Day 10, F 2/16/2024

Topic 5: Homogeneous, linear, constant coefficient DEs (day 2 of 2)
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1 Agenda

- All about e^{rt}
- Quick review
- Damped harmonic oscillators
- Decay rate of exponentials
- Pole diagrams
- Existence and uniqueness theorem (if time - it's in the reading)

2 Review

Solve $mx'' + bx' + kx = 0$ (m, b, k positive constants)

Solution: Characteristic equation: $mr^2 + br + k = 0$.

Characteristic roots: $r = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}$.

Basic solutions depend on the type of roots. For example:

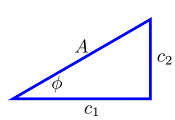
$$\begin{aligned} r = -2,\, -7 \quad &\longrightarrow x_1 = e^{-2t},\ x_2 = e^{-7t} \\ r = -2 \pm 7i \quad &\longrightarrow x_1 = e^{-2t}\cos(7t),\ x_2 = e^{-2t}\sin(7t) \\ r = -2,\, -2 \quad &\longrightarrow x_1 = e^{-2t},\ x_2 = te^{-2t} \end{aligned}$$

In all cases, the general solution is $x(t) = c_1x_1 + c_2x_2$, (c_1, c_2 constants).

2.1 Polar form of sinusoids

$$\underbrace{c_1\cos(\omega t) + c_2\sin(\omega t)}_{\text{rectangular form}} = \underbrace{\overset{\text{Amplitude}}{A}\cos(\omega t - \overset{\text{Phase lag}}{\phi})}_{\text{Polar form or amplitude-phase form}}$$

Relationship betweem c_1, c_2, A, ϕ:



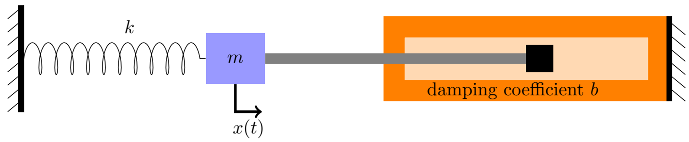
The figure shows $c_1 = \cos\phi$, $c_2 = \sin\phi$.

To see the two forms are equal use the cosine addition formula:

$$c_1\cos(\omega t) + c_2\sin(\omega t) = A\cos(\phi)\cos(\omega t) + A\sin(\phi)\sin(\phi) = A\cos(\omega t - \phi).$$

3 Damped harmonic oscillator

Here is one version: The ends of the spring and damper are fixed and there is no input driving the mass.



- k = spring constant
- b = linear damping constant
- m = mass
- x = displacement from equilibrium

Model: $mx'' + bx' + kx = 0$

Natural frequency (spring/mass): $\omega_0 = \sqrt{k/m}$, i.e., the frequency of the spring-mass with no damping: $mx'' + kx = 0$.

3.1 Solving $mx'' + bx' + kx = 0$

Roots: $r = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}$.

r real: overdamped

- 'b big', i.e., $b^2 - 4mk > 0$
- Roots are real and negative: $-r_2$, $-r_2$
- $x(t) = c_1 e^{-r_1 t} + c_2 e^{-r2t}$, – no oscillation, decays to $x = 0$, i.e., decays to equilibrium

r complex: underdamped

- 'b small', i.e., $b^2 - 4mk < 0$
- Roots are complex: $-\dfrac{b}{2m} \pm \beta i$, $\beta = \dfrac{\sqrt{|b^2 - 4mk|}}{2m}$
- Real parts are negative
- $x(t) = c_1 e^{-bt/2m} \cos(\beta t) + c_2 e^{-bt/2m} \sin(\beta t)$ – Oscillates, decays to $x = 0$ (equilibrium)

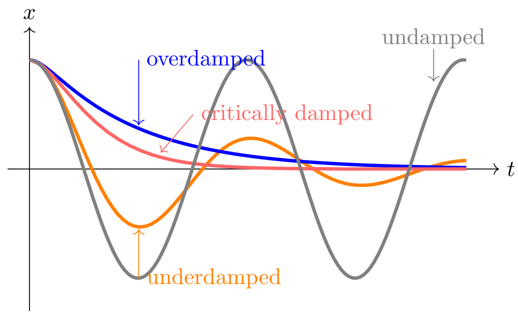
r repeated: critically damped

- 'b just right', i.e., $b^2 - 4mk < 0$
- Roots are real and negative: $-\dfrac{b}{2m}, -\dfrac{b}{2m}$
- $x(t) = c_1 e^{-bt/2m} + c_2 t e^{-bt/2m}$, – no oscillation, decays to $x = 0$ (equilibrium)

If initial velocity $x'(0) = 0$ (at rest)

- Overdamped: will not cross equilibrium for $t > 0$, i.e., $x(t) > 0$.
- Critically damped: same

- Underdamped: crosses equilibrium an infinite number of times



Damped harmonic oscillators starting from rest

4 Exponential decay rate

We know $e^{-t} \longrightarrow 0$ as $t \longrightarrow 0$.

e^{-3t}	decays to 0 like e^{-3t}
$e^{-3t}\cos t$	decays to 0 like e^{-3t}
te^{-3t}	decays to 0 like e^{-3t}
$e^{-3t} + e^{-2t}$	decays to 0 like e^{-2t}
$c_1e^{-2t} + c_2e^{-3t} + c_4e^{-4t}$	decays to 0 like e^{-2t}

5 Pole diagrams for linear, constant coefficient systems

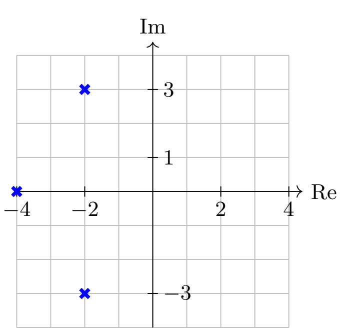
For the system $P(D)x = 0$ we can draw a pole diagram. This tells at a glance if solutions oscillate, if solutions go to 0 as t gets big and the decay rate of solutions.

Rules:

- In complex plane
- Put an $\times$ at each characteristic root
- The roots are also known as poles

Example 1. Suppose the roots are $-2 \pm 3i$, -4. Draw the pole diagram. Do solutions oscillate? Do they go to 0? How fast do the solutions decay?

Solution: We put an $\times$ at each of the roots (poles).



Complex roots $\longrightarrow$ the general solution is oscillatory.

All real parts < 0 (all poles in left half-plane) $\longrightarrow$ all solutions go to 0.

Decay determined by the root farthest to the right, i.e., solutions decay like e^{-2t}.

6 Existence and uniqueness

(Will do in class only if there is time.)

Why are 2 parameters enough to get all the solutions to a second-order DE?

Existence and uniqueness theorem: The DE with initial conditions:

$$mx'' + bx' + kx = 0, \quad x(t_0) = b_0,\, x'(t_0) = b_1$$

has a **unique** solution.

Proof. This makes physical sense. The mathematical analysis is challenging.

Important implication: What we called our general solution does, in fact, give us every possible solution.

Example 2. Consider $x'' + 8x' + 7x = 0$. Show that $x(t) = c_1e^{-t} + c_2e^{-7t}$ gives every possible solution.

Solution: The characteristic roots are -1, -7, so we know that $x(t) = c_1e^{-t} + c_2e^{-7t}$ are solutions. To show they give every solution, we have to show they cover every initial condtion.

So suppose we have initial conditions $x(t_0) = b_0$, $x'(t_0) = b_1$, then we have to find c_1 and c_2 to match these conditions. That is, we have to solve the algebraic system of equations

$$\begin{aligned} x(t_0) &= c_1e^{-t_0} + c_2e^{-7t_0} = b_0 \\ x'(t_0) &= -c_1e^{-t_0} - 7c_2e^{-7t_0} = b_1 \end{aligned}$$

The coefficient matrix $\begin{bmatrix} e^{-t_0} & e^{-7t_0} \\ -e^{-t_0} & -7e^{-7t_0} \end{bmatrix}$ is nonsingular (has an inverse). So there is always a solution to the equations. (In fact, exactly one solution.)

More generally, for n^{th} order DEs, we need n initial conditions. That is, the DE

$$a_nx^{(n)} + a_{n-1}x^{(n-1)} + ... + a_1x' + a_0x = 0$$

with initial conditions

$$x(t_0) = b_0,\, x'(t_0) = b_1,\, ...,\, x^{(n)}(t_0) = b_n.$$

has a unique solution.

This implies we need exactly n coefficients $c_1, ..., c_n$ in the general solution.