Solutions Day 1, M 2/5/2024

Topic 1: Introduction to differential equations

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Problem 1.

(a) *Use separation of variables to solve* $\dfrac{dy}{dx} = x(y-2)^2$.

Solution: Separate variables: $\dfrac{dy}{(y-2)^2} = x\,dx$.

Integrate: $\displaystyle\int \frac{dy}{(y-2)^2} = \int x\,dx \quad \Rightarrow -\frac{1}{y-2} = \frac{x^2}{2} + C.$

Algebra: $y = 2 - \dfrac{2}{x^2+2C}$. Even better, we can change the meaning of C and write $y = 2 - \dfrac{2}{x^2+C}$.

(b) *Verify that the solution $y(x) \equiv 2$ is also a solution. Explain why this solution was lost in the separation of variables in Part (a).*

Solution: Plug $y(x) = 2$ into both sides of the DE.

$$\left.\begin{array}{ll} \text{Left-hand side:} & \frac{dy}{dx} = 0 \\ \text{Right-hand side:} & x(y-2)^2 = x \cdot 0 = 0 \end{array}\right\} \text{the same!}$$

This was lost in Part (a) because we divided by $(y-2)^2$. If $y = 2$, this is division by 0, so the algebra is not valid.

(c) *Give the full solution to the DE.*

Solution: Full solution: $\boxed{\begin{cases} y(x) = 2 - \frac{2}{x^2+C}, & \text{where } C \text{ is any constant} \\ y(x) = 2 & \text{constant solution.} \end{cases}}$

(d) *Verify your solution is a solution.*

Solution: Plug $y(x) = 2 - \dfrac{2}{x^2+C}$ into the DE.

$$\left.\begin{array}{ll} \text{Left-hand side:} & \frac{dy}{dx} = 2(x^2+C)^{-2} \cdot 2x = \frac{4x}{(x^2+C)^2} \\ \text{Right-hand side:} & x(y-2)^2 = x\left(\frac{2}{x^2+C}\right)^2 = \frac{4x}{(x^2+C)^2} \end{array}\right\} \text{the same!}$$

Plug $y(x) = 2$ into the DE.

$$\left.\begin{array}{ll} \text{Left-hand side:} & \frac{dy}{dx} = 0 \\ \text{Right-hand side:} & x(y-2)^2 = x(2-2)^2 = 0 \end{array}\right\} \text{the same!}$$

Problem 2. *Solve* $\dfrac{dx}{dt} = 3x$.

Solution: Separate variables: $\dfrac{dx}{x} = 3\,dt$.

Integrate: $\ln|x| = 3t + c$.

Exponentiate: $|x| = e^c e^{3t}$.

If $x < 0$, then $x = -e^c e^{3t}$.
If $x > 0$, then $x = e^c e^{3t}$.
If $x = 0$, this is the lost solution.

More simply, $x(t) = \tilde{c}e^{3t}$, where $\tilde{c}$ can take any value.

Problem 3. *Give the DE modeling the effect of gravity on a falling mass m at height h above the Earth's surface. (h can be large.) Assume the mass is falling towards the center of the Earth.*

Solution: Newton's law of gravitation: $m\frac{d^2h}{dt^2} = -\frac{GmM_e}{(h+R_e)^2}$. Where,

M_e = mass of the Earth
R_e = radius of the Earth
G = gravitational constant

Note: Force is actually a vector. By assuming the mass is falling in the direction of the force, we can ignore other directions.