PROFESSOR: So here's where we were. So we were talking about this example of b to s gamma and a heavy to light current. In terms of this chi field, our current was chi, which is going to have some label, in a heavy quark field. And we did the one with diagrams in SET. We saw that after we take into account also the 0 bin contribution that it breaks up as follows.

There's a piece that I've underlined in orange that matches exactly with the IR of QCD. There's a piece that I underlined in blue which is the ultraviolet divergences in the effective theory. And so what we do with that is we add a counter-term for the operators and the effective theory to cancel these divergences.

And then there's whatever is left over after doing that, which is this pink. And the difference of the analog pink in the full theory and this pink will give you the matching. So when I say "matching," this is part of what goes into that matching. So what I want to spend today's lecture on is really discussing renormalization in the effective theory and summing logarithms in the effective theory.

It's kind of interesting and different than things we've encountered before, because we see even at one loop order, we have 1 over epsilon squared divergence. And that wasn't something that we encountered before. And so we have to see how that plays out.

But just at the simplistic level, let's just say or record for the record that we could write the following. There was a Wilson coefficient of this operator. And we can divide it into renormalize Wilson coefficient in a counterterm. And then we just think that this counterterm is canceling that divergence. And so the counterterm we need is the following.

So with this minus sign canceling that minus sign, you get plus. And so the counterterm for the operator therefore should have a minus and just all-- exactly these terms. Because of kinematics in this problem, this omega, which is a delta function in this operator, is equal to actually mb.

And I'll go through why that is a little later today. So take it for granted for now. It comes about just simply from kinematics. But I'm going to sometimes write the mb as omega because that reminds you that it was this label in the colinear field.

So this is the type of counterterm we have. It's got this 1 over epsilon squared. It's got this log, so it looks kind of different. And the notation here is that we have a coefficient that's inside an integral, where this thing meant that. So basically what I'm saying when I say omega is equal to mb is that this delta function becomes omega minus mb. We'll come back to that.

So if we have a counterterm like that, then we can do running. So we'll come back to this example. Let me first make a few general comments, and then we'll come back and see where this z leads us. So in general, you're going to have this structure-- some integral, some Wilson coefficients that depend on some omegas-- could be more than one of them-- and then some operator that has some delta functions in it that pick out those omegas.

And because of that integral, in general, you have to be careful, because that integral could also play a role in the renormalization. So that delta function is picking out the total momentum of the product of the quark field times the Wilson line. And in our example, it's fixed by external kinematics. So I'll come back to this and show you that not in every example is it fixed by external kinematics in a minute. But our example is special because the integral actually doesn't play an important role.

So there's in some sense two parts of this statement. There is the fact that it's external, and then there's the fact that it's fixed to something. So let me first deal with the issue of it being external. And what that means is it does not involve loop momenta.

You see, when we did the calculation, we were basically doing a calculation, not worrying so much about this delta function that was sitting inside our operator. We ignored it. We didn't write it down in our diagrams.

Technically, we should have written that delta function, this delta function, down, too, and included it in our calculation. But actually that's OK because that delta function just comes outside the loop integrals. That's what I want to show you. So it does not involve loop momenta.

So in all the diagrams we consider, that's trivial, except for one non-trivial example. And that non-trivial example is the example where we had a colinear vertex-type diagram like this. So we called the external momentum here p. This was p plus k.

And if I let this guy go this way, we can call it minus k. You see, the Wilson coefficient sits here at the vertex. And it acts on the colinear fields that come out of that vertex. But it gives some of the momentum of those colinear fields. So we get this guy, and then we get this guy.

And so we just kept the external guy. So from this notation, the same thing as saying this delta function. We can see that it only depends on the external p, on the n bar dot k. And that's because of the structure of this operator.

So that won't always be true. It could be the case that the Wilson-- in a more complicated example, that the Wilson coefficient could depend on the k. And then that would have some implications for renormalization. And we will treat an example of that sort in the near future.

The second thing to ask is, what does it fix to? So if we put a little more into our diagram, let's put the photon in. This is a b quark. This is a strange quark b to s gamma, for example. Then the kinematics of the external lines there is we have mbv coming in for the b quark, p of the photon, and p of the external strange quark. And the photon is light-like. p gamma squared is 0.

And in the setup here, we can just think of it as e gamma times n bar, because p is colinear to the n direction, and it's going-- we have the photon going this way. And it's back to back with a jet that's described by the colinear n direction, just as the beam goes on decaying to a photon in a back-to-back jet.

And so if you dot into this equation n bar, since n bar dot v is 1 in conventional coordinates, we just get that mb is 0 plus n bar dot p. And that's the statement that omega was just mb. So when we were doing our loop calculations last time, we kept writing mb.

We could have been more careful and written-- well, we couldn't have been more careful, because kinematics was demanding that this was true. But just in order to track what the Wilson coefficients can depend on, really what they can depend on is these labels of these colinear fields. And so I'm going to start writing omega for that reason because that's the more generic thing that happens.

Even if we're not in a situation where it's fixed to be some mass, we'll be able to get dependence on the values of these parameters of omega. So any questions about that? All right, so this Wilson coefficient just comes out, and we didn't have to worry about this delta function.

So how do we calculate the anomalous dimension? It's the usual story. So mu d by dbu of the bare coefficient is 0. That implies a renormalization group equation for the ms bar coefficient. And it's got this familiar form, where the gamma is products of z's and z inverses-- just this.

So this is an all-orders formula. If we plug in our one loop result here, then on the left, if we worked order by order in alpha, on the left-hand side, we can just take the 1 in the zc universe here. And the zc, when we take b by d by d mu, there's mu in the alpha s, and there's mu in this log. And we're going to have to differentiate both of those. So let me write it out.

So the minus sign canceled the minus sign. The zc inverse is just 1. And then I have b db dbu over the rest of this stuff, which includes both the alpha and the log mu over w. These are the type of contributions we get, where these guys here are coming from using the fact that mu d by d mu of alpha when you're in d dimensions is minus 2 epsilon alpha plus-- order alpha squared term's related to the beta function, which we can drop here. And then this guy comes from the explicit-- differentiating the explicit log mu over w. The divergent terms cancel, and we're left with a finite anomalous dimension.

And it's got two types of terms in it. So the type of anomalous dimensions that we're used to are like this guy, here-- this 5 times cf. That would be like a standard anomalous dimension for an operator. In this case, we have this extra piece that has this log of mu over w. And this guy is what's called cusp anomalous dimension, the guy that multiplies the log. So with that language, you could call this guy the regular anomalous dimension.

Now, if you're going to do renormalization group evolution, you're going to sum logs. You could think that you're starting at some hard scale like w. w is mb. That's the hard scale. And then you want to run down to some small-scale mu.

When you run down, if you take mu small, this term is larger than this term. So actually, this term here gives you the leading-- what are called the leading logs in this case. And this term here is actually only needed for next leading log.

Because this term here grows when you go down, and that term doesn't, this term here is actually giving you the leading logs. So I'll come back and talk a little bit more about leading log versus next-to-leading log. And also, I'll explain to you what this word "cusp" in this-- where's the cusp, since we haven't seen it yet? I'll come back and explain where that is in a minute. For now, let's just take the leading log anomalous dimension and see just to solve the anomalous dimension equation.

So we'll just take the leading long term and not worry about the other term, so just taking the cusp term. So plugging that back into our equation for c, we have this differential equation, which we can also write as d log c by d log mu. And that's an equation that we can basically just integrate since there's no c on the right-hand side. It's a homogeneous equation. We can integrate it with some boundary condition.

So we could specify the boundary condition at an arbitrary scale. Let me, for simplicity, use the following tree-level boundary condition. So at tree level, the Wilson coefficient was 1. That we have to pick some scale where we make that statement true.

And for convenience, I'm just going to take that scale to bw. That makes things a little simpler. In general, you could make this any hard scale, ie, any scale mu of order w. So you could call it something mu 0 and consider the renormalization group from that scale. But for convenience, I'm just taking that mu 0 to be w here.

So we take that down [INAUDIBLE] It's actually useful to look at what would happen if we didn't have a running coupling on the right-hand side. So say we were doing QED, and we didn't have any massless light fermions, so there was no running. Then this coupling would just be alpha, and it wouldn't run.

And then you'd just integrate this log, and you'd get a double log. You integrate one log by d log, you get log squared. For each coupling, you get log squared. This is the double log series that I promised you-- the Sudakov double logarithms that were going to come out of the renormalization group of SET. So QED with alpha fixed-- I'm going to take cf to be 1. When you take the exponential of what I just said, then this is your result for the Wilson coefficient.

And this is what's known as a Sudakov exponential. The leading log Wilson coefficient is the sum of these alpha times double log. So if I expanded this out, I'd have an infinite series, where for each pair of logarithms, I get one coupling. And that's the Sudakov double logarithms. So if we expand out the exponential, that's what we would get. So really, I should make my line go under the whole thing.

So why did we get something like this? So physically the reason that we got this Sudakov exponential or Sudakov form factor is related to the fact that the whole theory that we've developed put restrictions on radiation. So there was kinematic restrictions put on the radiation by the whole setup that we had. We had this jet, and it was colonnaded. And the radiation was forced to go inside that jet.

In the effective theory, we're seeing the Sudakov exponential come out of UV renormalization. The usual picture for it is that you're thinking about IR divergences. And you're thinking about the real radiation, not the virtual radiation. But as you may be familiar with what the effective theory does, is if you have an IR divergence in full QCD or full QED, then what the effective theory does is it takes the scale corresponding to-- so the IR divergences are logarithms.

And they're logarithms of something over something. And we've done in the effective theory is we've introduced a scale mu. We've introduced-- effectively, we've taken the part of that logarithm that was UV and taken it to infinity. And in this standard way, we've introduced a scale mu that splits things between Wilson coefficients and operators.

So there's this factorization scale that's splitting the hard physics from-- which is part of the logarithm that we're capturing. It's separating the hard physics from the infrared physics. And here, we're just looking at the hard physics. And we can see the Sudakov. We could also look at the real radiation and see the Sudakov by studying real radiation diagrams.

But in some sense, this is easier, because this is just coming out of a renormalization group equation. So the usual way of doing it would be to just look at in the full theory, and you'd see Sudakov logarithms from ratios of two things that you were thinking of as IR. Because the hard scale, which is mbw here, would also be an IR scale from the point of view of QCD. But here, it's become part of the Wilson coefficient. And now it can be treated by the renormalization group.

So QCD is not really any more complicated. We just have to deal with the running coupling. We've done that before. Let me remind you how it works because it's slightly more involved in this case.

So for the leading logs, we just need the beta 0 term. And we can switch variables from d log to d alpha in this equation, here. And so then we have to integrate this thing with respect to d alpha. And so what we can do is we can take this log, and we can also write the log as an integral over alpha. Alpha in this equation is some w variable. And so then just integrating both sides of this equation here, I can write it out like that.

So the log became this thing. And then one of these factors-- there was a minus sign. But then when I use this formula twice, the minus sign goes away, and I get two of those factors. This is the other measure here from the explicit d log mu. So we could underline things-- half of this guy, all of that guy, d log mu, other half of this guy and that guy. And then there's this explicit alpha cf over pi, which I've written as cf over pi out front, and this alpha here.

So we do that integral. We do this integral. Well, all the intervals are trivial. We do this integral, and then we get integration variable at fixed quantity. And then we can do this other integral. We get a solution. So I'll show you what the answer looks like. It's, again, an exponential, because we're solving for a log c. I'll write it like this.

So we're familiar that renormalization group equations, when you have a running coupling, can give you ratios of alphas. We saw that when we were running for fermion operators. Here, it's just a more complicated function, but it's again a function of that ratio. And this extra structure is coming about just from the running coupling. So the only difference between this and QED Sudakov is the running coupling.

So if we were to take this now, and we were to expand z, we were, say, to expand about alpha of w, then we would get logs of mu over w, right? And so what you should think that this z dependence encodes-- is an infinite series in this exponential. And so the structure of that infinite series, if I was to expand, would be alpha squared u over w alpha squared log cubed u over w, et cetera.

So in the exponent, a running of the coupling is only giving you one log for each alpha. So the series that I'm solving in the exponent by taking into account the running coupling is a series of this form, where I go down by one alpha and down by one log.

So from the structure of what we've been talking about here, it should become clear that the right thing to talk about, if you want to do counting, is log c, because the anomalous dimension equation was simple for log c. It was mu d by mu or d by d log mu of log c is equal to something. That something on the right-hand side is an expansion in alpha. And we could include higher-order terms in that expansion, but it would always be of that structure.

So to talk about what terms we're summing, we can write the following schematic equation for different orders. So if we have alpha s to the k log to the k plus 1, which is this series I've just denoted, we'll call that leading log. And if we add some higher-order term, like alpha s to the k log to the k, then that would be next-to-leading log. And if we had more alphas than logs, then we call that next-to-next-to leading log.

And the structure of the anomalous dimension equation always guarantees that if you solve it for this log, that this is what the higher-order terms would give you. So you should think of it that really, log counting is kind of like normal log counting, except for two caveats in this theory. You're doing it in the exponential. That's what taking the log did.

And you start out with one extra log, one more log than you're used to. But after that, it's kind of normal, because I'm summing here alpha times log, and I just increment for each alpha an extra log. And that's all the running coupling effects.

When I go to next-to-leading log, of course, we write in coupling effects, too, and other things that are causing the series. But it's down from this one because it doesn't have that enhanced log. And then this one is down again because it's, again, one less log. So this is the kind of terms that you could sum by having higher-order corrections in the anomalous dimensions. And for example, this cusp anomalous dimension is known at three loop orders. So certainly, next-to-next-to leading log is well within the realm of things that people talk about.

So let's ask that question. What coefficients, if we were to carry out this one new calculation, and we were to do higher loops, what coefficients do we need to compute? So remember the story when we were summing single log series was we would do tree-level matching one loop anomalous dimension. Then we'd do one loop matching, two loop anomalous dimension. What's the analog of that story here?

Let's make a little table. So what information do we need at tree level? What information do we need at one loop, two loops, three loops? So for a leading log, what we did is we put in tree-level matching. And at one loop, we only actually took the information that came from the coefficient of the 1 over epsilon squared, which is the cusp anomalous dimension. [INAUDIBLE] here.

If you go to one higher order, it turns out that this is the story to get this series. We would need the two-loop cusp anomalous dimension, the one loop non-cusp, and still the tree-level matching. So this next-to-leading log coefficient, this 5cf, that's this 1 over epsilon. In our example, this was the 5cf.

And only when you get to next-to-next-to-leading log, then the matching is one loop. We have the two-loop regular anomalous dimension, and then the 1 over epsilon squared cusp anomalous dimension from three loop. So it's kind of like it's the usual story, but we have an enhanced-- because of the double logarithms, we have an enhanced thing that we can talk about here, where we're not talking about the regular anomalous dimension, but this cusp anomalous dimension.

So this is the information that you would need to go to higher orders. And this cusp anomalous dimension is actually a universal thing. So given that it's been calculated, you can use it. And you don't have to recalculate it every time. These pieces you have to recalculate-- these 1 over epsilons, in general. Yeah?

STUDENT: Can you comment on the seemingly arbitrariness of the boundary condition you chose?

PROFESSOR: Yeah, so if I hadn't chose that boundary condition-- sure. Let me write something. So the way that you could think about the rg is as follows-- that you have c. And really what the rg is doing is determining some u. It allows you to run c from some point to some other point, like that.

So in this formula, mu 0 in some scale that's of order omega or w-- it doesn't have to be equal to w. It could be 2w, 1/2w. And the way that you're thinking about this equation is you're going to do perturbation theory for this thing. And then in that perturbation theory, you'll have alpha, and you'll have log squared of mu 0 over w.

But as long as you're saying mu 0's of order w, those aren't large logs. And this thing here in this formulation would have the double log series of alpha log squared of mu over mu0. But then mu0's of order w. And mu you can take much smaller, and that's the large logs. So for simplicity, I just took mu0 equal to w. Everything I've said, I could repeat, and this story would really be the same if I just used an arbitrary mu0.

And the reason you actually want to use an arbitrary mu0 in general is that you want to do scale variation to think about uncertainties. And so the mu0 dependence, just like in our-- when we were talking about electroweak operators and electric Hamiltonian, the mu0 dependence cancels between these things here. But that calculation is in order by order an alpha cancellation.

There's no large logs associated with that cancellation. And so what it allows you to do is if you've truncated this to some fixed order, and you've worked to some fixed order over there, then you can probe how much uncertainty you have by varying the mu0. This mu here will get tied up with the mu in the operator.

So we can talk more about where that mu goes later on. But at the level of this formula, it is-- in general, it's the case that you would do it with an arbitrary mu0, and then only fix mu0 at the end. And you'd even vary mu0 to get an idea of how much uncertainty you have in your leading log calculation, because varying mu0 would probe the next term in the series, here.

STUDENT: So when you say tree-level matching, you mean--

PROFESSOR: I mean this c.

STUDENT: --equal to 1.

PROFESSOR: That's right. I mean this guy, here. Right.

STUDENT: Right. And if you want to go to nnll, you'd take 1--

PROFESSOR: Then you'd have this. That wouldn't have this term. Yeah. All right, any other questions? OK, so what is this cusp that we've been saying the words for? I'll answer a couple of questions that have come up. Where's the cusp?

So if we look back at our operator, we had this operator. And we could make a field redefinition, if you remember, and put the ultrasoft effects into a Wilson line. Now for this heavy quark here, remember what that-- the theory for that was with hqet.

So the field theory for this heavy quark was an iv dot dhv. And if we actually make a similar field redefinition to the one we talked about to get this y, but on the heavy quark field, then we can actually, in terms of this guy here, get a free Lagrangian as well. So actually, all the effects from ultra soft gluons in this operator can be encoded in Wilson lines. So let me do that. So that just says that actually, when we talked about static sources, static sources could also be encoded as Wilson lines by a kind of similar manipulation to what we did when we talked about SET.

So if we look at the ultrasoft sector here, we have a Wilson line that actually has a path. So there's a Wilson line that comes from minus infinity along v to whatever position we put our operator at. Let's take it to be 0. And then we have a yn dagger which is extending out to plus infinity from 0, like that. And the cusp is this fact that there's a kink in this Wilson line. It's not a smooth path. It actually has a sharp angle. And that's the cusp.

So there's actually a general renormalization theory for renormalizing Wilson lines with cusps. So this is something that at some point in the history of QCD, people tried to reformulate QCD in terms of Wilson lines, entirely in terms of Wilson lines. Then they realized that they could have these cusps in the Wilson lines. And the renormalization of that theory became much more complicated than the renormalization of the QCD action because they couldn't prove that in general, these cusps wouldn't come out of the dynamics. And then people dropped it.

But along the way, anyway, they formulated a general renormalization theory of cusps in Wilson lines. And whenever you have a cusp like this, you end up having, in our language, this log, single log, in the anomalous dimension. So if one of the lines here is light-like, and one of them is, because n squared was 0, then the anomalous dimension for this cusp will have a single log like our log of mu over w.

So that actually holds not only to one loop, as we talked about, but actually to all orders in perturbation theory. If we'd gone to higher orders in perturbation theory, then what would happen is that the coefficient of that log would get corrected. But there'd still be only one log in our anomalous dimension. That's also something that you can argue from SET directly. But originally, it falls from this general theory of the renormalization of cusps of Wilson lines.

So if you were thinking about our calculation, you may remember actually that we got 1 over epsilon squared. It's from the ultrasoft diagrams. And that's in some sense what we've just talked about, because these are the ultrasoft diagrams, and they have 1 over epsilon squareds. There was also 1 over epsilon squareds from the colinear diagrams, right?

So if you like, this piece here also in some sense has kind of a cusp. And in this case, it's not-- it's actually what we're doing is we're taking a Wilson line along the n bar direction, which is our wn. And then we're attaching it to a quark field. So this is a full cn quark field. It's not a Wilson line.

So you can think of it as just ending on a quark field. But even just ending on a quark field, the quark field also has dynamics. It's not as simple as Wilson line dynamics, but it has dynamics. And we have interactions between these. And this is also kind of a cusp. Although it's not a simple Wilson line cusp, it's also kind of a cusp.

So even the colinear graphs with the Wilson lines ended on quarks also produced these q over epsilon squareds and have this kind of structure. And they actually have similar-- very similar relations to the cusps in regular Wilson lines. So that was one question-- what was this cusp?

Another question that came up was this fact that the w got fixed. And so we could ask the question, in general, when will that happen? So when will the w's that are showing up in our Wilson coefficients be fixed by external kinematics?

And actually, that's going to happen in the following case. We can actually state when it will happen quite generally. So imagine that we had an effective theory SET that had multiple colinear directions. So then we could build operators out of all those different colinear fields.

And we would do it with our building blocks, which are chi n or curley bn for each n. And if our operator only involves one building block for each direction, then it's going to be the case that the value of those labels or those omegas are always going to be fixed by external kinematics. So let me just write down another example.

Imagine we had something like an LHC process, where we had two gluons coming in and two quarks going out. But we'll think about this as protons colliding, coming in from two different directions. Let's call them n2 and n3-- and then going to two jets transverse to the axis by a large amount.

So there are also two different directions, which I can call n1 and n4. So this process of glu-glu goes to qq bar, which is really pp goes to dijets. As long as those dijets are well-separated from the b maxes, which is the case we're interested in, then we have this situation, where we have four different n's.

And so what kind of operator would you write down for that? Well, you have dw1, dw2, dw3, dw4. You'd have a Wilson coefficient that could depend on four different w's. And you'd then write down some operator out of building blocks, n1w1. I'm not going to worry about all the indices, but I'll worry about some of them.

That would look like that. So I just put down some operator that's got the right structure. It's got two quarks, two gluons. I make the gluons these b [INAUDIBLE]. I've chosen to contract them. Each of them has a different direction. This is in n1. And each of them also gets a corresponding large momentum. And the Wilson coefficient can depend on those large momenta. So this would be the operator that would describe that process.

But again, if you think about the renormalization, then if you think about the renormalization for a minute, the colinear diagrams aren't going to involve contractions between this guy and any of these other guys because these guys are totally independent. They're a different Lagrangian. So the contractions are, again, just like the kind of calculation that we did.

The colinear diagrams are just coming from this thing alone, and they don't care about that stuff. So it's actually the same calculation that we already did for b to s gamma if I wanted to do the colinear diagrams here. And then I would have to do the colinear diagrams for each of these guys, but it's kind of independent.

And again, the w1 is external for that calculation. So the anomalous dimension would just-- it would be outside it. It wouldn't involve-- it wouldn't appear and get thrown into some loop integral momentum.

And so if you think about what can fix these w's, it's really only external information. So what are the momentum fractions of the incoming quarks from the PDFs? What are the energies of the outgoing jets? That's the type of information that, even in this case, would fix the w's.

So if we were to calculate the Wilson coefficient for this operator and do its renormalization group, there would again-- it would again be of a product form. There's no convolution, just a product for this Wilson coefficient.

So when could that not be true? It could not be true if, for example, we had a chi bar and a chi that were in the same direction. So if we're in a situation where all the objects are different directions, that's what happens. And it's pretty simple.

But let's also do an example when is it not true. It's always good to have a counterexample. And it would not be true if we had an operator like this.

So here's an operator with just two quark building-block fields. I gave them different labels, because in general, the Wilson coefficient could depend on their large momenta. But now, they're in the same direction n, so they belong to the same colinear sector. And in this operator here, we would actually no longer be in the case where this thing would totally decouple from the loop integrals.

So if we think about inserting this operator, then some of these guys here will involve loop momenta. Actually, it turns out that one combination of the w's is still fixed by external kinematics. And so there's an overall-- think about the overall delta function on this product. That guy would be fixed. And then some difference of these w's would not be fixed. That's the true story about at least one combination that involve the loop momenta.

And what does that lead to? How does it complicate what we've already said? What that leads to is that when we formulate the anomalous dimension equations, they also involve integrals.

So for this guy here, for the one momentum that's not fixed-- so in general, there's two. And this is the one that's not fixed. We would get an equation that looks like this. Rather than the simple product form, there'd be an integral on the right-hand side.

And indeed, actually, if you do the renormalization of this operator that I wrote up there, just that simple quark operator, you actually reproduce a bunch of classical evolution equations that are of this form. So even just the simplest possible case we can think of where things are not fixed gives us a bunch of interesting results.

So for deep inelastic scattering, then what you're getting is the Altarelli-Parisi equation or DGLAP equation. And the evolution of the operator would be the evolution of the part-time distribution function. That's one thing that would be encoded in the renormalization of that equation.

There's other processes that are actually-- some of which we may talk about later on-- that are also encoded in the same type of operator, kind of a different projection of it. And one of them leads to something called the Brodsky-Lepage equation. And then there's another one that's showing up in something called-- a process called deeply virtual Compton scattering.

So the one that's most familiar is the renormalization of the PDF. And you may remember that the renormalization of the PDF has this form of having an integral. That'll come out of renormalization this operator. And we'll do that case.

So in general, the structure of the effective theory is leading us to find out what the structure of the renormalization is. And it can reproduce some well-known, classic things, but it comes out actually pretty easy. So I'm going to do this case, but I'm going to do this case in complete-- with all of the details filled in.

So first, I have to convince you that in deep inelastic scattering, you actually get that operator. And we'll see precisely what kind of matrix elements of that operator show up in deep inelastic scattering, why we get that operator. And then once we're convinced that we have that operator, and we'll actually drive a factorization theorem that involves that operator. Then we'll talk about its renormalization group evolution. And I'll show you that it has this form that I'm writing here.

So probably for the rest of today's lecture, certainly for the rest of today's lecture, we won't get to the running. We'll do that next time. But we'll at least get to the point, I think, where you'll see why that operator is showing up in DIS. So let's do DIS.

So I'm only going to talk in DIS about factorization, and then the renormalization group evolution. So no phenomenology, nothing like that-- we'll just talk about these two concepts. So DIS is electron-proton to electron anything, so just kinematics.

Think about a virtual photon exchange. Proton comes in. I'll say that the proton's momentum is a capital P. Gets blown apart. Call the stuff that's blown apart Px. And so Px is the sum of all the particles, all the final-state hadrons. Q squared of the virtual photon-- so this is q.

Little q squared is minus capital Q squared. And this thing is much bigger than on QCD. mu arc in x is capital Q squared divided by this dot product. And you can talk about Px, which is the Px mu. And by momentum conservation, that's the proton momentum, plus whatever momentum came in from the leptons q.

So Px squared-- if you square it, well, this guy is giving you mass of the protons squared. And then there's a dot-- a cross term, and this guy squared. And if you put those things together like this, so this is an exact equation. Px squared is that. And so there's actually different regions of DIS, and we're only going to talk about one of them. I have to enumerate what I'm talking about carefully. And I can do that by looking at Px squared or this factor 1 minus x over x, which I'll call 1 over x minus 1.

If this thing is of order q squared, then that means that this thing you're counting is of quarter 1. And in that case, it's what's called the inclusive operator product expansion. So this is the case that most books would deal with. And that's the one we'll deal with, actually, where effectively, we're not putting restrictions on x. We're just saying it's generic, and it's not approaching any endpoints.

There's also a situation where the Px gets smaller. And then this thing is close to an endpoint, lambda QCD over q. And that's called the endpoint region. And people talk about that. Usually when people say x goes to 1, this is what they mean-- that the 1 minus x minus 1 is of that size on the QCD over q.

There's even a third region, where the Px squared becomes hadronic lambda QCD squared. And that's like here, two powers. So taking that factor to be really small. And that's the resonance region. And that's the case where the final state x is just another proton or an excited state of a proton. So that's where elastic scattering would be. And that's an exclusive process. It's not inclusive anymore.

So actually, all three of these cases can be done with SET. And the way that it works is different in each case. And the case that we'll do is just the first one, which is kind of a classic one, and also the simplest one. So our Px squared is going to be of ordered q squared for our analysis.

We also need some partonic variables. So the struck quark carries some momentum fraction from the proton. This is the familiar language.

And for our analysis, what we're going to do is just take m bar dot little p of the quark to be something times n bar dot big P of the proton. So the fraction is this c variable, and it's the ratio of the quark momentum to the proton. But we'll do it in a very particular component, and we'll see why that's the right thing to do.

And if we have our picture here of the quark kinematics, then this P is the incoming P. And the outgoing P would be P prime, let's say. So I could think about an analog of this equation, where I square P prime. And if you do that, it's kind of similar to the hadronic case. The only difference is that the c shows up. And, of course, there's no mass of the proton. So P prime squared would be that. So we'll see how this variable c shows up.

Now, one thing that we have to decide about is frames of reference, because remember, when we were talking about degrees of freedom in SET, we had picked a frame to do that discussion. And it was almost always a center of mass frame so far in our discussions, or the rest frame of the initial state. And here, we're going to use a slightly different frame, which is the most convenient frame for deep elastic scattering. It's called a Breit frame.

So we're going to do our analysis in this frame. So what defines this frame? This frame is defined by taking q mu to just have a z component. Remember, it's space-like, so it has to be somewhere in the space-like column.

And we can choose it such that it's entirely in the z component, and nowhere else. And that's the Breit frame. If we want to write that in terms of our classic decomposition of nnn bar, is we can write it as a difference divided by 2. And that's giving the z component.

So in this frame what's happening is that your initial state proton is coming in with a very large momentum. And then it's being-- you're killing that momentum and then spitting it back out in a different direction. So you're spitting back out stuff in a different direction.

So the initial state proton is coming in with a large momentum in some direction. So if you work out the kinematics given that for what the proton would be, the proton's momentum would be in the following form. So it's got a large-- this is large, and this is small.

And if you have a large component in some light-like direction, that means it's colinear. So you could actually write it as-- using momentum conservation, you could write it like this. And it has a colinear scaling. So what we have in the deep inelastic scattering in the Breit frame is that the incoming proton is a linear proton.

And if we look at Px, and we just-- again, I'm not going through the details of this. But if we just decompose Px in terms of these coordinates, then we get this. And so as long as this factor 1 minus x over x is of order 1, you see that there's a large component in n and a large component in n bar. And that means it's hard.

So what would happen in these other cases here is that you would change that, right? It would no longer be hard. And that's why these cases here are different. But as long as we're in this first case, this is hard. And we can say that we have colinear modes and hard modes. And then we just want to write down an effective theory for those two things.

You could also do an analysis of DIS in the rest frame of the proton. That's another case. And actually, the final result that you would get would be the same. That's what we'll get from this frame. But this frame is actually a little easier.

So we're really talking about hard colinear factorization in some sense. Colinear describes the low-energy degree of freedom, which is the proton. And hard describes the off-shell final state and the hard fluctuations. And really, what we want to do in DIS in this classic case is just separate hard and colinear fluctuations, at least in this frame.

So we can do that for the cross-section. So let me remind you something about the cross-section in DIS. So just using nothing more than the fact that we're treating the leptons order by order in the photon, we can work the first-order in the electromagnetic coupling.

And then we can write down a formula like this, where we split it into electronic and hadronic tensor. And the hadronic tensor can be written as the imaginary part of some T, where T is the following thing. So this doesn't use anything about-- we haven't used any sort of perturbation theory or anything to write this-- what I'm telling you-- down. We just worked all orders, and basically used the optical theorem. So T is the time order product of the two curves.

Let me call this z. Actually, I don't know if that's-- before you start doing anything in QCD, that's how you could write this. And these are the electromagnetic currents for a quark.

STUDENT: But wouldn't you write it over that, though? [INAUDIBLE]

PROFESSOR: Thanks, yeah, because x was something else. Don't want to use x. All right, so for this T mu nu, we can also use current conservation and decompose it into two pieces. So this is a classic thing that we do in DIS. And nothing about the fact that we're using the effective theory really changes during this.

What you're after-- the effective theory is calculating these T's, which would be coefficients. And this part is all standard stuff for any analysis of DIS. So if you're not familiar with it, or you don't remember it, it's actually not that important.

But using the current conservation of T, the fact that when I dot a q into the T, you should get 0, because you're dotting a q into the current. The current is conserved. It's electromagnetic current. That tells you the possible structure of this thing and that there's two general terms if I'm doing a spin sum, which I am. There's the sum over spin up there. If we weren't summing over spin, if we were picking out particular spins of the proton, then the formula here could be a little more complicated.

So this satisfies all the symmetries. And in general, what you know is that you want the imaginary part of forward scattering graphs. So pictorially, it's sometimes useful to draw something. So here's a forward scattering graph.

And the thing that's different that we have-- know now-- and the information that we're going to use is that we're assigning the external guys to be colinear and the intermediate guy to be hard. So we want to integrate out the pink guy, as usual. The guy we want to integrate out is always pink. And we want to keep these colinear guys.

And so the operators in the effective theory-- we can already intuit what they should look like. They're just going to involve-- I have a current with two photons hanging out of it, and then colinear quarks. Those guys-- those are the external lines there.

And there's actually also an analogous thing with colinear gluons. And that's what the type of-- so I could have an operator at higher orders, where the external states here were gluons. And that's what the operators of the effective theory are going to look like. Just contract that pink line to a point, and that's what they look like.

So given that we know what they look like, we just have to write down the lowest dimension operators of that form. And the lowest dimension operators of that form are exactly the operator that I told you was going to be the one that comes in. So we can enumerate the lowest dimension, which where dimension here is counted as lambdas, right?

So it's the lowest order in the power counting operators, so not the lowest order in mass dimension. Let me write out a few things. Yeah, I already switched to this notation. Yeah.

So the type of operator we'd have that's order lambda squared-- this guy's order lambda. This guy's order lambda. That's lambda squared. And the most general thing we can think of is this. The reason why I wrote this out rather than writing it as a chi field is I wanted to emphasize that there's a flavor index.

So i is up quarks, down quarks, strange quirks, et cetera. And then we could also have gluons, and gluons are similar except in terms of our curly v field. And it is actually just a contraction of two curly v's, and then traced over. And then there's some Wilson coefficient here which is also just the Wilson coefficient for the gluon.

So this is, again, order lambda squared. The b perp field is order lambda squared. So we just write down the lowest dimension operators that have this form. And that's going to be the right answer-- the lowest order in lambda.

So it turns out actually that in this case, our lambda counting is exactly the corresponding twist expansion. I'm not going to--

STUDENT: All j's correspond to the operators that go into the T1's and T2's?

PROFESSOR: Yeah, right. So now what's the index j exactly? There's a T1 and a T2. And the T1 and the T2, although they have the same kind of quark and gluon field structure, they get different coefficients, and that's what the j is.

So when we think about doing a similar kind of decomposition, the effective theory, we can write it as follows. So as an 01 and an 02. And I think if I'm getting my mass detentions right, it looks like this. So there's a piece that multiplies a spin structure that's this g perp mu nu. That's transverse to Q.

And there's a piece that multiplies this guy, which is also transverse to Q in the Breit frame. So that's the kind of decomposition we would do in the effective theory. There'd be two different types of operators we could think about. But the only thing that actually tracks that-- because of the spin relations, the only thing that tracks that is the Wilson coefficient. And so that will differ in these two different cases.

So this guy here is going to be-- this will give the quark PDFs. And this Wilson coefficient will then be the thing that you convolute with the quark PDF in DIS. And then likewise, this guy is going to give the gluon PDF. So let's do the quark-- we'll do the quark contribution in detail. The gluon contribution is not really any harder.

So what I'm going to do is I'm not going to think about it in perturbation theory. I'm just going to think about it to all orders in perturbation theory. And really what that means is I'm not going to think about this c as expanded in perturbation theory or think about any of the diagrams here as expanded in perturbation theory. I'm just going to see, if I manipulate things, what does it lead to, using things like momentum conservation and stuff like that?

So let me write-- in order to do that, let me write it slightly differently than I just did, which I apologize for. But I'm going to-- I want the arguments of the c, just for a later convenience, instead of being w1 and w1 to be w plus and w-- minus the sum of the w's and the difference of the w's, but everything else the same. It's convenient to just talk about the sum and the difference rather than the individuals.

So what's the parton distribution function? Let me convince you that it's actually related to this operator. If you were in coordinate space, the coordinate space for us was z. We're just leaving x for something else. Then you could define the parton distribution function as a proton matrix element of quark fields with a Wilson line between them.

So they're on the light cone, and we have a formula like that. And also, one can convince oneself that if you want to write down for the antiquarks what the part-time distribution function is, it's the same formula as the quark formula, just with an overall minus sign. And z goes to minus z.

So that's an operator definition of the part-time distribution function. You can also define it by moments of the operator. That's the way that, for example, Peskin does it. But if you put all the information in those moments back into a single operator, then it becomes this thing. And if we Fourier transform this, then it becomes the operator that we have up there.

So there's something special about the matrix element we're taking here. And that is that this matrix element is forward. It has the same kinematics for the-- so this is a colinear proton, but we have the same momentum here and here. In the in state and the out state, you have the same momentum. And that leads to one kinematic restriction on what I'm about to write.

So what that imposes is that w1 should be equal to w2. So there's a delta function of w minus that comes from the momentum conservation or the restriction that it's a forward matrix element. But the sum of the two-- with my sign conventions, it's w minus. The sum of the two is unconstrained. Well, it's bounded, but unconstrained.

So we can write the unconstrained part as this interval of c parameter. The bounding just comes from the fact that the quark can't carry more momentum than the overall momentum of the proton. So that's why it's bounded. It can't carry negative momentum, and it can't carry-- negative physical momentum, it can't carry, and it can't carry less momentum in the proton. That's where the limits come from.

And then there's two pieces. The w plus could either be positive, or it could be negative. If it's positive, remember, positive labels-- that was our quarks. So this is the quark piece. And these are the antiquarks. They come with negative labels. Remember that we talked about that earlier. And so both the f and the f bar are hiding inside this formula.

So there is a simplification that there's this delta function here. And so basically what we're talking about then-- you can think of just doing the integral over that delta function. So then you're talking about effectively an operator where you don't worry about putting a label on this one. You have only one delta function left, which I can denote by putting a label on this one.

And this operator is actually exactly the operator that gives you the PDF. This operator is like a number operator for quarks, where you're thinking about momentum, a number operator with momentum omega. And if you want to think about there being some kind of field for a parton, this is about as close as you can get. So this quark field dressed by Wilson line is kind of like a parton in the parton model.

All right, so next time, we'll take this, put it together with this formula here, with the c, and just see where it leads us. And it will lead us directly to a factorization theorem for deep inelastic scattering involving parton distribution functions and some hard perturbatively calculable thing, which is actually a cross-section in the parton model. And then we'll talk about renormalization [INAUDIBLE]. You can already see that we're getting the operator that I promised you-- that DIS is described by this, by a linear operator with two different quark fields, and particular labeling of the momenta. Any questions?