that when you do the Fourier integral, the fact that it goes from minus infinity to x will give the i0. OK? But for these Wilson lines here, if you think about what's happening at the place where the i0 would actually matter, that's exactly the 0 bin. So it's going to be a little more-- Yeah. So you don't really care.

AUDIENCE: OK.

PROFESSOR: Yeah. I can only say that-- we'll explain what I just said to Ilia more when we come to it later on. OK. So this field here has-- I said, it's a momentum space, but then I wrote x. So really what I mean by momentum space is momentum is in the label momentum space. There's some guy, and the labels are these labels on the fields here. Right?

So there's still an x, and that x corresponds to the residual coordinate that we were using to do the multipole expansion. So in some sense, this isn't simply the Fourier transform of that, because of that. It's still knows about the separation between the large momentum and the soft momentum from that multipole expansion.

So this dependence on x allows us to think about the multipole expansion on these Wilson lines. It encodes all the residual momenta, and the fields in the Wilson line carry a residual momenta. So we're just putting our notation to use here. And if you really strictly want to connect to that formula up there, then you would take the residual momenta to be 0.

And then the Fourier transform with respect to just the large momentum Pl minus gives a line which is just depending on that one single coordinate. So if we just had a coordinate which is conjugate to the large momenta, which is the thing that's showing up here are the large momenta, then that would be the analog of what we were talking about up here. Where in general, this thing also has a dependence on the residual x, which we'll make use of in a minute.

OK. So let's think about, so I told you how this Wilson line transforms under Uc see. How does it transform under U ultrasoft? So under Uc, again using this matrix-type notation, it's just Uc. So this is a matrix multiplication in momentum space.

What about under an ultrasoft transformation? Well, under an ultrasoft transformation, this should be a derived-- in some sense, under a collinear transformation, it was a derived quantity, because we knew of the gauge field transform. The thing that gave us the transformation of the Wilson line is just the transformation of the gauge field. OK? So that's what led to, actually, this form, once we put in this fact. So under the ultrasoft transformation, we just take the transformation that we have for the collinear gauge field, and then we derive how the Wilson line should transform under the ultrasoft gauge symmetry. OK?

So how does this guy transform? This guy transforms by putting U's on the left and the right that are at x. But every field in this exponential has the same, U U dagger, multiply it, think about expanding that out. All the U U daggers that are next to each other cancel. You just get an overall U U dagger that comes right outside the exponential.

So this guy is just transforming with U ultrasoft of x. Wilson line back again, U ultrasoft dagger of x, and that's because this Wilson line with our notation is really a local thing with respect to the coordinate x. All the fields sit at x. OK? So it just transforms like that, because that's how the gauge field inside it transforms. So in some sense, both of these are derived from the transformation of An, but it's easier to think about how a Wilson line transforms and then think about how it should transform under collinear than just go through the An route and impose this condition.

OK. So let's come back now to our full current which was this, that had the Wilson line in it. And now if you ask how it transforms under Uc, you see what's going to happen. You get Uc dagger, Uc, and the ultrasoft guy doesn't transform, and then these cancel. So it's invariant under the transformations that have momenta of order of the ultrasoft.

It'll have a momentum of what are the linear scale. There's two objects in this thing that have momenta that can be of that size. The gluons, they're in W in this collinear quark field. Both of those things transform, but that transformation cancels between the two things.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Whoops, that's important. Thank you. And then under an ultrasoft transformation, we can also transform this guy, and then everybody's transforming. Gamma, of course, has not got color indices, and again, it's gauge invariant. OK? Missed a gamma. Gamma's not so important. All right?

So this current that we derived earlier is actually invariant under two different types of symmetries which are really momentum space splitting of a standard gauge symmetry into pieces where we have a large momenta and pieces where you have these larger momenta, collinear momenta and pieces where we have a smaller momenta, this ultrasoft momenta. But both of these types of momenta are momenta that we're allowing inside the effective theory. Yeah?

AUDIENCE: Just trying to remember that the ultrasoft scale was the same for the [INAUDIBLE].

PROFESSOR: Yeah. The momentum of the ultrasoft-- the ultrasoft momentum of the collinear quark, you pulled out mv, and then the remaining part was ultrasoft in the discussion for this particular thing. All right. So it's the Wilson line, the moral of this is that what we were doing before, where we're integrating out those collinear gluons, is actually connected to gauge symmetry. Because without that Wilson line, we wouldn't have a result that's invariant under the type of gauge symmetry we've been talking about.

So this Wilson line carries these n collinear gluons, and those gluons would be the gluons that in the full theory would give you a gauge symmetric result. They would combine. They would be the attachments to the heavy quark that combined with attachments to the light quark to give you a result that's invariant, if you looked at word identities, for example. But in the effective theory, these are just put into a Wilson line, because that's the leading order way that they can show up, but still we have a gauge invariant answer.

So the Wilson line is really needed for the gauge symmetry. And finally, the ultrasofts, as can be clear from the notation, the ultrasofts can just as well include the global transformation, if you like. So we don't really need three. We could put the global back with the ultrasofts. Then, we just have two.

All right. So that's our symmetry. That's an example of how it works. Now, let's talk about covariant derivatives.

So gauge symmetry ties together regular derivatives with the gauge field, and here, what it ties together at the following things. So because of the treatment of the A ultrasoft field as a background, it actually ties together both of these things. And then it does what you expect in a collinear sector. It ties together the derivative with the gauge field. And if you're acting on ultrasoft fields, then there's an ultrasoft covariant derivative which is just like that.

So now, you can ask the question if I just had this gauge symmetry, which is going to force me to use covariant derivatives-- which I was already thinking ahead to when I wrote down the Lagrangian earlier. Right? We already wrote it in terms of these kind of objects, so it seems like we're good to go. So you can ask the question if I just have power counting, the cn that we decided we're going to use, and gauge symmetry, is that enough? Am I done? Do I have everything I need for the leading order Lagrangian?

So power counting told us that we had this type of derivative, and that we could have two of these perp derivatives. OK? So those are the two type of terms we were getting. There's no other order lambda squared field structures that would have the right dimensions and that could do the job.

So these are really the only two that we have to think about. So you could try to think about maybe I can have something with some additional n bars floating around, but really, it boils down-- since you need to get a lambda squared, you need either two perp derivatives. You need two of them, because you still have a rotation symmetry around the direction of motion, or you could have an n dot D.

But there is one operator that is not rolled out, by gauge symmetry alone, and that is this operator. When we wrote the operator, we had D perp slash D perp slash. But nothing stops me from doing something similar in some way by just taking the D perps and contracting them, like that. That's a different operator than the D perp slash D perp operator that we got before. Oops, only one of those. OK?

So this operator is a different operator. In a priori, you could say, well, do loop connections generate this operator which is different than the D perp slash D perp slash operator. The answer is no, actually, but so far what we've done, it's not enough to see that. So we need actually this other symmetry that I alluded to, the reparameterization invariance, and that will actually rule out this guy.

So what's RPI? Well, when we formulated this theory, we needed a direction for the collinear line or for the collinear particles. And then we needed an auxiliary vector n bar, and we formulated the whole set up in terms of this n and n bar. But having vectors like that that we write down, that breaks Lorentz invariance in the same way that specifying v mu in HQET breaks Lorentz invariance.

So if you say that m mu nu is the set of Lorentz transformations, the usual six generators that are anti-symmetric, then the ones that are broken are these ones. And there's five of them, and the one that's not broken is this one which corresponds to rotations about the three, axis with the axis specified by n. OK? So those rotations would act in the components of these guys that are 0, if you like. So there's no issue there, and these guys are the guys that are connected to the pieces that were non-0 in general.

So there's going to be a larger, because there's five things here and because there's two vectors, it's going to be a larger set of reparameterization symmetries than in HQET. First, talk about just the n and n bar themselves. There's three types of reparameterization invariance that would leave n squared equals 0 and bar squared equals 0 and n dot n bar equals 2. And those were the formulas that we were using over and over again really. Everything else was just convention. OK?

So what are those three types? Well, we could take n, and we could change it by some amount in the perp direction and leave n bar unchanged. And since n bar dot something perp is 0, that would satisfy this, or we could do the opposite. That would be type two. So don't change n, and change n bar. And so this is two transformations there's two things specified by this perp guy and two here.

And then there's one more which is called three, and this one let me write it like this. It's a simultaneous transformation of these two guys, where I just do a multiplicative factor. Well, a multiplicative factor is not going to change the fact that's the square of something 0. The place where the normalization comes in is this n dot n bar, and if I just rescale them both by an opposite amount, then that remains satisfied as well.

So I formulated this in terms of an infinite-- I'm going to think about this is an infinitesimal transformation for delta perp, infinitesimal for epsilon perp. I could also expand this exponential and think of it as infinitesimal, but the finite one's easier. So let's just think of the finite transformation in the last one.

Now, this is an effective theory. So whenever we think about transformations, we should think about power counting. We just spent a lot of time thinking about that for gauge symmetry, and we should think about power counting here too. Did you have a question?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah. So what I was just about to talk about, how big things can be. So the power counting that's the right power counting for these guys, so these things are infinitesimal or finite, but they also have some counting in a different space which is the power counting space. And the right power counting for these guys is as follows. So this means I can make arbitrarily large transformations in a power counting sense of type two and type three. There's no constraint. That's what this means, so that maybe that's easier to swallow.

This one, there is a constraint, and the reason that you can think of there being a constraint, I can give you a simple example, and then you'll see how you would derive the other things as well. So think of n dot P. If you transform n dot P, that becomes n dot-- think of this is n mu, P mu. That becomes n mu, P mu plus delta perp dot P perp.

So this is order lambda squared, so this better be order lambda squared. But this momentum here is order lambda, so you need to have a power counting for the delta perp that makes it of order lambda. In order to keep the transform guy the same size as the guy you started with, because I don't want to consider transformations that would take me away from my power counting, that would violate it. So imposing that this thing is of order lambda squared says that this is true.

If you go through the same logic for the other guys, the n bar component was order lambda 0, so you can make the epsilon perp of order lambda 0. It doesn't cost you anything, because there's no constraint. You're not making something that would mess up the power counting.

All right. So what does this correspond to physically? So type three is actually pretty simple. Type three is like a boost along-- if you think about our back-to-back vectors, type three is like the analog of a boost, but you're transforming it in the passive sense of transforming the axis. And what the outcome is is very simple. It just implies that any operator that has an n must have a corresponding n bar sitting next to it, effectively.

So I'll give you some examples. So really, if you have-- let's just say it this way-- if you have an n bar mu in the numerator, then you either have a corresponding n in the numerator, or-- well, you have an n bar in the denominator. Since we could have n bars in the denominator and that was showing up in some places. Those are the two possible ways of compensating for the transformation. So it's just like a simple counting that you have.

So if you look at our lc 0, we had various terms. One of them was n bar slash 1 over i n bar over D. So here, we have an n bar in both the numerator and the denominator, and that compensates for the transformation. And then in another term, we had an n bar slash n dot D, and then that again is invariant in this type of transformation.

What about type one and two? So type one is the following. So think about these guys as in this kind of language they would be back to back. We're trying to describe the physics in this cone for the collinear particles, and n was the vector that was pointing inside that cone.

What this is saying, type one, is that I can rotate that guy, and as long as I'm not rotating it by too much-- i.e. I'm rotating it by a small amount of order lambda, which you can think of as staying inside the cone-- then I can describe everything equally well by some other vector that lives in that cone, and I can decompose the momenta and the modes in terms of that vector, and it will work equally well. OK. So that's what type one is physically doing, and so you're not making a large transformation here, because you want to still be inside the cone. You still want something that's pointing in the collinear direction.

Type two is related to the fact that this n bar vector was just an auxiliary vector that we used to decompose things. We didn't really care about it. It didn't have a strong physical motivation. It was just needed because we're using light cone coordinates. So I can make a very large transformation of that guy.

So here's a large transformation, and I can use some other guy. And we gave you an example earlier, which was 3, 2, 2, 1 as a possible value for the n bar, and that's something that you could think of getting to by a finite type two reparameterization transformation. OK? So that's the picture for what these transformations are, and we will finish up talking about them next time and will show that actually this additional term that we could write down in the Lagrangian is actually ruled out reparameterization symmetries.