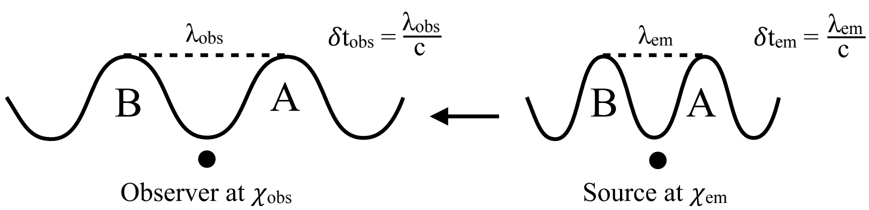
1.C Observational cosmology

Goal: relate the cosmological parameters to observations.

Redshift:
Redshift z is defined by the difference in observed wavelength and emitted wavelength of light:

$$\frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = 1 + \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} \equiv 1 + z \tag{306}$$

In cosmology, this is due to the expansion of space. Light travels from the source at $(t_{\text{em}}, a_{\text{em}}, z_{\text{em}})$ to the observer at $(t_{\text{obs}}, a_{\text{obs}}, z_{\text{obs}})$.



The spatial hypersurface can shrink or expand depending on $a(t)$, so λ_{obs} is not necessarily equal to λ_{em}. Photons always travel along the shortest path, so for light we have:

$$\mathrm{d}s = 0 \Rightarrow c\mathrm{d}t = a(t)\mathrm{d}\chi \tag{307}$$

$$\begin{aligned}
\text{Pulse A}: \int_{\chi_{\text{em}}}^{\chi_{\text{obs}}} \mathrm{d}\chi &= \int_{t_{\text{em}}}^{t_{\text{obs}}} \frac{c\mathrm{d}t}{a(t)} \\
\text{Pulse B}: \int_{\chi_{\text{em}}}^{\chi_{\text{obs}}} \mathrm{d}\chi &= \int_{t_{\text{em}}+\delta t_{\text{em}}}^{t_{\text{obs}}+\delta t_{\text{obs}}} \frac{c\mathrm{d}t}{a(t)} = \int_{t_{\text{em}}}^{t_{\text{obs}}} ... + \int_{t_{\text{obs}}}^{t_{\text{obs}}\delta t_{\text{obs}}} ... - \int_{t_{\text{em}}}^{t_{\text{em}}+\delta t_{\text{em}}} \\
&\approx \int_{t_{\text{em}}}^{t_{\text{obs}}} \frac{c\mathrm{d}t}{a(t)} + \frac{c\delta t_{\text{obs}}}{a(t_{\text{obs}})} - \frac{c\delta t_{\text{em}}}{a(t_{\text{em}})}
\end{aligned} \tag{308}$$

then

$$\frac{c\delta t_{\text{obs}}}{a(t_{\text{obs}})} = \frac{c\delta t_{\text{em}}}{a(t_{\text{em}})} \Rightarrow \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = \frac{a_{\text{obs}}}{a_{\text{em}}} \tag{309}$$

so

$$\frac{a_{\text{obs}}}{a_{\text{em}}} = 1 + z\ . \tag{310}$$

For $a_0 = 1$ (observing today) we have:

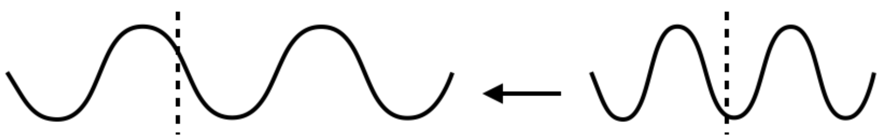
$$\begin{gathered}
\frac{1}{a} = 1 + z \\
\boxed{a = \frac{1}{1+z}}\ .
\end{gathered} \tag{311}$$

Note: the change of luminosity $L = \frac{\text{energy of photons}}{\text{time}}$ is affected "twice" by expansion since

$$\begin{aligned}\frac{\lambda_{\rm obs}}{\lambda_{\rm em}} &= \frac{\delta t_{\rm obs}}{\delta t_{\rm em}} = \frac{a_{\rm obs}}{a_{\rm em}} \\ \Rightarrow L_{\rm obs} &= \frac{h\nu_{\rm obs}}{\delta t_{\rm obs}} = \frac{h\nu_{\rm em}}{\delta t_{\rm em}}\left(\frac{a_{\rm em}}{a_{\rm obs}}\right)^2 = L_{\rm em}\left(\frac{a_{\rm em}}{a_{\rm obs}}\right)^2 .\end{aligned} \tag{312}$$

If $a_{\rm obs} = 1, z_{\rm obs} = 0$ and $a_{\rm em} = a, z_{\rm em} = z$, we have

$$\begin{aligned}L_{\rm obs} &= L_{\rm em}\left(\frac{1}{1+z}\right)^2 \\ \Rightarrow &\boxed{L_{\rm obs} = \frac{L_{\rm em}}{(1+z)^2}} .\end{aligned} \tag{313}$$



Distance measures:
Question: what is the distance between a source at (z, t, a) and an observer?
In static Euclidean space, we can measure a unique distance in different ways. For a source with luminosity L and size l, we have:

- luminosity distance D_L: $F = \frac{L}{4\pi D_L^2}$
- angular diameter distance D_A: $\varphi = \frac{l}{D_A}$

Note that $D_L \neq D_A$ in expanding space!

Comoving distance:
Comoving coordinates move with space as it expands.

$$\begin{aligned}\chi(t_{\rm em}, t_{\rm obs}) &= \int_{t_{\rm em}}^{t_{\rm obs}} \frac{c\mathrm{d}t}{a(t)} \\ &\quad (\text{for light, } \mathrm{d}s = 0 \Rightarrow a\mathrm{d}\chi = c\mathrm{d}t) \\ = \chi_{\rm em}^{\rm obs} &= c\int_{a_{\rm em}}^{a_{\rm obs}} \frac{\mathrm{d}a}{\dot{a}a} = \frac{c}{H_0}\int_{a_{\rm em}}^{a_{\rm obs}} \frac{\mathrm{d}a}{a^2 E(a)}\end{aligned} \tag{314}$$

The comoving distance is not measurable through observations, but it is useful theoretically.

We also define a function that depends on the curvature k of space that is helpful in writing the metrics:

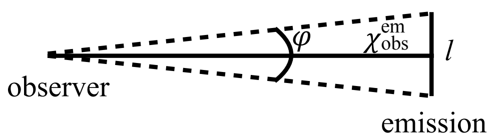
$$f_k(\chi) = \begin{cases} \frac{1}{\sqrt{k}} \sin\left(\chi\sqrt{k}\right), & k > 0 \\ \chi, & k = 0 \\ \frac{1}{\sqrt{|k|}} \sinh\left(\chi\sqrt{|k|}\right), & k < 0 \end{cases} \tag{315}$$

Angular diameter distance:
The *angular diameter distance* D_A is defined such that

$$\varphi = \frac{l}{D_A} \tag{316}$$

for an object that has angular size φ. Then the endpoints of l have the same (χ, θ, t):

$$\begin{aligned} l &= a_{\rm em} f_k\left(\chi_{\rm em}^{\rm obs}\right)\varphi \\ &= \frac{1}{1+z} f_k\left(\chi_{\rm em}^{\rm obs}\right)\varphi \end{aligned} \tag{317}$$

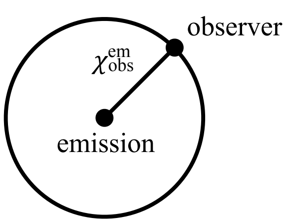


so

$$\boxed{D_A = \frac{1}{1+z} f_k\left(\chi_{\rm em}^{\rm obs}\right)}\,. \tag{318}$$

Luminosity distance:
The *luminosity distance* D_L is defined such that

$$F = \frac{L}{4\pi D_L^2}\,. \tag{319}$$



Then the observed surface, for $a_{\rm obs} = 1$, is

$$4\pi a_{\rm obs}^2 f_k^2\left(\chi_{\rm em}^{\rm obs}\right) = 4\pi f_k\left(\chi_{\rm em}^{\rm obs}\right)\,. \tag{320}$$

Furthermore, we can relate the observed and emitted luminosities

$$\begin{aligned} L_{\rm obs} &= \frac{1}{(1+z)^2} L_{\rm em} = \frac{1}{(1+z)^2} L \\ \Rightarrow F &= \frac{L}{4\pi f_k^2\left(\chi_{\rm obs}^{em}\right)(1+z)^2} = \frac{L}{4\pi\left[f_k^2\left(\chi_{\rm obs}^{em}\right)(1+z)\right]^2} \end{aligned} \tag{321}$$

so

$$\boxed{D_L = (1+z) f_k\left(\chi_{\text{em}}^{\text{obs}}\right)}. \tag{322}$$

This also gives us the relation

$$D_A = \frac{1}{(1+z)^2} D_L \tag{323}$$

so $D_A \approx D_L$ if $z \ll 1$.

Notes:

- The simplest Einstein-de Sitter case is:

$$\begin{aligned}
\Omega_{\Lambda,0} = \Omega_{\kappa,0} = \Omega_{r,0} = 0,\ \Omega_{m,0} = 1 \\
\Rightarrow D_A = \frac{2c}{H_0}\frac{1}{1+z}\left(1 - \frac{1}{\sqrt{1+z}}\right) \\
D_L = \frac{2c}{H_0}(1+z)\left(1 - \frac{1}{\sqrt{1+z}}\right)
\end{aligned} \tag{324}$$

  For $z \ll 1$:

$$D_A \approx D_L \approx \frac{2c}{H_0}\left(1 - \left(1 - \frac{1}{2}z\right)\right) \tag{325}$$

$$= \frac{c}{H_0} z \tag{326}$$

  Furthermore:

$$\chi_{\text{em}}^{\text{obs}} = \frac{c}{H_0}\int_{a_{\text{em}}}^{a_{\text{obs}}} \frac{\mathrm{d}a}{a^2 E(a)} = \frac{c}{H_0}\int_{a_{\text{em}}}^{1} \frac{\mathrm{d}a}{a^2 E(a)} \tag{327}$$

$$\tag{328}$$

  For $E(a) \approx 1$ and $a \approx 1 - z$:

$$\chi_{\text{em}}^{\text{obs}} \approx \frac{c}{H_0}\frac{1}{(1-z)^2}\int_{a_{\text{em}}}^{1} \mathrm{d}a \tag{329}$$

$$= \frac{c}{H_0}\frac{1}{(1-z)^2}(1-a) \tag{330}$$

$$\approx \frac{c}{H_0}\frac{1}{(1-z)^2} z \tag{331}$$

$$\approx \frac{c}{H_0} z \tag{332}$$

  So we get $D_A = D_L = \chi_{\text{em}}^{\text{obs}}$ for $z \ll 1$, i.e. agreement for low z.

- The general flat case is: $k = 0 \Rightarrow f_k(\chi) = \chi$.

$$\begin{aligned}
D_A &= \frac{1}{1+z}\chi_{\text{em}}^{\text{obs}} \\
&= \frac{c}{H_0}\frac{1}{1+z}\int_{a_{\text{em}}}^{a_{\text{obs}}} \frac{\mathrm{d}a}{a^2 E(a)} \\
&= \frac{c}{H_0}\frac{1}{1+z}\int_{a_{\text{em}}}^{1} \frac{\mathrm{d}a}{a^2 E(a)}
\end{aligned} \tag{333}$$

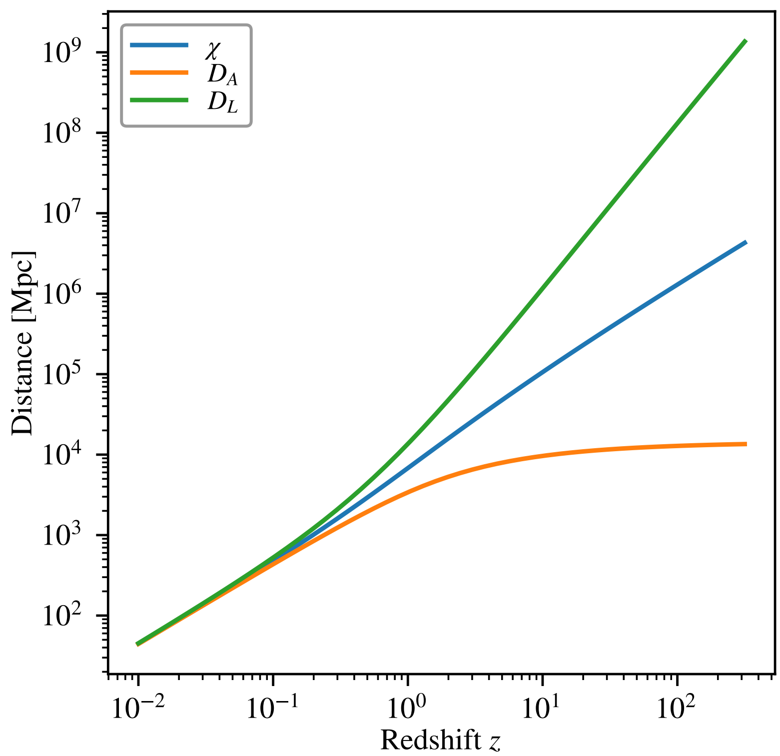
Since $a = \frac{1}{1+z}$ and $\mathrm{d}a = -\frac{1}{1+z}^2 \mathrm{d}z$, then

$$D_A = \frac{c}{H_0}\frac{1}{1+z}\int_0^z [\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0}]^{-\frac{1}{2}}\mathrm{d}z \tag{334}$$

So generally for $\Omega_{\kappa,0} = 0$ (i.e. flat):

$$D_A = \frac{c}{H_0}\frac{1}{1+z}\int_0^z [\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0}]^{-\frac{1}{2}}\mathrm{d}z \tag{335}$$

$$D_L = \frac{c}{H_0}(1+z)\int_0^z [\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0}]^{-\frac{1}{2}}\mathrm{d}z\ . \tag{336}$$



To the left is a plot comparing the angular diameter, luminosity, and comoving distances for a flat universe with $\Omega_m = 0.315$ and $H_0 = 67.4\,\mathrm{km/s/Mpc}$. By $z \sim 1$, there is a significant difference between the distance measures.

Comoving volume element:
We want to measure the number of objects in a given angle $\mathrm{d}\Omega$ on the sky in a redshift range $(z, z+\mathrm{d}z)$. What is the comoving number density of those objects, i.e. what is the corresponding comoving volume?

In general:

$$V = \mathrm{d}A\mathrm{d}r \tag{337}$$

where dA is area and dr is depth. So

$$\mathrm{d}V_\chi = \underbrace{\underbrace{(D_A{}^2\mathrm{d}\Omega)}_{\text{proper area}}(1+z)^2}_{\text{comoving area}} \cdot \underbrace{\mathrm{d}\chi}_{\text{comoving depth}} \tag{338}$$

$$A_{\text{proper}} = a^2 A_{\text{comoving}} \tag{339}$$

$$\Rightarrow A_{\text{comoving}} = \frac{1}{a^2}A_{\text{proper}} = (1+z)^2 A_{\text{proper}} \tag{340}$$

$\mathrm{d}\chi = \frac{c}{H_0}\frac{\mathrm{d}z}{E(z)}$, so

$$\mathrm{d}V_\chi = (D_A{}^2\mathrm{d}\Omega)(1+z)^2\frac{c}{H_0}\frac{1}{E(z)}dz \tag{341}$$

and finally:

$$\mathrm{d}V_\chi = \frac{c}{H_0}\frac{(1+z)^2 D_A{}^2}{E(z)}\mathrm{d}\Omega\mathrm{d}z\ . \tag{342}$$

Plug in $D_A = \frac{1}{1+z}f_\kappa(\chi)$:

$$\mathrm{d}V_\chi = \frac{c}{H_0}\frac{f_\kappa^2(\chi)}{E(z)}\mathrm{d}z\mathrm{d}\Omega \tag{343}$$

$$= f_k^2(\chi)rd\Omega\frac{\mathrm{d}\chi}{\mathrm{d}z}\mathrm{d}z\ . \tag{344}$$

1.D Inflation

So far, dynamics have been described by the Friedmann equations with some mass-energy content of the Universe: $\Omega_m, \Omega_r, \Omega_k, \Omega_\Lambda$. Is this sufficient to explain all data?

Problems:

- Horizon problem:

  $$\rho_r \propto (1+z)^4 \text{ and } \rho_r \propto T^4 \Rightarrow T \propto (1+z) \tag{345}$$

  The Universe cools and at some z_{recomb}, it consists of neutral hydrogen atoms (recombination). We get the balancing equation

  $$H^+ + e \rightleftharpoons H^0 + \chi \tag{346}$$

  where $\chi = 13.6$ eV is the ionization energy. We also have:

  $$\begin{aligned} x &= \frac{\text{number density of free e}^-}{\text{number density of protons}} \\ \eta &= \frac{n_b}{n_\gamma} = \frac{\text{baryon number density}}{\text{photon number density}} \approx 5\times10^{-10}\left(\frac{\Omega_{b,0}h^2}{0.01}\right) \end{aligned} \tag{347}$$